EXPANSIONS OF REAL NUMBERS IN NON-INTEGER BASES

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ABSTRACT. The works of Erdős et al. about expansions of 1 with respect to a non-integer base \( q \), referred to as \( q \)-expansions, are investigated to determine how far they continue to hold when the number 1 is replaced by a positive number \( x \). It is found that most results about \( q \)-expansions for real numbers greater than or equal to 1 are in somewhat opposite direction to those for real numbers less than or equal to 1. The situation when a real number has a unique \( q \)-expansion, and when it has exactly two \( q \)-expansions are studied. The smallest base number \( q \) yielding a unique \( q \)-expansion is determined and a particular sequence is shown, in certain sense, to be the smallest sequence whose corresponding base number \( q \) yields exactly two \( q \)-expansions.

1. Introduction

Let \( q \in (1, 2] \). By a \( q \)-expansion of 1, we mean a sequence \((e_i)_{i \geq 1}\) of integers in \( \{0, 1\} \) satisfying the equality \( 1 = \sum_{i=1}^{\infty} e_i/q^i \). Such an expansion is not unique in general. There exist two particular expansions, constructed via the so-called greedy and lazy algorithms. In the greedy algorithm, we choose the biggest possible value for \( e_i \), while in the lazy algorithm, we choose the smallest possible value for \( e_i \).

In 1990, Erdős, Joo, and Komornik [4] began the work about characterizing the unique \( q \)-expansion of 1 for non-integer base \( q \). In 1991, Erdős, Horváth, and Joo [3] showed that for almost all \( q \in (1, 2] \), there are uncountably many different \( q \)-expansions, and surprisingly, there exist as well uncountably many exceptional \( q \in (0, 1) \) for which there is only one \( q \)-expansion. In 1998, Komornik and Loreti [5] determined the smallest base \( q \in (1, 2] \) for which the \( q \)-expansion of 1 is unique. In 1999, Komornik and Loreti [6] gave a sufficient condition for which the number 1 has exactly two different \( q \)-expansions as well as using this information to construct the smallest base \( q \) for which the number 1 has exactly two different \( q \)-expansions. In 2002, Dajani and Kraaikamp [2] studied the ergodic properties of non-greedy series expansions to non-integer

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bases $\beta > 1$. It was shown that the so-called lazy expansion is isomorphic to the greedy expansion. Furthermore, a class of expansions to bases $\beta > 1$, $\beta \not\in \mathbb{Z}$, in between the lazy and the greedy expansions are introduced and studied. These expansions are of the form $Tx = \beta x + \alpha \mod 1$. A more recent article with contents related to this work is [7].

In this paper, our overall objective is to investigate how far the results mentioned above, excluding the cardinality and the ergodicity ones, continue to hold for the positive number $x$ replacing the number $1$. In the next section, general results about greedy and lazy $q$-expansions are derived. It is found that most results about $q$-expansions for real numbers greater than or equal to $1$ are in somewhat opposite direction to those for real numbers less than or equal to $1$, which illustrate the remarkable standing of the number $1$ in this regard. Through the concept of U-sequences, we then investigate the situation when a real number has unique $q$-expansion and determine the smallest such base. Finally, the situation with exactly two $q$-expansions is studied and a particular sequence, first treated in [6], which becomes in certain sense the smallest sequence for certain positive number with corresponding base $q$ yielding exactly two $q$-expansions is considered.

Let $q \in (1, 2]$. By an expansion with respect to $q$, or $q$-expansion, of a positive real number $x$ we mean a sequence $(e_i)_{i \geq 1} \subseteq \{0, 1\}$ satisfying

$$\sum_{i=1}^{\infty} e_i q^i = x.$$ 

It is easily checked that $x$ has an expansion if and only if $0 \leq x \leq 1/(q-1)$.

The lexicographical order $\prec$ is defined as follows: given two real sequences $(a_i)$ and $(b_i)$, we write $(a_i) \prec (b_i)$ or $a_1 a_2 \cdots \prec b_1 b_2 \cdots$ if there exists a positive integer $n$ such that $a_i = b_i$ for all $i < n$, but $a_n < b_n$. It is easily checked that this is a complete ordering.

Using this lexicographical order, we now define three special sequences, termed D-, U- and T-sequences. The notions of these three sequences were first considered by Komornik and Loreti [6].

A sequence $(a_i)_{i \geq 1} \subseteq \{0, 1\}$ is called a D-sequence if

$$\sum_{i=1}^{\infty} e_i \epsilon_i q^i = x.$$ 

where for brevity we write $\epsilon_i$ for $1-e_i$ and $s$ for $\epsilon_1 \epsilon_2 \cdots$ if $s = (\varepsilon_i) \subseteq \{0, 1\}$.

If $(a_i)$ begins with $N$ ($\geq 2$) consecutive 1 digits and if there are neither $N$ consecutive 1 digits, nor $N$ consecutive 0 digits later, then it is easily checked that $(a_i)$ is a U-sequence.
A sequence \((e_i)_{i \geq 1} \subseteq \{0, 1\}\) is called a T-sequence if the following three conditions hold:

1. \((e_{n+i}) \prec (e_i)\) whenever \(e_n = 0\) (i.e., \((e_i)\) is D-sequence);
2. there exists a positive integer \(m\) such that \(e_m = 1\), and
3. there exists a sequence \((e_i)_{i \geq 1} \subseteq \{0, 1\}\) defined by \(e_{i+m} + \varepsilon_i \in \{0, 1\}\) \((i \geq 1)\), such that if the sequence \((\delta_i)_{i \geq 1} \subseteq \{0, 1\}\) is defined by

\[
\delta_i = \begin{cases} 
  e_i & \text{if } i < m \\
  0 & \text{if } i = m \\
  e_i + \varepsilon_{i-m} & \text{if } i > m,
\end{cases}
\]

then the following three requirements hold:

1. \((\delta_{n+1}\delta_{n+2} \cdots \prec e_1e_2 \cdots)\) whenever \(\delta_n = 1\),
2. \((\delta_{n+1}\delta_{n+2} \cdots \prec e_1e_2 \cdots)\) whenever \(e_n = 1\) and \(n > m\),
3. \((\delta_{n+1}\delta_{n+2} \cdots \prec e_1e_2 \cdots)\) whenever \(\delta_n = 0\) and \(n > m\).

Komornik and Loreti [6] showed that if \((e_i)\) is a T-sequence with \(e_i = \varepsilon_i\), then there exists a \(q \in (1, 2]\) such that 1 has exactly two expansions.

A real number \(q \in (1, 2]\) is called a T-base number if there exists a positive real number \(x\) with exactly two different \(q\)-expansions.

As a general, preliminary result, we have:

**Theorem 1.1.** Let \((e_i) \subseteq \{0, 1\}\). Then the map

\[
q \mapsto \sum_{i \geq 1} e_i/q^i
\]

is continuous and strictly decreasing from the interval \((1, 2]\) onto the interval \(\left[\sum_{i \geq 1} e_i/2^i, \sum_{i \geq 1} e_i\right]\).

