POSITIVE SOLUTIONS FOR A SYSTEM OF SINGULAR SECOND ORDER NONLOCAL BOUNDARY VALUE PROBLEMS

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Abstract. Sufficient conditions for the existence of positive solutions for a coupled system of nonlinear nonlocal boundary value problems of the type

\[-x''(t) = f(t, y(t)), \quad t \in (0, 1),
\]
\[-y''(t) = g(t, x(t)), \quad t \in (0, 1),
\]
\[x(0) = y(0) = 0, \quad x(1) = \alpha x(\eta), \quad y(1) = \alpha y(\eta),
\]
are obtained. The nonlinearities \(f, g : (0, 1) \times (0, \infty) \rightarrow (0, \infty)\) are continuous and may be singular at \(t = 0, t = 1, x = 0,\) or \(y = 0.\) The parameters \(\eta, \alpha\) satisfy \(\eta \in (0, 1), 0 < \alpha < 1/\eta.\) An example is provided to illustrate the results.

1. Introduction

Nonlocal boundary value problems (BVPs) arise in different areas of applied mathematics and physics. For example, the vibration of a guy wire composed of \(N\) parts with a uniform cross section and different densities in different parts can be modeled as a nonlocal boundary value problem [18]; problems in the theory of elastic stability can also be modeled as nonlocal boundary value problems [19].

The study of nonlocal BVPs for linear second order ordinary differential equations was initiated by Il’in and Moiseev in [10, 11] and extended to nonlocal linear elliptic boundary value problems by Bitsadze and Samarskii, [2, 3, 4]. Existence theory for nonlinear three-point boundary value problems was initiated by Gupta [9]. Since then the study of nonlinear regular multi-point BVPs has attracted the attention of many researchers; see for example, [5, 9, 13, 14, 15, 17, 18, 20] for scalar equations, and for systems of ordinary differential equations, see [6, 7, 12].

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Recently, the study of singular BVPs has also attracted some attention. An excellent resource with an extensive bibliography was produced by Agarwal and O’Regan [1]. Recently, S. Xie and J. Zhu [21] applied topological degree theory in a cone to study the following two point BVP for a coupled system of nonlinear fourth-order ordinary differential equations

$$-x^{(4)} = f_1(t, y), \quad t \in (0, 1),$$
$$-y'' = f_2(t, x), \quad t \in (0, 1),$$
$$x(0) = x(1) = x''(0) = x''(1) = 0,$$
$$y(0) = y(1) = 0.$$  

(1.1)

In [21], the nonlinearities $f_i \in C((0, 1) \times \mathbb{R}^+, \mathbb{R}^+)$ satisfy $f_i(t, 0) \equiv 0 \ (i = 1, 2)$ and may be singular at $t = 0$ or $t = 1$ only.

More recently, Y. Zhou and Y. Xu [23] studied the following nonlocal BVP for a system of second order regular ordinary differential equations

$$-x''(t) = f(t, y), \quad t \in (0, 1),$$
$$-y''(t) = g(t, x), \quad t \in (0, 1),$$
$$x(0) = 0, \quad x(1) = \alpha x(\eta),$$
$$y(0) = 0, \quad y(1) = \alpha y(\eta),$$

(1.2)

where $\eta \in (0, 1)$, $0 < \alpha < 1/\eta$, $f, g \in C([0, 1] \times [0, \infty), [0, \infty))$, $f(t, 0) \equiv 0$, $g(t, 0) \equiv 0$. The above system was extended to the singular case by B. Liu, L. Liu, and Y. Wu [16], where the nonlinearities $f, g$ were assumed to be singular at $t = 0$ or $t = 1$ together with the assumption that $f(t, 0) \equiv 0$, $g(t, 0) \equiv 0$, $t \in (0, 1)$.

In this paper, we generalize the system (1.2) by allowing $f, g$ to be singular at $t = 0$, $t = 1$, or $x = 0$, or $y = 0$ and obtain sufficient conditions for the existence of a positive solution of the BVP for the system of singular equations, (1.2). By singularity we mean that the functions $f(t, u)$ or $g(t, u)$ are allowed to be unbounded at $t = 0$, $t = 1$, or $u = 0$. In general, the assumption that there exist singularities with respect to the dependent variable is not new; see [1, 6], for example. However, in the case of nonlocal boundary conditions and coupled systems of ordinary differential equations, we believe this assumption is new.

