LIE BIALGEBRAS ARISING FROM POISSON BIALGEBRAS

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Abstract. It gives a method to obtain a natural Lie bialgebra from a Poisson bialgebra by an algebraic point of view. Let \( g \) be a coboundary Lie bialgebra associated to a Poission Lie group \( G \). As an application, we obtain a Lie bialgebra from a sub-Poisson bialgebra of the restricted dual of the universal enveloping algebra \( U(g) \).

Introduction

Assume throughout that \( G \) denotes a connected and simply connected Lie group with Lie algebra \( g \), \( O(G) \) the coordinate ring of \( G \) and \( U(g) \) the universal enveloping algebra of \( g \).

If \( G \) is a Poisson Lie group, then \( O(G) \) is a Poisson Hopf algebra and \( g \) becomes a Lie bialgebra. Conversely, if \( g \) has a Lie bialgebra structure, then \( G \) becomes a Poisson Lie group by [2, Chapter 1]. On the other hand, if \( U(g) \) has a co-Poisson Hopf structure with co-Poisson bracket \( \delta \), then \( (g, \delta_g) \) becomes a Lie bialgebra. Conversely if \( (g, \delta) \) is a Lie bialgebra, then the cobracket \( \delta \) extends uniquely to a Poisson co-bracket on \( U(g) \), which makes \( U(g) \) into a co-Poisson Hopf algebra (see [2, Proposition 6.2.3]). Moreover, the coordinate ring \( O(G) \) is isomorphic as a Hopf algebra to the restricted dual \( U^\circ(g) \) of \( U(g) \) and it is sometimes more convenient to work on \( U^\circ(g) \) than to do on \( O(G) \). For instance, Hodges and his colleagues worked on restricted duals to obtain mathematical properties of a quantum group in [3] and [4]. Hence it makes sense mathematically to study a relationship between Lie bialgebras and restricted duals of their enveloping algebras.

Let \( (A, \iota, m, \{\cdot, \cdot\}, \epsilon, \Delta) \) be a Poisson bialgebra and \( m = \ker \epsilon \). In 1.5, we prove by an algebraic point of view that the pair \( ((m/m^2)^*, m/m^2) \) is a natural Lie bialgebra obtained from \( (A, \iota, m, \{\cdot, \cdot\}, \epsilon, \Delta) \).

Let \( g \) be a coboundary Lie bialgebra. The restricted dual \( A \) of \( U(g) \) is the vector space spanned by all coordinate functions \( c^M_{f,v} \), where \( M \) is a finite
dimensional left $U(g)$-module and $f \in M^*, v \in M$. Here we give an explicit Poisson bracket on $A$ that is the Sklyanin Poisson bracket. Let $B$ be a sub-Poisson bialgebra of the restricted dual $A$. Then, as an application of 1.5, we obtain a Lie bialgebra $((m_B/m_B^2)^*, m_B/m_B^2)$ arising from $B$, where $m_B$ is the kernel of the counit in $B$.

Assume throughout that $k$ denotes a field of characteristic zero, all vector spaces considered here are over $k$ and if $A$ is a bialgebra with comultiplication $\Delta$, then we use Sweedler’s notation

$$\Delta(a) = \sum_{(a)} a' \otimes a'', \quad a \in A.$$ 

Recall that a Poisson algebra $A$ is a $k$-algebra with $k$-bilinear map $\{\cdot, \cdot\}$, called a Poisson bracket, such that

1. $A$ is a Lie algebra over $k$.

2. $\{ab, c\} = a\{b, c\} + \{a, c\}b$ for all $a, b, c \in A$. (Leibniz rule)

**1. Lie bialgebra arising from Poisson bialgebra**

**Definition 1.1.** A Poisson algebra $A$ with Poisson bracket $\{\cdot, \cdot\}$ is said to be a Poisson bialgebra if $A$ is also a bialgebra $(A, \iota, m, \epsilon, \Delta)$ over $k$ such that

1. $\Delta(\{a, b\}) = \{\Delta(a), \Delta(b)\}_{A \otimes A}$

for all $a, b \in A$, where the Poisson bracket $\{\cdot, \cdot\}_{A \otimes A}$ on $A \otimes A$ is defined by

$$\{a \otimes b, c \otimes d\}_{A \otimes A} = \{a, c\} \otimes bd + ac \otimes \{b, d\}$$

for all $a, b, c, d \in A$.

A Poisson bialgebra $A$ is often denoted by $A = (A, \iota, m, \{\cdot, \cdot\}, \epsilon, \Delta)$. If a Poisson bialgebra $A$ is a Hopf algebra, then $A$ is called a Poisson Hopf algebra (see [2, 6.2.1] and [1, III.5.3]).

**Lemma 1.2.** If $(A, \iota, m, \{\cdot, \cdot\}, \epsilon, \Delta)$ is a Poisson bialgebra, then $\epsilon(\{a, b\}) = 0$ for all $a, b \in A$.