**Proof.** Let \(q \in (1, 2]\) and \(F(q) = \sum_{i \geq 1} e_i/q^i\). That this map is strictly decreasing is clear. If \(q_1 < q_2\), then

\[
|F(q_1) - F(q_2)| = \sum_{i \geq 1} \left| \frac{e_i (q_1^i - q_2^i)}{q_1^i q_2^i} \right| \leq \left| q_1 - q_2 \right| \sum_{i \geq 1} \frac{i}{q_1^i q_2^i},
\]

showing that \(F\) is continuous. That this map is onto follows from the intermediate value theorem for continuous functions. \(\Box\)

2. Greedy expansions

Let \(q \in (1, 2]\) and \(x \in [0, 1/(q - 1)]\). We define the greedy \(q\)-expansion \((a_i) \subseteq \{0, 1\}\) of \(x\) as follows: if for some positive integer \(n\), the numbers \(a_i\) are defined for all \(i < n\), then set \(a_n = 1\) whenever

\[
\sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n} \leq x,
\]
and $a_n = 0$ otherwise, where the summation is taken as 0 if $n = 1$.

Our next result reveals some intrinsic relations between the greedy $q$-expansion of a number in $[1, 1/(q - 1)]$ and that of any non-negative real number in $[0, 1/(q - 1)]$.

**Theorem 2.1.** Let $q \in (1, 2]$, $y \geq 1$ and let $(e_i)$ be the greedy $q$-expansion of $y$.

(a) The greedy $q$-expansion, $(a_i)$, of any $x \in [0, 1/(q - 1)]$ satisfies

$$a_{n+1}a_{n+2} \cdots < e_1e_2 \cdots$$

whenever $a_n = 0$.

(b) If the sequence $(e_i)$ is finite with a last nonzero digit $e_k$, then no greedy $q$-expansion is eventually periodic with the period $e_1e_2 \cdots e_{k-1}(e_k - 1)$.

**Proof.** (a) Assume that $a_n = 0$. If $(a_{n+i}, a_{n+1}) > (e_i)$, then there exists an integer $k$ such that $a_{n+i} = e_i$ for $i = 1, 2, \ldots, k - 1$, but $a_{n+k} > e_k$. Thus $e_k = 0$ and $a_{n+k} = 1$ and so, by the definition of greedy $q$-expansion of $y$,

$$\sum_{i=1}^{k-1} \frac{e_i}{q^i} + \frac{1}{q^k} > y.$$

Thus

$$\sum_{i=1}^{\infty} \frac{a_{n+i}}{q^i} \geq \frac{\sum_{i=1}^{k-1} a_{n+i}}{q^i} + \frac{1}{q^k} = \frac{\sum_{i=1}^{k-1} e_i}{q^i} + \frac{1}{q^k} > y,$$

and so

$$x = \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{a_n}{q^n} + \frac{1}{q^n} \left( \frac{a_{n+1}}{q} + \frac{a_{n+2}}{q^2} + \cdots \right) > \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{a_n}{q^n} + \frac{y}{q^n} \geq \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n},$$

contradicting the definition of the greedy $q$-expansion of $x$ (because $a_n = 0$).

If $(a_{n+i} = e_i)$, then $a_{n+i} = e_i$ for all $i \geq 1$. Thus

$$x = \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{a_n}{q^n} + \frac{1}{q^n} \left( \frac{a_{n+1}}{q} + \frac{a_{n+2}}{q^2} + \cdots \right)$$

$$= \sum_{i=1}^{n-1} \frac{e_i}{q^i} + \frac{e_n}{q^n} + \frac{1}{q^n} \left( \frac{e_1}{q} + \frac{e_2}{q^2} + \cdots \right)$$

$$= \sum_{i=1}^{n-1} \frac{e_i}{q^i} + \frac{e_n}{q^n} + \frac{y}{q^n}$$

$$\geq \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n},$$

again contradicting the definition of the greedy $q$-expansion of $x$ (because $a_n = 0$).
(b) Assume on the contrary that the greedy \( q \)-expansion \((a_i)\) of some \( x \in [0,1/(q-1)] \) is eventually periodic with period \( e_1 e_2 \cdots e_{k-1} (e_k - 1) \). Since
\[
y - \frac{1}{q^k} = e_1 q^{-1} + \cdots + e_{k-1} q^{-k-1},
\]
we have
\[
x = \left( \frac{a_1}{q} + \cdots + \frac{a_r}{q^r} \right) + \frac{1}{q^r} \left( \frac{e_1}{q} + \cdots + \frac{e_{k-1}}{q^{k-1}} \right) + \frac{1}{q^{r+k}} + \cdots
\]
\[
= \left( \frac{a_1}{q} + \cdots + \frac{a_r}{q^r} \right) + \frac{1}{q^r} \left( \frac{e_1}{q} + \cdots + \frac{e_{k-1}}{q^{k-1}} \right) + \left( y - \frac{1}{q^k} \right) \left( \frac{1}{q^{r+k}} \right)
\]
\[
\geq \left( \frac{a_1}{q^r} + \cdots + \frac{a_r}{q^r} \right) + \frac{1}{q^r} \left( \frac{e_1}{q^r} + \cdots + \frac{e_k}{q^k} \right) = \sum_{i=1}^{r+k-1} \frac{a_i}{q^i} + \frac{1}{q^{r+k}},
\]
contradicting the definition of the \( q \)-greedy expansion of \( x \) (because \( a_{r+k} = 0 \)). □

**Remarks.**
1) The case where \( y = 1 \) is Lemmas 2(a) in [2] and Lemma 1.4(a) in [5].
2) The converse of Theorem 2.1(a) is not true, i.e., there exists an \( x \in [0,1/(q-1)] \), whose \( q \)-expansion, \((a_i)\), satisfies the condition (2.1), but this expansion is not the greedy \( q \)-expansion of \( x \), as seen in the following example.

**Example.** Take \( q = \frac{9}{5} \) and \( x = y = \frac{27199387096045}{22876792454961} = 27199387096045/914 \geq 1 \). Here
\[
(e_i) = (1,1,1,1,1,0,0,1,0,0,0,0,0,0,0,0,0,0,\ldots),
\]
the expression holding up to the first eighteen digits, is the greedy \( q \)-expansion of \( x = y \) and
\[
(a_i) = (1,1,1,1,1,0,0,0,0,1,1,0,1,1,1,1)
\]
is a finite \( q \)-expansion of \( x = y \) which satisfies the condition (2.1), but \((a_i)\) is not a greedy \( q \)-expansion.

Next we derive more characterizations of greedy \( q \)-expansions.