Throughout this paper, we shall assume that

$$f, g : (0, 1) \times (0, \infty) \rightarrow (0, \infty)$$

are continuous and may be singular at $t = 0$, $t = 1$, or $u = 0$. We also assume that $f(t, 0), g(t, 0)$ are not identically 0. Let $N > \max\{\frac{1}{\eta}, \frac{1}{1-\eta}, \frac{2-\alpha}{1-\alpha \eta}\}$ denote a fixed positive integer. Assume that the following conditions hold:

($A_1$) there exist $K, L \in C((0, 1), (0, \infty))$ and $F, G \in C((0, \infty), (0, \infty))$ such that

$$f(t, u) \leq K(t)F(u), \quad g(t, u) \leq L(t)G(u), \quad t \in (0, 1), \ u \in (0, \infty)$$
and

\[ a := \int_0^1 (1-t)K(t)\,dt < +\infty, \quad b := \int_0^1 (1-t)L(t)\,dt < +\infty; \]

(A2) there exist \( \alpha_1, \alpha_2 \in (0, \infty) \) with \( \alpha_1 \alpha_2 \leq 1 \) such that

\[ \lim_{u \to \infty} \frac{F(u)}{u^{\alpha_1}} \to 0, \quad \lim_{u \to \infty} \frac{G(u)}{u^{\alpha_2}} \to 0; \]

(A3) there exist \( \beta_1, \beta_2 \in (0, \infty) \) with \( \beta_1 \beta_2 \geq 1 \) such that

\[ \liminf_{u \to 0^+} \min_{t \in [\eta, 1]} \frac{f(t, u)}{u^{\beta_1}} > 0, \quad \liminf_{u \to 0^+} \min_{t \in [\eta, 1]} \frac{g(t, u)}{u^{\beta_2}} > 0; \]

(A4) \( f(t, u), G(u) \) are non-increasing with respect to \( u \) and for each fixed \( n \in \{ N, N+1, N+2, \ldots \} \), there exists a constant \( M_1 > 0 \) such that

\[ f \left( \frac{1}{n}, b \mu_n G \left( \frac{1}{n} \right) \right) \geq M_1 \left( \nu_n \int_{\eta}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)\,ds \right)^{-1}; \]

(A5) \( F(u), g(t,u) \) are non-increasing with respect to \( u \) and for each fixed \( n \in \{ N, N+1, N+2, \ldots \} \), there exists a constant \( M_2 > 0 \) such that

\[ F \left( \nu_n \int_{\eta}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)g(s, M_2)\,ds \right) \leq \frac{M_2 - \frac{1}{n}}{a \mu_n}. \]

The parameters \( \mu_n \) and \( \nu_n \) in (A4) and (A5) are given by

\[ \mu_n = \max \{ 1, \alpha \} \left( \frac{1}{1 - \frac{2}{n} + \frac{2}{n} - \alpha \eta} \right), \quad \nu_n = \min \{ 1, \alpha \} \left( \frac{\min \{ \eta - \frac{1}{n}, 1 - \frac{1}{n} - \eta \}}{1 - \frac{2}{n} + \frac{2}{n} - \alpha \eta} \right). \]

Since \( N > \max \{ \frac{1}{n}, \frac{1}{n}, \frac{2-n}{n} \} \), \( \mu_n, \nu_n > 0 \).

We state the main results of this paper here.

**Theorem 1.1.** Assume that (A1) – (A3) hold. Then the system (1.1) has at least one positive solution.

**Theorem 1.2.** Assume that (A1), (A2) and (A4) hold. Then the system (1.1) has at least one positive solution.

**Theorem 1.3.** Assume that (A1), (A3) and (A5) hold. Then the system (1.1) has at least one positive solution.

**Theorem 1.4.** Assume that (A1), (A4) and (A5) hold. Then the system (1.1) has at least one positive solution.
2. Preliminaries