**Proof.** By (2), we have that

$$\{a, b\} = m \circ (\epsilon \otimes \text{id}_A) \circ \Delta(\{a, b\})$$

$$= m \circ (\epsilon \otimes \text{id}_A)(\sum \{a', b'\} \otimes a''b'' + a'b' \otimes \{a'', b''\})$$

$$= \sum \epsilon(\{a', b'\})a''b'' + \sum \epsilon(a'b')\{a'', b''\}$$

$$= \sum \epsilon(\{a', b'\})a''b'' + \{a, b\}$$

for $a, b \in A$ and thus we have $\sum \epsilon(\{a', b'\})a''b'' = 0$. Hence

$$0 = \epsilon(\sum \epsilon(\{a', b'\})a''b'') = \sum \epsilon(\{a', b'\})\epsilon(a'')\epsilon(b'') = \epsilon(\{a, b\}),$$
as claimed. □

**Corollary 1.3.** In a Poisson bialgebra \((A, \iota, m, \{\cdot, \cdot\}, \epsilon, \Delta)\), set \(\ker \epsilon = m\). Then \(m/m^2\) is a Lie algebra with Lie bracket

\[
[a + m^2, b + m^2] = [a, b] + m^2, \quad a, b \in m.
\]

**Proof.** The Lie bracket (3) is well-defined by Lemma 1.2. Clearly \((m/m^2, [\cdot, \cdot])\) is a Lie algebra.

1.4. Let \((A, \iota, m, \epsilon, \Delta)\) be a bialgebra and set

\[A^\circ = \{f \in A^* \mid f(I) = 0\} \text{ for some ideal } I \text{ of } A \text{ such that } \dim(A/I) < \infty.\]

Then \(A^\circ\), called the restricted dual of \(A\), becomes a bialgebra with bialgebra structure: For \(f, g \in A^\circ\) and \(a, b \in A\),

\[(fg)(a) = \sum f(a')g(a''), \quad \Delta(f)(a \otimes b) = f(ab).\]

Denote

\[P_\epsilon(A^\circ) = \{f \in A^\circ \mid f(ab) = \epsilon(a)f(b) + f(a)\epsilon(b), \quad \forall a, b \in A\}.\]

That is, \(P_\epsilon(A^\circ) = \{f \in A^\circ \mid \Delta(f) = \epsilon \otimes f + f \otimes \epsilon\}\). It is well-known that \(P_\epsilon(A^\circ)\) is a Lie algebra with Lie bracket

\[[f, g] = fg - gf\]

for all \(f, g \in P_\epsilon(A^\circ)\).

Denote \(m = \ker \epsilon\) and let \(i : m \rightarrow A\) be the canonical injection. Then \(i^*\) is a surjection of \(A^*\) onto \(m^*\). Let \(f \in \ker i^*\). Then \(f(a - \epsilon(a)) = 0\) for all \(a \in A\) since \(f(m) = 0\) and \(a - \epsilon(a)1 \in m\). Thus \(f = f(1)\epsilon\) for \(f \in \ker i^*\). It follows that \(\ker i^* = k\epsilon\). Given \(f, g \in m^*\), choose representatives \(f' = i^{-1}(f), g' = i^{-1}(g)\). If \(f'_1 = i^{-1}(f), g'_1 = i^{-1}(g)\), then \(f'_1 = f' + \alpha \epsilon, g'_1 = g' + \beta \epsilon\) for some \(\alpha, \beta \in k\).

Thus

\[
f'_1g'_1 - g'_1f'_1 = (f' + \alpha \epsilon)(g' + \beta \epsilon) - (g' + \beta \epsilon)(f' + \alpha \epsilon)
\]

\[
= (f'g' + \beta f' + \alpha g' + \alpha \beta \epsilon) - (g'f' + \beta f' + \alpha g' + \alpha \beta \epsilon)
\]

\[
= f'g' - g'f'
\]

since \(\epsilon\) is the multiplicative identity in \(A^*\). Hence \([f, g] = i^*(f'g' - g'f')\) is independent of representatives and defines a Lie bracket on \(m^*\). Identifying \(\{f \in m^* \mid f(m^2) = 0\}\) with \((m/m^2)^*\), \((m/m^2)^*\) is a Lie subalgebra of \(m^*\) by [8, 2.1.2].

**Lemma.** The linear map

\[i^*|_{P_\epsilon(A^\circ)} : P_\epsilon(A^\circ) \rightarrow (m/m^2)^*, \quad f \mapsto i^*|_{P_\epsilon(A^\circ)}(f) = f|_m\]

is a Lie isomorphism.
Proof. Note that $A = k_1A \oplus m$ and if $f \in P(A^\circ)$, then $f(m^2) = 0$. Hence $i^*|_{P(A^\circ)}$ is well-defined. If $f \in \ker(i^*|_{P(A^\circ)})$, then
$$f(a_1A + a) = af(1_A) + f(a) = 0$$
for all $a \in k$ and $a \in m$. It follows that $i^*|_{P(A^\circ)}$ is injective. If $f \in m^*$ such that $f(m^2) = 0$, then $f$ is extended to $A$, denoted by $f'$, by setting
$$f'(k1_A) = 0, \quad f'|_m = f.$$ Then, for any $a, b \in k$ and $a \in m$,
$$f'((\alpha 1_A + a)(\beta 1_A + b)) = f(ab) + f(\beta a)$$
$$= \epsilon(\alpha 1_A + a)f'(\beta 1_A + b) + f'(\alpha 1_A + a)\epsilon(\beta 1_A + b).$$
Hence $f' \in P(A^\circ)$ and thus $i^*|_{P(A^\circ)}$ is surjective. Now $i^*|_{P(A^\circ)}$ is a Lie isomorphism by the definition of Lie brackets.