**Theorem 2.2.** Let \( q \in [1,2] \). A sequence \((a_i)\) is the greedy \( q \)-expansion of \( x \) if and only if \( \sum_{i=1}^{\infty} a_{k+1}/q^i < 1 \) whenever \( a_k = 0 \).

**Proof.** Let \((a_i)\) be the greedy \( q \)-expansion of \( x \) and assume \( a_k = 0 \). By definition,
\[
\sum_{i=1}^{k-1} \frac{a_i}{q^i} + \frac{1}{q^k} > x,
\]
and so
\[ \frac{1}{q^k} = \sum_{i=k}^{\infty} \frac{a_i}{q^i} = \sum_{i=1}^{\infty} \frac{a_i}{q^i} = \sum_{i=1}^{\infty} a_{k+i}. \]

The required inequality follows after multiplying by \( q^k \).

Assume \( \sum_{i=1}^{\infty} a_{k+i}/q^i < 1 \) whenever \( a_k = 0 \). If \( a_n = 1 \), then
\[ x = \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n} + \sum_{i>n} \frac{a_i}{q^i} \geq \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n}. \]

If \( a_n = 0 \), then \( \sum_{i=1}^{\infty} a_{n+i}/q^i < 1 \), and so \( \sum_{i=1}^{\infty} a_{n+i}/q^{n+i} < 1/q^n \). Thus
\[ x = \sum_{i=1}^{\infty} \frac{a_i}{q^i} = \sum_{i=1}^{\infty} \frac{a_i}{q^i} + \sum_{i=1}^{\infty} a_{n+i}/q^{n+i} < \sum_{i=1}^{\infty} \frac{a_i}{q^i} + \frac{1}{q^n}, \]
i.e., \( (a_i) \) is the greedy \( q \)-expansion of \( x \). \( \square \)

**Remark.** Theorem 2.2 is Lemma 1(a) in [2], but the proof here is different.

**Theorem 2.3.** Let \( q \in (1, 2] \).

(a) Let \((e_i)\) be an infinite \( q \)-expansion of \( y \in [0, 1] \) and let \((a_i)\) be a \( q \)-expansion of \( x \in [0, 1/(q-1)] \). If the condition (2.1) holds, then \((a_i)\) is the greedy \( q \)-expansion of \( x \).

(b) Let \((e_i)\) be a \( q \)-expansion of \( y \in [0, 1] \) and let \((a_i)\) be a finite \( q \)-expansion of \( x \in [0, 1/(q-1)] \). If the condition (2.1) holds, then \((a_i)\) is the greedy \( q \)-expansion of \( x \).

(c) Let \((e_i)\) be a finite \( q \)-expansion of \( y \in [0, 1] \) and denote by \( e_k \) its last nonzero element. Let \((a_i)\) be a \( q \)-expansion of \( x \in [0, 1/(q-1)] \). Assume (2.1) holds.

(c.1) If \( y < 1 \), then \((a_i)\) is the greedy \( q \)-expansion of \( x \).

(c.2) If \( y = 1 \) and assume that \((a_i)\) is not eventually periodic with period \( e_1 \cdots e_{k-1}(e_k - 1) \), then \((a_i)\) is the greedy \( q \)-expansion of \( x \).

**Proof.** There is nothing to prove if \( a_n = 1 \), while for those \( n \) with \( a_n = 0 \), the results follow from Theorem 2.2 if we can show that
\[ \sum_{i=1}^{\infty} \frac{a_{n+i}}{q^i} < 1. \]

From (2.1), there is a sequence of integers \( n = k_0 < k_1 < \cdots \) satisfying the conditions: with \( j \in \mathbb{N}, \)
\[ a_{k_{j-1}+i} = e_i \text{ for all } 1 \leq i < k_j - k_{j-1} \text{ and } a_{k_j} = e_{k_j - k_{j-1}}. \]

(a) If the sequence \((e_i)\) is infinite, then
\[ \frac{1}{q^n} \sum_{i=1}^{\infty} \frac{a_{n+i}}{q^i} = \sum_{j=1}^{\infty} \frac{k_j - k_{j-1} - 1}{q^{k_{j-1}+i}} = \sum_{j=1}^{\infty} \left( \frac{1}{q^{k_{j-1}+i}} - \frac{1}{q^{k_j}} \right) \]
Suppose that the conclusion is false. We have two possible cases.

(b) If the sequence \((a_i)\) is finite, assume that there exists a positive integer \(m\) satisfying \(a_i = 0\) for all \(i > k_m\). Now

\[
\frac{1}{q^m} \sum_{i=1}^{m} \frac{a_{n+i}}{q^i} = \sum_{j=1}^{m} \left( \frac{k_j - k_{j-1}}{q^{k_{j-1}+1}} \right) = \sum_{j=1}^{m} \left( \frac{e_i}{q^{k_{j-1}+1}} - \frac{1}{q^{k_j}} \right)
\]

proving (2.2).

(c) If the sequence \((e_i)\) is finite, proceeding as in the proof of (a) leads to (2.3) with strict inequality being now non-strict. Observe that \(e_{k_j} - e_{k_{j-1}} = 1\) so \(k_j - k_{j-1} \leq k\). A closer inspection of the proof reveals that we obtain equality exactly when \(y = 1\) and \(k_j - k_{j-1} = k\) for every \(j\), i.e., when the sequence \((a_{n+i})\) is periodic with period \(e_1 \cdots e_{k-1}(e_k - 1)\). This contradicts the fact that \((a_i)\) is not eventually periodic with period \(e_1 \cdots e_{k-1}(e_k - 1)\). Hence, \((a_i)\) is the greedy \(q\)-expansion of \(x\).

\[\square\]

Remains. 1) Theorem 2.3 (a), (b) is Lemma 3 in [2] and the proofs given here are the same. Lemma 1.5 (a) in [5] is a special case of Theorem 2.3 (c.2) above. 2) The converse of Theorem 2.3 (c.1) is not true, i.e., there exist \(y\) with finite \(q\)-expansion \((e_i)\), and \(x\) with greedy \(q\)-expansion \((a_i)\), such that \((a_i)\) does not satisfy the condition (2.1) as seen in the following example.

Example. Take \(q = 4/3\). We have

\((a_i) = (1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)\)

is the greedy expansion of \(x = \frac{68001788610072039914841}{75557863725914323419136}\). Taking \(y = x < 1\), we get \((a_i) = (e_i)\). Note that \((a_i)\) does not satisfy the condition (2.1).

Theorem 2.4. Let \(q, q' \in (1, 2]\), \(x \in [0, 1/(q-1)] \cap [0, 1/(q'-1)]\). Let \((e_i)\) and \((e'_i)\) be the greedy \(q\)-expansion, respectively, greedy \(q'\)-expansion of \(x\). If \(q < q'\), then \((e_i) \prec (e'_i)\).

Proof. Suppose that the conclusion is false. We have two possible cases.