For each \( x \in C[0,1] \) we write \( \|x\| = \max \{|x(t)| : t \in [0,1] \} \). Clearly, \( C[0,1] \) with the norm \( \| \cdot \| \) is a Banach space. For \( n \geq N \), define a cone \( P \), and a cone \( K_n \) of \( C[\frac{1}{n}, 1 - \frac{1}{n}] \) as follows:

\[
P = \{ x \in C[0,1] : x(t) \geq 0, t \in [0,1] \},
\]
\[
P_n = \{ x \in P : x \text{ is concave on } [0,1], \min_{t \in [\frac{1}{n}, 1 - \frac{1}{n}]} x(t) \geq \frac{1}{n} \},
\]
\[
K_n = \{ x \in C[\frac{1}{n}, 1 - \frac{1}{n}] : x \text{ is concave on } [0,1] \}.
\]

For any real constant \( r > 0 \), define

\[
\Omega_r = \{ x \in C[0,1] : \|x\| < r \}
\]
as an open neighborhood of \( 0 \in C[0,1] \) of radius \( r \). \((x(t), y(t))\) is called a positive solution of (1.1) if

\[
(x, y) \in (C[0,1] \cap C^2(0,1)) \times (C[0,1] \cap C^2(0,1)),
\]
\[
x(t) > 0, y(t) > 0 \text{ on } (0,1) \text{ and } (x, y) \text{ satisfies (1.1)}.
\]

The proofs of our main results (Theorems 1.1-1.4) are based on the Guo-Krasnosel’skii fixed-point theorem.

**Lemma 2.1** ([8, Guo Krasnosel’skii Fixed-Point Theorem]). Let \( K \) be a cone of a real Banach space \( E \), and let \( \Omega_1, \Omega_2 \) be bounded open neighborhoods of \( 0 \in E \), and assume \( \Omega_1 \subset \Omega_2 \). Suppose that \( T : K \cap (\Omega_2 \setminus \Omega_1) \to K \) is completely continuous such that one of the following conditions holds:

(i) \( \|Tx\| \leq \|x\| \) for \( x \in \partial \Omega_1 \cap K \); \( \|Tx\| \geq \|x\| \) for \( x \in \partial \Omega_2 \cap K \);

(ii) \( \|Tx\| \leq \|x\| \) for \( x \in \partial \Omega_2 \cap K \); \( \|Tx\| \geq \|x\| \) for \( x \in \partial \Omega_1 \cap K \).

Then, \( T \) has a fixed point in \( K \cap (\Omega_2 \setminus \Omega_1) \).

For fixed \( n \geq N \) and \( z \in C[0,1] \), the linear boundary value problem

\[
-u''(t) = z(t), \quad t \in [\frac{1}{n}, 1 - \frac{1}{n}],
\]
\[
u(\frac{1}{n}) = \frac{1}{n}, \quad u(1 - \frac{1}{n}) = au(\eta) + \frac{1-a}{n},
\]
has a unique solution

\[
u(t) = \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t, s)z(s)ds,
\]
where $H_n : \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \times \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \to [0, \infty)$ is an associated Green’s function and is defined by

\begin{equation}
H_n(t, s) = \begin{cases}
(t - \frac{1}{n})((1 - \frac{1}{n} - s) - \alpha(q - s)) - (t - s), & \frac{1}{n} \leq t \leq 1 - \frac{1}{n}, s \leq \eta, \\
(t - \frac{1}{n})((1 - \frac{1}{n} - s) - \alpha(q - s)) - (t - s), & \frac{1}{n} \leq t \leq 1 - \frac{1}{n}, s \leq \eta, \\
(t - \frac{1}{n})(1 - \frac{1}{n} - s) - (t - s), & \frac{1}{n} \leq s \leq t \leq 1 - \frac{1}{n}, s \geq \eta.
\end{cases}
\end{equation}

We note that $H_n(t, s) \to H(t, s)$ as $n \to \infty$, where

\begin{equation}
H(t, s) = \begin{cases}
(t - \frac{1}{n})\frac{(1 - s) - \alpha(t - \frac{1}{n} - s)}{1 - \alpha q} - (t - s), & 0 \leq s \leq t \leq 1, s \leq \eta, \\
(t - \frac{1}{n})\frac{(1 - s) - \alpha(t - \frac{1}{n} - s)}{1 - \alpha q}, & 0 \leq t \leq s \leq 1, s \leq \eta, \\
(t - \frac{1}{n})\frac{(1 - s)}{1 - \alpha q}, & 0 \leq t \leq s \leq 1, s \geq \eta, \\
(t - \frac{1}{n}) - (t - s), & 0 \leq s \leq t \leq 1, s \geq \eta,
\end{cases}
\end{equation}

is the Green’s function corresponding to the boundary value problem

\[-u''(t) = z(t), \quad t \in [0, 1],
\]
\[u(0) = 0, \quad u(1) = \alpha u(\eta),\]

with

\[u(t) = \int_0^1 H(t, s)z(s)ds,\]

as its integral representation. We need the following properties of the Green’s function $H_n$ in the sequel. For the proof, see [22].