1.5. Let us recall the definition for Lie bialgebra in [2, 1.3] and [9, 2.1.1]. A Lie bialgebra is a pair $(\mathfrak{g}, \psi)$, where $\mathfrak{g}$ is a Lie algebra and $\psi : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$, called cobracket, satisfying the following conditions:

(a) The dual map $\psi^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ makes $\mathfrak{g}^*$ a Lie algebra.

(b) The cobracket $\psi : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ is a 1-cocycle on $\mathfrak{g}$ with respect to the $\mathfrak{g}$-module structure on $\mathfrak{g} \wedge \mathfrak{g}$ given by the adjoint action. In other words, we have that for any $a, b \in \mathfrak{g}$,
$$\psi([a, b]) = a \cdot \psi(b) - b \cdot \psi(a),$$
where
$$a \cdot (b \otimes c) = [a \otimes 1 + 1 \otimes a, b \otimes c] = [a, b] \otimes c + b \otimes [a, c].$$

In a Lie bialgebra $(\mathfrak{g}, \psi)$, a Lie ideal $\mathfrak{b}$ of $\mathfrak{g}$ is said to be a Lie bialgebra ideal if $\psi(\mathfrak{b}) \subseteq \mathfrak{g} \otimes \mathfrak{b} + \mathfrak{b} \otimes \mathfrak{g}$. A Lie homomorphism $\varphi : (\mathfrak{g}, \psi) \rightarrow (\mathfrak{g}', \psi')$ is said to be a Lie bialgebra homomorphism if $(\varphi \otimes \varphi) \circ \psi = \psi' \circ \varphi$. Note that if $\mathfrak{b}$ is a Lie bialgebra ideal of $(\mathfrak{g}, \psi)$, then $(\mathfrak{g}/\mathfrak{b}, \overline{\psi})$ is also a Lie bialgebra. A Lie bialgebra $(\mathfrak{g}, \psi)$ is frequently denoted by $(\mathfrak{g}, \mathfrak{g}^*)$.

**Theorem.** Let $(A, \iota, m, \{\cdot, \cdot\}, \epsilon, \Delta)$ be a Poisson bialgebra and let $m = \ker \epsilon$. Then $(\mathfrak{m}/\mathfrak{m}^2)^*, \mathfrak{m}/\mathfrak{m}^2$ is a Lie bialgebra.

**Proof.** We will show that the pair $((\mathfrak{m}/\mathfrak{m}^2)^*, \psi)$ is a Lie bialgebra, where $\psi : (\mathfrak{m}/\mathfrak{m}^2)^* \rightarrow (\mathfrak{m}/\mathfrak{m}^2)^* \wedge (\mathfrak{m}/\mathfrak{m}^2)^*$ is defined by
$$\psi(f)(z_1 \otimes z_2) = f([z_1, z_2])$$
for all $z_1, z_2 \in \mathfrak{m}/\mathfrak{m}^2$. It is enough to prove that $\psi$ is a 1-cocycle on $(\mathfrak{m}/\mathfrak{m}^2)^*$. The natural $\mathfrak{k}$-bilinear form $\langle \cdot, \cdot \rangle$ defined by
$$\langle \cdot, \cdot \rangle : (\mathfrak{m}/\mathfrak{m}^2)^* \times \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{k}, \quad \langle f, a + \mathfrak{m}^2 \rangle = f(a + \mathfrak{m}^2)$$
is a nondegenerate $k$-bilinear form. Identifying $(m/m^2)^*$ to $P_+(A^0)$ by 1.4, we have that, for $f, g \in (m/m^2)^*$ and $a, b \in m$,

$$
\langle \psi([f, g]), (a + m^2) \otimes (b + m^2) \rangle = \langle [f, g], \{a, b\} + m^2 \rangle = (g\{a, b\}) - (g f)(\{a, b\}) = \sum f(a'b')g(\{a'', b''\}) + f(\{a', b'\})g(\{a'' b''\})$$

by (2). Let

$$
\psi(f) = \sum f_1 \otimes f_2, \quad \psi(g) = \sum g_1 \otimes g_2.
$$

Then, by (4), we have

$$
f(\{a, b\}) = \langle \psi(f), (a + m^2) \otimes (b + m^2) \rangle = \sum f_1(a)f_2(b),$$

$$
g(\{a, b\}) = \langle \psi(g), (a + m^2) \otimes (b + m^2) \rangle = \sum g_1(a)g_2(b)
$$

for all $a, b \in m$. Hence

$$
(f \cdot \psi(g) - g \cdot \psi(f), (a + m^2) \otimes (b + m^2)) = \sum [f, g_1] \otimes g_2 + g_1 \otimes [f, g_2] - [g, f_1] \otimes f_2 - f_1 \otimes [g, f_2],$$

$$(a + m^2) \otimes (b + m^2)) = \sum f(a')g_1(a'')g_2(b) - g_1(a')f(a'')g_2(b)$$

$$+ g_1(a)f(b')g_2(b'') - g_1(a)g_2(b')f(b'')$$

$$- g(a')f_1(a'')f_2(b) + f_1(a'g(a''))f_2(b)$$

$$- f_1(a)g(b')f_2(b'') + f_1(a)f_2(b')g(b'')$$

$$= \sum f(a')g(\{a'', b\}) - g(\{a', b\})f(a'')$$

$$+ f(b')g(\{a', b''\}) - g(\{a', b'\})f(b'')$$

$$- g(a')f(\{a'', b\}) + f(\{a', b\})g(a'')$$

$$- g(b')f(\{a, b''\}) + f(\{a, b'\})g(b'').$$
Thus we have \( \psi([f, g]) = f \cdot \psi(g) - g \cdot \psi(f) \) for all \( f, g \in (m/m^2)^* \) and so \( \psi \) is a 1-cocycle as claimed. \( \square \)