Case 1: \((e_i) = (e'_i)\). Thus \(x = \sum_{i=1}^{\infty} e'_i/(q')^i = \sum_{i=1}^{\infty} e_i/q^i = x\), a contradiction.

Case 2: \((e_i) \succ (e'_i)\). Thus there exists an integer \(n\) such that \(e_i = e'_i\) for all \(1 \leq i < n\) but \(e_n > e'_n\). We must have \(e_n = 1\) and \(e'_n = 0\). By the definition of the greedy \(q\)-expansion,

\[
\sum_{i=1}^{n-1} e'_i/q^n + \frac{1}{q^n} \leq \sum_{i=1}^{n-1} e_i/q^i + \frac{1}{q^n} < \sum_{i=1}^{n-1} e_i/q^i + \frac{1}{q^n} \leq x,
\]

contradicting the definition of greedy \(q'\)-expansion of \(x\) as \(e'_n = 0\). \[\square\]
3. Lazy expansions

Let \( q \in (1, 2] \), \( y \in [0, 1/(q - 1)] \). The lazy \( q \)-expansion \((b_i)\) of \( y \) is defined as follows: if for some positive integer \( n \) the numbers \( b_i \) are defined for all \( i < n \), then set \( b_n = 0 \) whenever

\[
\sum_{i=1}^{n-1} \frac{b_i}{q^{i}} + \sum_{i>n} \frac{1}{q^{i}} \geq y,
\]

and set \( b_n = 1 \) otherwise, where the summation is taken as 0 if \( n = 1 \).

Lazy \( q \)-expansions enjoy two simple properties which we now describe.

**Property L1.** A real number \( y \in [0, 1/(q - 1)] \) has \((b_i)\) as its lazy \( q \)-expansion if and only if the sequence \((a_i) := (1 - b_i)\) is the greedy \( q \)-expansion of \( \frac{1}{q-1} - y \) (This “duality” property implies that every \( y \in [0, 1/(q - 1)] \) has a lazy \( q \)-expansion).

**Proof.** First observe that

\((b_i)\) is a \( q \)-expansion of \( y \) if and only if \( \sum_{i=1}^{\infty} b_i/q^i = y \) if and only if \( \sum_{i=1}^{\infty} (1 - b_i)/q^i = \frac{1}{q-1} - y \) if and only if \((1 - b_i)\) is a \( q \)-expansion of \( \frac{1}{q-1} - y \).

Assume that \((b_i)\) is the lazy \( q \)-expansion of \( y \). If \( 1 - b_n = 0 \), then

\[
y > \sum_{i=1}^{n-1} \frac{b_i}{q^{i}} + \sum_{i>n} \frac{1}{q^{i}},
\]

and so

\[
\frac{1}{q-1} - y < \frac{1}{q-1} - \sum_{i=1}^{n-1} \frac{b_i}{q^{i}} - \sum_{i>n} \frac{1}{q^{i}} = \sum_{i=1}^{n-1} \frac{1 - b_i}{q^{i}} + \frac{1}{q^{n}}.
\]

If \( 1 - b_n = 1 \), then \( y \leq \sum_{i=1}^{n-1} b_i/q^i + \sum_{i>n} 1/q^i \), and so

\[
\frac{1}{q-1} - y \geq \frac{1}{q-1} - \sum_{i=1}^{n-1} \frac{b_i}{q^{i}} - \sum_{i>n} \frac{1}{q^{i}} = \sum_{i=1}^{n-1} \frac{1 - b_i}{q^{i}} + \frac{1}{q^{n}}.
\]

Thus \((1 - b_i)\) is the greedy \( q \)-expansion of \( \frac{1}{q-1} - y \).

Assume that \((1 - b_i)\) is the greedy \( q \)-expansion of \( \frac{1}{q-1} - y \). If \( b_n = 0 \), then

\[
\frac{1}{q-1} - y \geq \sum_{i=1}^{n-1} (1 - b_i)/q^i + 1/q^n,
\]

and so

\[
y \leq \frac{1}{q-1} - \sum_{i=1}^{n-1} \frac{1 - b_i}{q^{i}} - \frac{1}{q^n} = \sum_{i=1}^{n-1} \frac{b_i}{q^{i}} + \sum_{i>n} \frac{1}{q^{i}}.
\]

If \( b_n = 1 \), then \( \frac{1}{q-1} - y < \sum_{i=1}^{n-1} (1 - b_i)/q^i + 1/q^n \), and so \( y > \sum_{i=1}^{n-1} b_i/q^i + \sum_{i>n} 1/q^i \). Thus \((b_i)\) is the lazy \( q \)-expansion of \( y \). \( \square \)

**Property L2.** If \((a_i)\) and \((b_i)\) are the greedy and lazy \( q \)-expansions, respectively, of \( x \), and if there exists another \( q \)-expansion \((c_i)\) of \( x \), then

\[
(b_i) \preceq (c_i) \preceq (a_i)
\]
(In other words, the greedy $q$-expansion is the greatest $q$-expansion and the lazy $q$-expansion is the smallest $q$-expansion of $x$ lexicographically).

**Proof.** Let $(a_i)$ and $(b_i)$ be the greedy, respectively, lazy $q$-expansions of $x$ and let $(c_i)$ be another $q$-expansion of $x$.

To show that $(b_i) \preceq (c_i)$, assume $(b_i) \succ (c_i)$. Then there exists an integer $n$ such that $b_i = c_i$ for all $1 \leq i < n$ but $b_n > c_n$. Thus $b_n = 1$ and $c_n = 0$. By the definition of lazy $q$-expansion, we have

$$\sum_{i=1}^{n-1} \frac{b_i}{q^i} + \sum_{i=n+1}^{\infty} \frac{1}{q^i} < x = \sum_{i=1}^{\infty} \frac{c_i}{q^i} = \sum_{i=1}^{n-1} \frac{c_i}{q^i} + \sum_{i=n}^{\infty} \frac{c_i}{q^i}.$$ 

Thus

$$\sum_{i=n+1}^{\infty} \frac{1}{q^i} < \sum_{i=n}^{\infty} \frac{c_i}{q^i},$$

contradicting the definition of the sequence $(c_i) \subseteq \{0, 1\}$.

To show that $(c_i) \preceq (a_i)$, assume $(c_i) \succ (a_i)$. Then there exists an integer $n$ such that $c_i = a_i$ for all $1 \leq i < n$ but $c_n > a_n$. Thus $c_n = 1$ and $a_n = 0$. By the definition of greedy $q$-expansion, we have

$$\sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n} > x = \sum_{i=1}^{\infty} \frac{c_i}{q^i} = \sum_{i=1}^{n-1} \frac{c_i}{q^i} + \sum_{i=n}^{\infty} \frac{c_i}{q^i},$$

which implies $0 > \sum_{i=n+1}^{\infty} \frac{c_i}{q^i}$, again contradicting the definition of the sequence $(c_i) \subseteq \{0, 1\}$. □

**Remark.** Properties L1 and L2 are well known and have appeared in several articles and with quite short proofs, e.g. [6] and [1], where in the latter paper simple and short dynamical proofs are given. We give the above proofs for two reasons; first, they are elementary and second, to make this exposition self-contained.

We next derive further characterizations of lazy $q$-expansions.

**Theorem 3.1.** Let $q \in (1, 2]$, $x \in [0, 1/(q-1)]$. Then $(b_i)$ is the lazy $q$-expansion of $x$ if and only if $\sum_{i=1}^{\infty} (1 - b_{k+1})/q^i < 1$ whenever $b_k = 1$.

**Proof.** Let $(b_i)$ be the lazy $q$-expansion of $x$. Assuming $b_k = 1$, we get

$$\sum_{i=1}^{k} \frac{b_i}{q^i} + \sum_{i=k+1}^{\infty} \frac{1}{q^i} < x + \frac{1}{q^k} = \sum_{i=1}^{\infty} \frac{b_i}{q^i} + \frac{1}{q^k},$$

and so

$$\sum_{i=1}^{\infty} \frac{1 - b_{k+i}}{q^i} < 1.$$
Conversely, assume $\sum_{i=1}^{\infty} (1 - b_{k+i})/q^i < 1$ whenever $b_k = 1$. If $b_n = 0$, then
\[
x = \sum_{i=1}^{n-1} \frac{b_i}{q^i} + \sum_{i=n+1}^{\infty} \frac{b_i}{q^i} \leq \sum_{i=1}^{n-1} \frac{b_i}{q^i} + \sum_{i=n+1}^{\infty} \frac{1}{q^i}.
\]
If $b_n = 1$, then from the assumption we have $\sum_{i=n}^{\infty} (1 - b_{n+i})/q^{n+i} < 1/q^n$, and so $\sum_{i=1}^{\infty} 1/q^{n+i} + \sum_{i=0}^{n-1} b_i/q^i < x + 1/q^n$, i.e., $\sum_{i=1}^{n-1} b_i/q^i + \sum_{i=n+1}^{\infty} 1/q^i < x$, showing that the $q$-expansion is lazy.

**Remark.** Theorem 3.1 is Lemma 1(b) in [2], but the proof here is different.

**Theorem 3.2.** Let $(e_i)$ be an infinite $q$-expansion of $y \leq 1$. If another infinite $q$-expansion $(b_i)$ of $x \in [0, 1/(q - 1)]$ satisfies the condition
\[
(3.1) \quad (1 - b_{n+i}) < (e_i) \text{ whenever } b_n = 1,
\]
then $(b_i)$ is the lazy $q$-expansion of $x$.

**Proof.** By Theorem 3.1, it suffices to show that if $b_k = 1$, then
\[
(3.2) \quad \sum_{i=1}^{\infty} \frac{1 - b_{k+i}}{q^i} < 1.
\]
Let $b_k = 1$. By hypothesis, there is a sequence of integers $k = k_0 < k_1 < \ldots$ satisfying for each $j = 1, 2, \ldots$ the conditions
\[
1 - b_{k_j-i} = e_i \text{ when } 1 \leq i < k_j - k_{j-1}, \text{ and } 1 - b_{k_j} < e_{k_j-k_{j-1}}.
\]
We have
\[
\frac{1}{q^k} \sum_{i=1}^{\infty} \frac{1 - b_{k+i}}{q^i} = \sum_{j=1}^{\infty} \left( \sum_{i=1}^{k_j-k_{j-1}} \frac{1 - b_{k_{j-1}+i}}{q^{k_{j-1}+i}} \right) = \sum_{j=1}^{\infty} \left( \sum_{i=1}^{k_j-k_{j-1}} \frac{e_i}{q^{k_{j-1}+i}} - \frac{1}{q^{k_j}} \right)
\]
\[
= \sum_{j=1}^{\infty} \left( \frac{1}{q^{k_j-1}} \sum_{i=1}^{k_j-k_{j-1}} \frac{e_i}{q^i} - \frac{1}{q^{k_j}} \right) < \sum_{j=1}^{\infty} \left( \frac{y}{q^{k_{j-1}}} - \frac{1}{q^{k_j}} \right) \leq \frac{1}{q^{k_0}},
\]
and the desired result follows at once. □

**Remark.** Proposition 2.1 in [5] is a special case of Theorem 3.2 when $y = 1$.

**Theorem 3.3.** Let $q \in (1, 2]$, $(e_i)$ be a finite $q$-expansion of $y \leq 1$ and denote by $e_i$, its last nonzero digit. If an infinite $q$-expansion $(b_i)$ of $x \in [0, 1/(q - 1)]$ satisfies the condition (3.1) and
\[
L > \min\{k; \text{ for each } i \in \mathbb{N}, \text{ if } b_i = 1, \text{ then } b_{i+j} \neq e_j \text{ when } 1 \leq j < k
\]
\[
(3.4) \quad \text{and } b_{i+k} = e_k = 1,
\]
then $(b_i)$ is the lazy $q$-expansion of $x$. 
Proof. Proceeding exactly as in the proof of Theorem 3.2, we end up at (3.3) but the strict inequality now becomes non-strict. If (3.3) is an equality, then \( y = 1 \) and more importantly, \( k_j - k_{j-1} = L \) for each \( j \) but the condition (3.4) prevents this from happening. \( \square \)

Remarks. Theorem 3.3 is new and complements Theorem 3.2. The condition (3.1) is not needed when \( x = 0 \). For then \( x \) has only a unique \( q \)-expansion which must then be \( (0) \) violating (3.1).

4. Numbers with unique \( q \)-expansion and smallest base

In this section, we first find conditions for which the greedy and lazy \( q \)-expansions of a fixed real number coincide, i.e., conditions for which the \( q \)-expansion is unique.

Theorem 4.1. If the number \( \sigma \geq 1 \) has a unique \( q \)-expansion, \( (\varepsilon_i) \), for a given \( q \in (1, 2) \), then this unique \( q \)-expansion is a U-sequence.

Proof. Let \( \sigma \geq 1 \) and \( (\varepsilon_i) \) be unique, and so is a greedy \( q \)-expansion. We deduce from Theorem 2.1, using \( x = y = \sigma \), that \( (\varepsilon_{n+i}) \prec (\varepsilon_i) \) whenever \( \varepsilon_n = 0 \). Since \( (\varepsilon_i) \) is also the lazy \( q \)-expansion of \( \sigma \), by Property L1, the \( q \)-expansion \( (1-\varepsilon_i) \) is the greedy \( q \)-expansion of \( \frac{1}{q-1} - \sigma \). Taking \( x = \frac{1}{q-1} - \sigma \), \( y = \sigma \) in Theorem 2.1, we get \( (1-\varepsilon_{n+i}) \prec (\varepsilon_i) \) whenever \( 1 - \varepsilon_n = 0 \), which shows that \( (\varepsilon_i) \) is U-sequence. \( \square \)

Remark. Theorem 4.1 is Lemma 2(b) in [2], but the proof here is different.