**Example 1.6.** Let \( q \) be an indeterminate over \( k \). By [1, I.2.2], the coordinate ring of quantum \( n \times n \)-matrices, denoted by \( O_q(M_n(k)) \), is the \( k[q^{\pm 1}] \)-algebra generated by \( x_{ij} \), \( 1 \leq i, j \leq n \), subject to the relations

\[
x_{ij}x_{rs} - x_{rs}x_{ij} = \begin{cases} 
q x_{rs}x_{ij} & i = r \text{ and } j < s, \\
q^{-1} x_{js}x_{ri} & i < r \text{ and } j > s, \\
0 & i < r \text{ and } j = s, \\
q^{-1}(q - 1)(q + 1)x_{is}x_{jr} & i < r \text{ and } j < s.
\end{cases}
\]

Hence \( O_q(M_n(k))/\langle q - 1 \rangle \) is the commutative \( k \)-algebra \( k[x_{ij} \mid i, j = 1, \ldots, n] \). Moreover \( O_q(M_n(k))/\langle q - 1 \rangle \) is a Poisson algebra with Poisson bracket

\[
\{ x_{ij}, x_{rs} \} = (q - 1)^{-1}(x_{ij}x_{rs} - x_{rs}x_{ij})
\]

by [1, III.5.4]. More precisely, we have that

\[
\{ x_{ij}, x_{rs} \} = \begin{cases} 
\frac{x_{rs}}{q} x_{ij} & i = r \text{ and } j < s, \\
\frac{x_{js}}{q} x_{ri} & i < r \text{ and } j = s, \\
0 & i < r \text{ and } j > s, \\
2x_{is}x_{jr} & i < r \text{ and } j < s.
\end{cases}
\]

The coordinate ring of \( n \times n \)-matrices is the commutative \( k \)-algebra

\[
k[x_{ij} \mid i, j = 1, \ldots, n],
\]

denoted by \( O(M_n(k)) \), which is a bialgebra with the coalgebra structure

\[
\epsilon(x_{ij}) = \delta_{ij}, \quad \Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}.
\]

The algebra \( O(M_n(k)) \) is also a Poisson algebra with Poisson bracket

\[
\{ x_{ij}, x_{rs} \} = \begin{cases} 
x_{ij}x_{rs} & i = r \text{ and } j < s, \\
x_{ij}x_{rs} & i < r \text{ and } j = s, \\
0 & i < r \text{ and } j > s, \\
2x_{is}x_{jr} & i < r \text{ and } j < s.
\end{cases}
\]

by the above paragraph. Moreover \( O(M_n(k)) \) is a Poisson bialgebra since

\[
\Delta(\{ x_{ij}, x_{rs} \}) = \{ \Delta(x_{ij}), \Delta(x_{rs}) \}
\]

for all \( i, j, r, s \), any Poisson bracket satisfies the Leibniz rule and \( \Delta \) is an algebra homomorphism.
In \( m/m^2 \), set
\[
e_{ij} = x_{ij} + m^2, \quad e_{kk} = (x_{kk} - 1) + m^2, \quad i \neq j, \quad 1 \leq k \leq n.
\]
Then \( e_{ij}, i, j = 1, \ldots, n \), form a \( k \)-basis of \( m/m^2 \) and satisfy
\[
[e_{ii}, e_{is}] = e_{is} \quad i < s,
[e_{ii}, e_{is}] = -e_{is} \quad i > s,
[e_{ij}, e_{is}] = 0 \quad i \neq j, i \neq s,
[e_{ii}, e_{ri}] = e_{ri} \quad i < r,
[e_{ii}, e_{ri}] = -e_{ri} \quad i > r,
[e_{ij}, e_{rj}] = 0 \quad i \neq j, r \neq j,
[e_{ij}, e_{rs}] = 0 \quad i < r, j < s, i \neq s, r \neq j.
\]
by (5). The dual \( (m/m^2)^* \) has the dual basis \( e^*_{ij} \) for \( e_{ij}, i, j = 1, \ldots, n \), satisfying
\[
[e^*_{ij}, e^*_{rs}] = \delta_{jr} e^*_{is} - \delta_{is} e^*_{jr}
\]
for all \( i, j, r, s \). That is, \( (m/m^2)^* \) is isomorphic to the general linear Lie algebra \( gl_n(k) \). Moreover the pair \(( (m/m^2)^*, m/m^2 ) \) is a Lie bialgebra by 1.5. Now the cobracket \( \psi : (m/m^2)^* \rightarrow (m/m^2)^* \wedge (m/m^2)^* \) is given by
\[
\psi(e^*_{i}) = 0
\]
\[
\psi(e^*_{ij}) = e^*_{ij} \wedge e^*_{ij} + e^*_{ij} \wedge e^*_{ij} + \sum_{i < k < j} 2e^*_{ik} \wedge e^*_{kj} \quad i < j,
\]
\[
\psi(e^*_{ij}) = e^*_{ij} \wedge e^*_{ii} + e^*_{ij} \wedge e^*_{ij} + \sum_{j < k < i} 2e^*_{kj} \wedge e^*_{ik} \quad i > j.
\]

**Example 1.7.** Let \( b \) denote the Lie ideal \( k(\sum e^*_{ii}) \) of \( (m/m^2)^* \) in Example 1.6. Then \( b \) is a Lie bialgebra ideal since \( \psi(b) \subseteq (m/m^2)^* \otimes b + b \otimes (m/m^2)^* \) and thus \( (m/m^2)^*/b \) is also a Lie bialgebra. In fact, it is checked immediately that the Lie bialgebra \( (m/m^2)^*/b \) is isomorphic to the well-known Lie bialgebra \( (sl_n(k), \delta) \), where \( \delta : sl_n(k) \rightarrow sl_n(k) \wedge sl_n(k) \) is given by
\[
\delta(h_i) = 0, \quad \delta(E_{ii+1}) = h_i \wedge E_{i+1}, \quad \delta(E_{i+1,i}) = h_i \wedge E_{i+1,i},
\]
where \( E_{ii} \) is the \( n \times n \)-matrix with 0 for all positions except \( (i, j) \)-position and 1 and for \( (i, j) \)-position and \( h_i = E_{ii} - E_{i+1,i+1} \) for \( i = 1, \ldots, n - 1 \). (The cobracket \( \delta \) is uniquely determined by (6) since \( \delta \) is a 1-cocycle and \( sl_n(k) \) is generated by \( h_i, E_{ii+1}, E_{i+1,i}, i = 1, \ldots, n - 1 \).) The cobracket \( \delta \) in (6) is the standard Lie bialgebra structure in \( sl_n(k) \) (see \([2, 1.3.8]\) ).
2. Application

2.1. Let \( A = (A, \iota, m, \epsilon, \Delta) \) be a bialgebra. Note that the dual \( A^* \) is an \( A-A \) bimodule:

\[
(a\varphi b)(x) = \varphi(bax), \quad \varphi \in A^*, \quad a, \ b, \ x \in A.
\]

For a left \( A \)-module \( M \), the dual space \( M^* \) is a right \( A \)-module with structure

\[
(fa)(x) = f(ax), \quad a \in A, \ f \in M^*, \ x \in M.
\]

Let \( \mathcal{C} \) be a class of finite dimensional left \( A \)-modules which is closed under finite direct sums and finite tensor products. For any \( M \in \mathcal{C} \), \( f \in M^* \) and \( v \in M \), the coordinate function \( c_{M,f,v} \in A^* \) is defined by

\[
c_{M,f,v}(x) = f(xv), \quad x \in A.
\]

Then \( c_{M,f,v} \) is an element of the restricted dual \( A^\circ \) of \( A \) since the annihilator \( I \) of \( M \) is an ideal of \( A \) such that the dimension of \( A/I \) is finite and \( c_{M,f,v}(I) = 0 \). It is well-known that the vector space \( A^\mathcal{C} \) spanned by all coordinate functions \( c_{M,f,v} \), \( M \in \mathcal{C} \), \( f \in M^* \), \( v \in M \), is a sub-bialgebra of \( A^\circ \) with structure

\[
\Delta(c_{M,f,v}) = \sum_i c_{M,f_i,v_i} \otimes c_{M,f_i,v_i}, \quad \epsilon(c_{M,f,v}) = f(v),
\]

where \( \{v_i\} \) and \( \{f_i\} \) are dual bases for \( M \) and \( M^* \) (see [1, I.7]). Moreover if \( A \) is a Hopf algebra and \( \mathcal{C} \) is closed under duals, then \( A^\mathcal{C} \) is a Hopf algebra with antipode \( S \) defined by

\[
S(c_{M,f,v}) = c_{M,f,v}^*, \quad M \in \mathcal{C}, \ f \in M^*, \ v \in M.
\]

Observe that \( A^\mathcal{C} \) has a left and right \( A \)-action induced by (7):

\[
a \cdot c_{M,f,v} = c_{M,f,av}, \quad c_{M,f,v} \cdot a = c_{M,f,va}, \quad a \in A.
\]

2.2. Let \((\mathfrak{g}, \psi)\) be a Lie bialgebra and let \( \Delta \) be the comultiplication of \( U(\mathfrak{g}) \). The cobracket \( \psi \) is extended uniquely to a \( \Delta \)-derivation \( \overline{\psi} \) from \( U(\mathfrak{g}) \) into \( U(\mathfrak{g}) \otimes U(\mathfrak{g}) \). That is,

\[
\overline{\psi}: U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})
\]

is a \( k \)-linear map such that \( \overline{\psi}_\mathfrak{g} = \psi \) and \( \overline{\psi}(xy) = \overline{\psi}(x)\Delta(y) + \Delta(x)\overline{\psi}(y) \) for all \( x, y \in U(\mathfrak{g}) \).