Theorem 4.2. If the greedy \( q \)-expansion \( (\varepsilon_i) \) of \( \sigma \in [0, 1] \) with \( q \in (1, 2] \) is an U-sequence, then \( \sigma \) has a unique \( q \)-expansion for this given \( q \).

Proof. Assume the \( q \)-expansion \( (\varepsilon_i) \) is a U-sequence. Then \( (1-\varepsilon_{n+i}) \prec (\varepsilon_i) \) whenever \( 1 - \varepsilon_n = 0 \). Since \( (\varepsilon_i) \) is a \( q \)-expansion of \( \sigma \), by the first part of the proof of Property L1, \( (1-\varepsilon_i) \) is a \( q \)-expansion of \( \frac{1}{q-1} - \sigma \). Being a U-sequence, \( (\varepsilon_i) \) is infinite. Taking \( y = \sigma \in [0, 1] \), \( x = \frac{1}{q-1} - \sigma \) in Theorem 2.3(a), we deduce that \( (1-\varepsilon_i) \) is the greedy \( q \)-expansion of \( \frac{1}{q-1} - \sigma \). By Property L1, \( (\varepsilon_i) \) is the lazy \( q \)-expansion of \( \sigma \). Since \( (\varepsilon_i) \) is both greedy and lazy, the number \( \sigma \) has a unique \( q \)-expansion. \( \square \)

Remark. Taking \( \sigma = 1 \) in Theorems 4.1 and 4.2, we get Theorem 2.2 in [5], which shows how special the number 1 is.

For certain real number \( y \leq 1 \), among base numbers \( q \) for which \( y \) has unique \( q \)-expansions, it is possible to determine the smallest such base \( q \), which we now show.

Theorem 4.3. Let \( (\delta_i) \subseteq \{0, 1\} \) be defined recursively as follows:
- First set \( \delta_1 = 1 \).
- If \( n \geq 0 \) and if \( \delta_1, \ldots, \delta_{2^n} \) are already defined, set \( \delta_{2^n+k} = 1 - \delta_k \) for \( 1 \leq k < 2^n \) and \( \delta_{2^n+1} = 1 \).
If \( y \in \left[ \sum_{i=1}^{\infty} \delta_i/2^i, 1 \right] \), then there is a smallest base \( q \in (1, 2] \) for which \( y \) has a unique \( U \)-sequence \( q \)-expansion. This \( q \) is the unique positive solution of the equation

\[
y = \sum_{i=1}^{\infty} \frac{\delta_i}{q^i}.
\]

**Proof.** From Theorem 3 in [3], \((\delta_i)\) is the smallest \( U \)-sequence. For fixed \( y \in \left[ \sum_{i=1}^{\infty} \delta_i/2^i, 1 \right] \), by Theorem 1.1, using \((\delta_i) = (e_i)\), there exists a unique \( q \in (1, 2] \) for which \( y = \sum_{i=1}^{\infty} \delta_i/q^i \). Using Theorem 2.3 (a) with \( x = y \), \((e_i) = (a_i) = (\delta_i)\), it follows that \((\delta_i)\) is the greedy \( q \)-expansion and so by Theorem 4.2, \( y \) has a unique \( q \)-expansion.

If \( y \) has another \( U \)-sequence \( q' \)-expansion \((e_i)\), which is also unique by the previous arguments, since \((\delta_i)\) is the smallest \( U \)-sequence, then \((e_i) \succeq (\delta_i)\) and Theorem 2.4 implies \( q' \geq q \).

\[\square\]

5. **Numbers with exactly two \( q \)-expansions and smallest sequence**

We now proceed to find conditions for which there are exactly two \( q \)-expansions, which must then be greedy and lazy, of a positive number \( y \leq 1 \). Let \((e_i)\) be an infinite T-sequence. Since \((e_i)\) is also a D-sequence, then \( e_1 = 1 \); for otherwise applying (1.1) we would get \((e_i) = (0)\), contradicting (1.1). From Theorem 1.1, for \( y \in \left[ \sum_{i=1}^{\infty} e_i/2^i, \sum_{i=1}^{\infty} e_i \right) \), there exists a unique \( q \in (1, 2] \) satisfying

\[
\sum_{i=1}^{\infty} \frac{\epsilon_i}{q^i} = y.
\]

By Theorem 2.3 (a), \((e_i)\) is the greedy \( q \)-expansion of \( y \). Let \( m \), \((\epsilon_i)\), \((\delta_i)\) be as defined in the definition of T-sequence. Assume further that \((e_i)\) is a \( q \)-expansion of \( 1 \). Thus

\[
\sum_{i=1}^{\infty} \frac{\delta_i}{q^i} = \sum_{i<m} \frac{\epsilon_i}{q^i} + \sum_{i>m} \frac{\epsilon_i + \epsilon_{i-m}}{q^i} = \sum_{i \neq m} \frac{\epsilon_i}{q^i} + \frac{1}{q^m} \sum_{i=1}^{\infty} \frac{\epsilon_i}{q^i} = \sum_{i=1}^{\infty} \frac{\epsilon_i}{q^i} = y,
\]

showing that \((\delta_i)\) is also a \( q \)-expansion of \( y \). Notice that the \( q \)-expansions \((e_i)\) and \((\delta_i)\) are different because \( e_m = 1 \) but \( \delta_m = 0 \).

**Theorem 5.1.** Let \((e_i)\) be an infinite T-sequence with corresponding \( m \), \((\epsilon_i)\), \((\delta_i)\). For \( y \in \left[ \sum_{i=1}^{\infty} e_i/2^i, 1 \right] \), let \( q \in (1, 2] \) be the unique base, as guaranteed by Theorem 1.1, such that \( y = \sum_{i=1}^{\infty} e_i/q^i \). Assume \((e_i)\) is a \( q \)-expansion of \( 1 \). Then \( y \) has exactly two different \( q \)-expansions, given by (5.1) and (5.2).

**Proof.** From what mentioned above, \((e_i)\) is the greedy \( q \)-expansion of \( y \). On the other hand, from (1.3) and Theorem 3.2, we see that \((\delta_i)\) is the lazy \( q \)-expansion of \( y \leq 1 \). It remains to verify that if a sequence \((\rho_i) \subseteq \{0, 1\}\) satisfies the strict
inequalities \((\delta_i) \prec (\rho_i) \prec (e_i)\), then

\[
\sum_{i=1}^{\infty} \frac{\rho_i}{q^i} = y.
\]

Fix such a sequence \((\rho_i)\). Then \(\rho_i = \delta_i = e_i\) for all \(i < m\). Since \(\delta_m = 0\) and \(e_m = 1\), we have either \(\rho_m = \delta_m = 0\) or \(\rho_m = e_m = 1\).