Let \((\mathfrak{g}, \psi)\) be a coboundary Lie bialgebra such that the cobracket \( \psi \) determined by a classical \( r \)-matrix \( r = \sum_i a_i \otimes b_i \). That is, \( r \) satisfies the modified classical Yang-Baxter equation and \( \psi \) is defined by

\[
\psi(x) = x \cdot r = \sum_i [x, a_i] \otimes b_i + a_i \otimes [x, b_i] = [\Delta(x), r]_{U(\mathfrak{g}) \otimes U(\mathfrak{g})}
\]
for all \(x \in \mathfrak{g}\) (refer to [2, 2.1] and [9, §4.1] for the definition of a coboundary Lie bialgebra). Then the extension map \(\bar{\psi}\) of \(\psi\) to \(U(\mathfrak{g})\) is given by \(\bar{\psi}(x) = [\Delta(x), \tau]|_{U(\mathfrak{g})} \otimes U(\mathfrak{g})\) for all \(x \in U(\mathfrak{g})\).

**Theorem.** Let \((\mathfrak{g}, \psi)\) be a coboundary Lie bialgebra such that the cobracket \(\psi\) is determined by a classical \(r\)-matrix \(r\). Fix a class \(\mathcal{C}\) of finite dimensional left \(U(\mathfrak{g})\)-modules which is closed under finite direct sums and finite tensor products. Denote by \(A(\mathcal{C})\) the vector space spanned by all coordinate functions \(c_{f,v}^M, M \in \mathcal{C}, f \in M^*, v \in M\). Then \(A(\mathcal{C})\) is a Poisson bialgebra with Poisson bracket

\[
\{c_{f,v}^M, c_{g,w}^N\}(x) = (\bar{\psi}(x), c_{f,v}^M \otimes c_{g,w}^N)
\]

for all \(x \in U(\mathfrak{g})\).

**Remark.** Observe that, in the above theorem, \(A(\mathcal{C})\) is a sub-Poisson bialgebra of the restricted dual \(U(\mathfrak{g})^\circ\) and we obtain a Lie bialgebra \((\mathfrak{m}/\mathfrak{m}^2)^*, \mathfrak{m}/\mathfrak{m}^2)\) by applying 1.5 to \(A(\mathcal{C})\), where \(\mathfrak{m}\) is the kernel of the counit in \(A(\mathcal{C})\).

**Proof of Theorem.** We have already known that \(A(\mathcal{C})\) is a sub-bialgebra of the restricted dual \(U(\mathfrak{g})^\circ\) with structure (8) by 2.1.

Denote \(r = \sum_i a_i \otimes b_i\). Then

\[
\bar{\psi}(x) = [\Delta(x), \tau]|_{U(\mathfrak{g})} \otimes U(\mathfrak{g}) = \sum (x^a a_i \otimes x^b b_i - a_i x^a \otimes b_i x^b)
\]

for all \(x \in U(\mathfrak{g})\), thus

\[
\{c_{f,v}^M, c_{g,w}^N\}(x) = \sum_i \sum (c_{i,v}^M(x^a a_i) c_{g,w}^N(x^b b_i) - c_{i,v}^M(a_i x^a) c_{g,w}^N(b_i x^b))
\]

\[
= \sum_i (c_{i,v}^M c_{g,w}^N(a_i b_i) - c_{i,v}^M c_{g,w}^N b_i a_i)(x).
\]

Hence

\[
\{c_{f,v}^M, c_{g,w}^N\} = \sum_i (c_{i,v}^M c_{g,w}^N a_i b_i) - \sum_i (c_{i,v}^M c_{g,w}^N b_i a_i) \in A(\mathcal{C}),
\]

that is, the Poisson bracket (10) is well-defined.

Let \(\tau: U(\mathfrak{g}) \otimes U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})\) be the flip. Since \(U(\mathfrak{g})\) is cocommutative and \(\tau(r) = -r\), we have that \(\tau \bar{\psi}(x) = -\bar{\psi}(x)\) for all \(x \in U(\mathfrak{g})\), thus we have immediately that \(\{c_{f,v}^M, c_{g,w}^N\} = -\{c_{g,w}^N, c_{f,v}^M\}\) for all \(c_{f,v}^M, c_{g,w}^N \in A(\mathcal{C})\) by (10).

For distinct numbers \(s, t = 1, 2, 3\), denote by \(r_{st} \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}\) the element with \(a_i\) for \(s\)-component, \(b_i\) for \(t\)-component and 1 for the other component. For instance, \(r_{12} = \sum a_i \otimes b_i \otimes 1\) and \(r_{31} = \sum b_i \otimes 1 \otimes a_i\). Note that \(r_{st} = -r_{ts}\) for all distinct numbers \(s, t = 1, 2, 3\), by the skew symmetry of \(r\). Since \(\Delta(a) =\)
$$a \otimes 1 + 1 \otimes a$$ for all $a \in g$, we have

$$\{\{c_{f,v}^M, c_{g,w}^N\}, c_{h,u}^L\}(x) = (\Delta^2(x)(r_{13} + r_{23})r_{12}, c_{f,v}^M \otimes c_{g,w}^N \otimes c_{h,u}^L)$$

$$- (r_{12}\Delta^2(x)(r_{13} + r_{23}), c_{f,v}^M \otimes c_{g,w}^N \otimes c_{h,u}^L)$$

$$- (r_{12}(r_{13} + r_{23})\Delta^2(x) r_{12}, c_{f,v}^M \otimes c_{g,w}^N \otimes c_{h,u}^L)$$

$$+ (r_{12}(r_{13} + r_{23})\Delta^2(x), c_{f,v}^M \otimes c_{g,w}^N \otimes c_{h,u}^L)$$

for $x \in U(g)$, where $\Delta^2 = (\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$, by (10). Hence, by $r_{st} = -r_{ts}$ for all $s, t = 1, 2, 3$ and the coassociativity of $\Delta$, we have that

$$\{(\{c_{f,v}^M, c_{g,w}^N\}, c_{h,u}^L) + \{(c_{f,v}^M, c_{g,w}^N) + \{c_{f,v}^M, c_{g,w}^N\}\}(x)$$

$$= (\Delta^2(x)(r_{13} + r_{23})r_{12}, c_{f,v}^M \otimes c_{g,w}^N \otimes c_{h,u}^L)$$

$$- (r_{12}\Delta^2(x)(r_{13} + r_{23}), c_{f,v}^M \otimes c_{g,w}^N \otimes c_{h,u}^L)$$

$$- (r_{12}(r_{13} + r_{23})\Delta^2(x) r_{12}, c_{f,v}^M \otimes c_{g,w}^N \otimes c_{h,u}^L)$$

$$+ (r_{12}(r_{13} + r_{23})\Delta^2(x), c_{f,v}^M \otimes c_{g,w}^N \otimes c_{h,u}^L)$$

$$= 0$$

for any $c_{f,v}^M, c_{g,w}^N, c_{h,u}^L \in A(C)$ and $x \in U(g)$ since

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$$

$$= (r_{12}r_{13} - r_{13}r_{12}) + (r_{12}r_{23} - r_{23}r_{12}) + (r_{13}r_{23} - r_{23}r_{13})$$

$$= \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j + \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j + \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j]$$

is $g$-invariant. Hence the Poisson bracket (10) satisfies the Jacobi identity.
By (11), we have
\[ \{\ell^M_{f,v}, \ell^N_{g,w}\} = \sum c_{f,a,v}^M(c_{g,b,y}^N\ell^L_{b,u} + c_{g,b,u}^N\ell^L_{b,v}) - \sum c_{f,a,v}^M(c_{g,b,u}^N\ell^L_{b,v} + c_{g,b,v}^N\ell^L_{b,u}) = \{\ell^M_{f,v}, \ell^N_{g,w}\} \ell^L_{h,u} + c_{g,w}^N\{\ell^M_{f,v}, \ell^L_{h,u}\}. \]

It follows that the Poisson bracket (10) satisfies the Leibniz rule.

Let us prove that \( \Delta(\{\ell^M_{f,v}, \ell^N_{g,w}\}) = \{\Delta(\ell^M_{f,v}), \Delta(\ell^N_{g,w})\} \) for all elements \( \ell^M_{f,v}, \ell^N_{g,w} \in A(C) \). Note that \( \Delta(\ell^M_{f,v}) = \sum_{j,k} \ell^M_{f,v_j} \otimes \ell^N_{g,wh_k} \Delta(\ell^M_{f,v}) = \sum_k \ell^N_{g,wh_k} \otimes \ell^N_{g,w} \), where \( \{v_j\}, \{f_j\} \) are dual bases for \( M \) and \( M^* \) and \( \{w_k\}, \{g_k\} \) are dual bases for \( N \) and \( N^* \). Now, for any \( x, y \in U(g) \),
\[
\Delta(\{\ell^M_{f,v}, \ell^N_{g,w}\})(x \otimes y) = \langle \psi(x), \ell^M_{f,v} \otimes \ell^N_{g,w} \rangle
\]
\[
= \langle \psi(x)\Delta(y), \ell^M_{f,v} \otimes \ell^N_{g,w} \rangle + \langle \Delta(x)\psi(y), \ell^M_{f,v} \otimes \ell^N_{g,w} \rangle
\]
\[
= \sum_{j,k} \langle \psi(x), \ell^M_{f,v_j} \otimes \ell^N_{g,wh_k} \rangle \langle \Delta(y), \ell^M_{f,v} \otimes \ell^N_{g,w} \rangle
\]
\[
+ \sum_{j,k} \langle \Delta(x), \ell^M_{f,v_j} \otimes \ell^N_{g,wh_k} \rangle \langle \psi(y), \ell^M_{f,v} \otimes \ell^N_{g,w} \rangle
\]
\[
= \sum_{j,k} \{\ell^M_{f,v_j}, \ell^N_{g,wh_k}\} \otimes \{\ell^M_{f,v}, \ell^N_{g,w}\}(x \otimes y)
\]
\[
+ \sum_{j,k} \{\ell^M_{f,v_j}, \ell^N_{g,wh_k}\} \otimes \{\ell^M_{f,v}, \ell^N_{g,w}\}(x \otimes y)
\]
\[
= \{\Delta(\ell^M_{f,v}), \Delta(\ell^N_{g,w})\}(x \otimes y).
\]
Hence we have \( \Delta(\{\ell^M_{f,v}, \ell^N_{g,w}\}) = \{\Delta(\ell^M_{f,v}), \Delta(\ell^N_{g,w})\} \) for all elements \( \ell^M_{f,v}, \ell^N_{g,w} \in A(C) \). This completes the proof. \( \square \)

**Proposition 2.3.** Let \((g, \psi)\) be a coboundary Lie bialgebra such that \( g \) is connected and simply connected and let \( C \) be the set of all finite dimensional left \( U(g) \)-modules. Then \( A(C) \) is the restricted dual \( U(g)^\circ \). Moreover the given Lie bialgebra \((g, \psi)\) is isomorphic to \((m/m^2)^*, m/m^2\), where \( m \) is the kernel of the counit \( \epsilon \) of \( A(C) \).