Case 1: \(\rho_m = 0\). Then there is an integer \(n > m\) such that \(\rho_i = \delta_i\) for all \(i < n\) and \(\delta_n = 0 < 1 = \rho_n\). Using the properties of \(T\)-sequence and the same arguments as in the proof of Theorem 2.3 (a) up to equation (2.3), we deduce \(\sum_{i=1}^{\infty} \delta_{n+i}/q^i < 1\). Thus

\[
\sum_{i=1}^{\infty} \frac{\rho_i}{q^i} - y = \frac{1}{q^n} + \sum_{i>n} \frac{\rho_i - \delta_i}{q^i} \geq \frac{1}{q^n} - \sum_{i>n} \frac{\delta_i}{q^i} = \frac{1}{q^n} \left(1 - \sum_{i=1}^{\infty} \frac{\delta_{n+i}}{q^i}\right) > 0,
\]

proving (5.3).

Case 2: \(\rho_m = 1\). Then there is an integer \(n > m\) such that \(\rho_i = e_i\) for all \(i < n\) and \(\rho_n = 0 < 1 = e_n\). Using the properties of \(T\)-sequence and the same arguments as in the first case, we have \(\sum_{i=1}^{\infty} \frac{\epsilon_{n+i}}{q^i} < 1\). Hence

\[
\sum_{i=1}^{\infty} \frac{\rho_i}{q^i} - y = -\frac{1}{q^n} + \sum_{i>n} \frac{\rho_i - e_i}{q^i} \leq -\frac{1}{q^n} + \sum_{i>n} \frac{e_i}{q^i} = -\frac{1}{q^n} \left(1 - \sum_{i=1}^{\infty} \frac{\epsilon_{n+i}}{q^i}\right) < 0,
\]

again implying (5.3).

**Remark.** Theorem 3.1 in [5] is a special case of Theorem 5.1 above when \(y = 1\).

**Theorem 5.2.** Let \((e_i)\) be a finite \(T\)-sequence with \(e_L\) being its last nonzero digit and corresponding \(m\), \((\epsilon_i)\), \((\delta_i)\). For \(y \in \left[\sum_{i=1}^{\infty} e_i/2^i, 1\right]\), let \(q \in (1, 2]\) be the unique base such that \(y = \sum_{i=1}^{\infty} e_i/q^i\). Assume that \((e_i)\) is the greedy \(q\)-expansion of 1 and that

\[
L > \min\{k; \text{for each } i \in \mathbb{N}, \text{ if } \delta_i = 1, \text{ then } \delta_{i+j} \neq e_j \text{ when } 1 \leq j < k \text{ and } \delta_{i+k} = e_k = 1\}.
\]

Then \(y\) has exactly two different \(q\)-expansions, given by (5.1) and (5.2).

**Proof.** The proof proceeds exactly as in Theorem 5.1, except that now, at the beginning of the proof, we appeal to Theorem 3.3 instead of Theorem 3.2.

**Remark.** Theorem 5.2 is new and complements Theorem 5.1.

As an example for Theorem 5.2, take

\[
y = \frac{3902563888221395449817251061561905663982412670490}{39141444333903073791808962606796280957916632792441}
\]

and \(q = 1.9\). Here

\[(e_i) = (1, 1, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)\]
Let \( \delta \) denote the period.

By Theorem 2.4, it suffices to show that (5.3)

\[ \text{Theorem 5.3.} \]

\[ \text{Case 2.} \]

Proof. By Theorem 2.4, it suffices to show that (\( e_i \)) is an infinite T-sequence \( y \) is the greedy \( q \)-expansion of 1, \( m = 10 \) and \( e_{i+m} + e_i \in \{0, 1\} \) (\( i \geq 1 \)).

Let \( (e_i) \) be an infinite T-sequence with corresponding \( m, (e_i) \) and \( (\delta_i) \). For a given real number \( y \) in an appropriate range, under the hypotheses of Theorem 5.1, \( y \) has exactly two \( q \)-expansions, namely the greedy \( (e_i) \) and the lazy \( (\delta_i) \). The corresponding base \( q \) is then a T-base number. We now ask the question: given the real number \( y \) in an appropriate range what is its smallest, with respect to lexicographic order, T-sequence? An answer is given in the next theorem.

**Theorem 5.3.** Let \( (e_i) = 111 \ 001 \), the symbol 001 denoting the period 001 of a periodic sequence. If \( (e_i) \) is an infinite T-sequence \( q \)-expansion of \( y \in \left[ \sum_{i=1}^{\infty} e_i / 2^i, 1 \right] \cap \left[ \sum_{i=1}^{\infty} e_i' / 2^i, 1 \right] \) which begins with 111 with corresponding \( m > 3 \) not a multiple of 3, and \( (e_i) \) being the greedy \( q \)-expansion of 1, then \( q \geq q' \), where \( q' \in (1, 2] \) is the unique real number satisfying \( \sum_{i=1}^{\infty} e_i' / (q')^i = y \).

Proof. By (1.2) the sequence \( (e_i) \) also begins with 111, i.e., \( \delta_3 = 111001 \cdots \). Applying (1.3), we conclude that \( \delta_{k+6} = 1 \) (because \( \delta_{k+3} = 1 \)). Therefore, the sequence \( (e_i) \) also begins with 111 001 \cdots 01 000 11. We distinguish two cases.

**Case 1:** If \( e_{k+6} = 1 \), then \( \delta_{k+6} = 1 \) (since \( m \) cannot be a multiple of 3). From (1.1), \( e_{k+9} = e_{k+10} = e_{k+11} = 0 \) (because \( e_{k+3} = 0 \)). The step now repeats as in Case 1.

**Subcase 1.1:** If \( e_{k+9} = 1 \), then \( \delta_{k+9} = 1 \) (since \( m \) cannot be a multiple of 3). From (1.1), \( e_{k+10} = e_{k+11} = 0 \) (because \( e_{k+3} = 0 \)). The step now repeats as in Case 1.

**Subcase 1.2:** If \( e_{k+9} = 0 \), then \( \delta_{k+9} = 0 \) (because of (5.2))

The step then repeats as in Case 2.

**Case 2:** If \( e_{k+6} = 0 \), then \( \delta_{k+6} = 0 \) (because of (5.2)). From (1.3), \( \delta_{k+7} = \delta_{k+8} = 1 \) (because \( \delta_{k+3} = 1 \)). Thus \( e_{k+7} = e_{k+8} = 0 \) (because of (5.2)). The step now repeats as in Case 1.