**Proof.** Note that the set of all finite dimensional left \( U(g) \)-modules is closed under finite direct sums and finite tensor products. Since every element of the restricted dual \( U(g)^\circ \) is represented by a coordinate function \( c_{f,a,v}^M \) for some finite dimensional left \( U(g) \)-module \( M \), we have immediately that \( A(C) \) is the restricted dual \( U(g)^\circ \). Moreover \((m/m^2)^*, m/m^2\) is a Lie bialgebra by 2.2 and 1.5, and \( g = (m/m^2)^* \) by [7, 7.11]. Thus \( g^* \) is equal to \( m/m^2 \) as a Lie algebra by (3) and (10). It follows that the Lie bialgebra \((m/m^2)^*, m/m^2\) is equal to \((g, \psi) = (g, g^*)\). \( \square \)
Example 2.4. In the symplectic Lie algebra $\mathfrak{sp}_4$, set

\[
\begin{align*}
 h_1 &= E_{11} - E_{22} - E_{33} + E_{44}, \\
 e_1 &= E_{12} - E_{13}, \\
 f_1 &= E_{21} - E_{34}, \\
 e_2 &= E_{24}, \\
 f_2 &= E_{42}, \\
 e_3 &= E_{14} + E_{23}, \\
 f_3 &= E_{41} + E_{32}, \\
 e_4 &= E_{13}, \\
 f_4 &= E_{31}
\end{align*}
\]

(see [5, 8.3] for $\mathfrak{sp}_4$). Let $H$ be the subspace of $\mathfrak{sp}_4$ spanned by $h_1, h_2$ and let $\alpha_1, \alpha_2 \in H^*$ be defined by

\[
\begin{align*}
 \alpha_1(h_1) &= 2, & \alpha_2(h_1) &= -2, \\
 \alpha_1(h_2) &= -1, & \alpha_2(h_2) &= 2.
\end{align*}
\]

Then $e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4$ are weight vectors with weights

\[
\begin{align*}
 \text{wt}(e_1) &= \alpha_1, & \text{wt}(e_2) &= \alpha_2, & \text{wt}(e_3) &= \alpha_1 + \alpha_2, & \text{wt}(e_4) &= 2\alpha_1 + \alpha_2, \\
 \text{wt}(f_1) &= -\alpha_1, & \text{wt}(f_2) &= -\alpha_2, & \text{wt}(f_3) &= -(\alpha_1 + \alpha_2), & \text{wt}(f_4) &= -(2\alpha_1 + \alpha_2).
\end{align*}
\]

Hence $\alpha_1, \alpha_2$ are positive simple roots. It is well-known that

\[
\psi(e_i) = 0, \quad \psi(f_i) = 0, \quad \psi(e_i) = e_i \wedge h_1, \quad \psi(e_i) = 2e_i \wedge h_2, \\
\psi(e_i) = e_i \wedge h_1 + 2e_3 \wedge h_2 - 4e_1 \wedge e_2, \quad \psi(e_i) = 2e_4 \wedge h_1 + 2e_4 \wedge h_2 - 2e_1 \wedge e_3, \\
\psi(f_i) = f_i \wedge h_1, \quad \psi(f_i) = 2f_i \wedge h_2, \\
\psi(f_i) = f_i \wedge h_1 + 2f_3 \wedge h_2 - 4f_1 \wedge f_2, \quad \psi(f_i) = 2f_4 \wedge h_1 + 2f_4 \wedge h_2 - 2f_1 \wedge f_3
\]

(see [9, Exercise 4.1.11]).

The weight lattice $\mathbf{P}$ in $\mathfrak{sp}_4$ is a free abelian group with basis consisting of the fundamental dominant integral weights $\lambda_1, \lambda_2$, where $\lambda_i(h_j) = \delta_{ij}$ for $i, j = 1, 2$. Hence

\[
\alpha_1 = 2\lambda_1 - \lambda_2, \quad \alpha_2 = -2\lambda_1 + 2\lambda_2.
\]

The natural $\mathfrak{sp}_4$-module $V = \mathbf{k}^4$ is an irreducible highest weight module with highest weight $\lambda_1$. In fact, set

\[
\begin{align*}
 v_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
 v_2 &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \\
 v_3 &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \\
 v_4 &= \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}
\end{align*}
\]

Then $v_1$ is a highest weight vector with highest weight $\lambda_1$ and

\[
v_1 \in V_{\lambda_1}, v_2 = f_1 v_1 \in V_{-\lambda_1 + \lambda_2}, v_3 = f_2 v_2 \in V_{\lambda_1 - \lambda_2}, v_4 = f_1 v_3 \in V_{-\lambda_1}.
\]

Here we simply write $e_{f,v}$ for $e^V_{f,v}, v \in V, f \in V^*$. Observe that
where $\psi(x_2) = x_2 \wedge h_1 + 2x_2 \wedge h_2$,
\[ \psi(y_1) = y_1 \wedge h_1, \quad \psi(y_2) = y_2 \wedge h_1 + 2y_2 \wedge h_2. \]

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