**Subcase 2.1:** If \( e_{k+9} = 1 \), then \( \delta_{k+9} = 1 \) (since \( m \) cannot be a multiple of 3).

**Subcase 2.2:** If \( e_{k+9} = 0 \), then \( \delta_{k+9} = 0 \) (because of (5.2)).

From (1.3), \( \delta_{k+10} = \delta_{k+11} = 1 \) (because \( \delta_{k+3} = 1 \)). The step repeats as in Case 2.

Continuing in the same manner, we deduce that \( m \) must be arbitrarily large, which is impossible. □
Theorem 4.1 in [5] is a special case of Theorem 5.3 when \( y = 1 \).

As an example of Theorem 5.3, let

\[
y = \frac{3902563888221395449817251061561905663982412670490}{39141443390307379180896260679628095791663279241}
\]

and \( q = 1.9 \). The unique positive solution of the equation \( \sum_{i=1}^{\infty} e'_i/(q')^i = y \) is \( q' \approx 1.874535175 \). From Theorem 4.1 in [5], when \( y = 1 \) we have \( q' = 1.871349313 \).

The last two results show that for certain \( y \leq 1 \), the sequence \((e'_i)\) with base \( q' \) yields a unique \( q' \)-expansion whose base is an accumulation point of, yet smaller than, other T-base numbers \( q \) of \( y \) with exactly two \( q \)-expansions.

**Theorem 5.4.** Let \((e'_i) = 111 \ 001 \), for \( y \in \left[ \sum_{i=1}^{\infty} e'_i/2^i, 1 \right] \), there is a unique \( q' \in (1, 2] \) such that \((e'_i)\) is a \( q' \)-expansion of \( y \) and this \( q' \)-expansion is always unique.

**Proof.** Taking both sequences to be \((e'_i)\) in Theorem 2.3 (a), we have that \((e'_i)\) is the greedy \( q' \)-expansion of \( y \). Since \((e'_i)\) is also a U-sequence, Theorem 4.2 infers that \( y \) has a unique \( q' \)-expansion. \( \square \)

**Theorem 5.5.** Let \((e'_i) = 111 \ 001, y \in \left[ \sum_{i=1}^{\infty} e'_i/2^i, 1 \right] \). By Theorem 1.1, there exists a unique \( q' \in (1, 2] \) such that \( y = \sum_{i=1}^{\infty} e'_i/(q')^i \). Let \( k \in \mathbb{N} \) and let

\[
\left( e^{(k)}_i \right) := 111 \ 001 \ 1 \ 001 \ 10 \cdots 001 \ 001 \ 001 \ 001 \cdots
\]

be the sequence obtained by inserting the block \( 10 \cdots 0 \) (one 1 followed by \( (3k + 4) \) 0’s) between the \( k \)th and \((k+1)\)th block of \( 001 \) of \((e'_i)\). By Theorem 1.1, there exists a unique \( q_k \in (1, 2] \) such that \( y = \sum_{i=1}^{\infty} e^{(k)}_i/q_k^i \). Let \( e^{(k)}_i \) be the greedy \( q_k \)-expansion of 1. Assume there are infinitely many \( k \) such that

\[
(5.6) \quad \left( e^{(k)}_{n+1} \right) \prec (e^{(k)}_n) \quad \text{whenever } e^{(k)}_n = 1 \text{ and } 1 \leq n \leq 3k + 3,
\]

\[
(5.7) \quad (e^{(k)}_{n+1}) \prec (e^{(k)}_n) \quad \text{whenever } e^{(k)}_n = 0 \text{ and } 1 \leq n \leq 3k + 4,
\]

\[
(5.8) \quad e^{(k)}_{3k+4+3n} = 0 \text{ where } 3n = \{3t; t \in \mathbb{N}\},
\]

\[
(5.9) \quad e^{(k)}_{3k+2+3n} = e^{(k)}_{3k+3+3n} \neq 11,
\]

\[
(5.10) \quad e^{(k)}_{3k+2+3n} = e^{(k)}_{3k+3+3n} = e^{(k)}_{3k+5+3n} = e^{(k)}_{3k+6+3n} \neq 10, \ 01.
\]

Then \( q' \) is an accumulation point of the set of \( T \)-base numbers.

**Proof.** We start by verifying that \((e^{(k)}_i)\) is a \( T \)-sequence with \( m = 3k + 4 \) and \((\delta^{(k)}_i) \subseteq \{0, 1\} \) so constructed as in (1.2) with corresponding \((e^{(k)}_i)\) and \((\delta^{(k)}_i)\); such construction is valid by (5.8).
From the shape of the sequence \((e^{(k)}_i)\), we see that \((e^{(k)}_i)\) is a D-sequence and 
\[ e^{(k)}_{3k+4} = 1. \]
There remains to check the requirements (1.3), (1.4) and (1.5). The requirement (1.4) follows immediately from the shape of the sequence \((e^{(k)}_i)\).

When \(i < m = 3k + 4\), since \(\delta^{(k)}_i = e^{(k)}_i\), the requirement (1.3) holds for these \(i\). From (5.6), respectively (5.7), together with the definition (1.2), \((\delta^{(k)}_i)\) satisfies (1.3), respectively (1.5), when \(m + 1 \leq n \leq m + 3k + 4\).

For \(n \geq m + 3k + 5\), (1.3) holds by the definition (1.2) and the shape of \((e^{(k)}_i)\). As for (1.5), we distinguish four cases.

Case 1: \(\delta^{(k)}_{m+3k+2+3N} \neq 00\). From (5.9),
\[ \delta^{(k)}_{m+3k+5+3N} \delta^{(k)}_{m+3k+6+3N} \neq 11, \]
i.e., \((\delta^{(k)}_i)\) satisfies (1.5).

Case 2: \(\delta^{(k)}_{m+3k+2+3N} \delta^{(k)}_{m+3k+3+3N} = 01\). From (5.9),
\[ \delta^{(k)}_{m+3k+5+3N} \delta^{(k)}_{m+3k+6+3N} \neq 11, \]
while from (5.10), \(\delta^{(k)}_{m+3k+5+3N} \delta^{(k)}_{m+3k+6+3N} \neq 10\), i.e., \((\delta^{(k)}_i)\) satisfies (1.5).

Case 3: \(\delta^{(k)}_{m+3k+2+3N} \delta^{(k)}_{m+3k+3+3N} = 10\). From (5.9),
\[ \delta^{(k)}_{m+3k+5+3N} \delta^{(k)}_{m+3k+6+3N} \neq 11, \]
i.e., \((\delta^{(k)}_i)\) satisfies (1.5).

Case 4: \(\delta^{(k)}_{m+3k+2+3N} \delta^{(k)}_{m+3k+3+3N} = 11\). That \((\delta^{(k)}_i)\) satisfies (1.5) follows at once from (5.9).

Since \((e^{(k)}_i)\) is a T-sequence, taking \(k \to \infty\), we have \((e^{(k)}_i) \to (e^*_i)\) and the corresponding base numbers \(q_k \to q^*\), which completes the proof. □

References

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