**Lemma 2.2.** The function $H_n$ can be written as

\begin{equation}
H_n(t, s) = G_n(t, s) + \frac{\alpha \left(t - \frac{1}{n}\right)}{1 - 2/n + \eta - \alpha \eta}G_n(\eta, s),
\end{equation}

where

\begin{equation}
G_n(t, s) = \frac{n}{n - 2}\left\{\begin{array}{ll}
(s - \frac{1}{n}) \left(1 - \frac{1}{n} - t\right), & \frac{1}{n} \leq s \leq t \leq 1 - \frac{1}{n}, \\
(t - \frac{1}{n}) \left(1 - \frac{1}{n} - s\right), & \frac{1}{n} \leq t \leq s \leq 1 - \frac{1}{n}.
\end{array}\right.
\end{equation}
Lemma 2.3. Let
\[ \mu_n = \max\{1, \alpha\} \left\lfloor 1 - \frac{2}{n} + \frac{2}{\alpha - \alpha n} \right\rfloor, \quad \nu_n = \min\{1, \alpha\} \min\{\eta - \frac{1}{n}, 1 - \frac{1}{n} - \eta\} \left\lfloor 1 - \frac{2}{n} + \frac{2}{\alpha - \alpha n} \right\rfloor. \]

Then
\[ (i) \ H_n(t, s) \leq \mu_n \left( s - \frac{1}{n} \right) \left( 1 - \frac{1}{n} - s \right), \quad (t, s) \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \times \left[ \frac{1}{n}, 1 - \frac{1}{n} \right], \]
\[ (ii) \ H_n(t, s) \geq \nu_n \left( s - \frac{1}{n} \right) \left( 1 - \frac{1}{n} - s \right), \quad (t, s) \in \left[ \eta, 1 - \frac{1}{n} \right] \times \left[ \frac{1}{n}, 1 - \frac{1}{n} \right]. \]

Now consider the system of nonlinear non-singular BVPs
\[ -x''(t) = f(t, \max\left\{ \frac{1}{n}, y(t) \right\}), \quad t \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right], \]
\[ -y''(t) = g(t, \max\left\{ \frac{1}{n}, x(t) \right\}), \quad t \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right], \]
\[ x\left( \frac{1}{n} \right) = \frac{1}{n}, \quad x(1 - \frac{1}{n}) = \alpha x(\eta) + \frac{1 - \alpha}{n}, \]
\[ y\left( \frac{1}{n} \right) = \frac{1}{n}, \quad y(1 - \frac{1}{n}) = \alpha y(\eta) + \frac{1 - \alpha}{n}, \]

where \( n > N \). Write (2.6) as an equivalent system of integral equations
\[ x(t) = \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t, s) f(s, \max\left\{ \frac{1}{n}, y(s) \right\}) ds, \]
\[ y(t) = \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t, s) g(s, \max\left\{ \frac{1}{n}, x(s) \right\}) ds. \]

Thus, \((x, y)\) is a solution of (2.6) if and only if
\[ (x, y) \in C\left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \times C\left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \]
and \((x, y)\) is a solution of (2.7).

Define operators \( A_n, B_n, T_n : K_n \to K_n \) by
\[ (A_n y)(t) = \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t, s) f(s, \max\left\{ \frac{1}{n}, y(s) \right\}) ds, \]
\[ (B_n x)(t) = \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t, s) g(s, \max\left\{ \frac{1}{n}, x(s) \right\}) ds, \]
\[ (T_n x)(t) = (A_n(B_n x))(t). \]

If \( u_n \in K_n \) is a fixed point of \( T_n \), then the system of BVPs (2.6) has a solution \((x_n, y_n)\) given by
\[ \begin{cases} x_n(t) = u_n(t), \\ y_n(t) = (B_n u_n)(t). \end{cases} \]

By construction, the system of BVPs (2.6) is regular and so the following lemma is standard.

Lemma 2.4. Assume \( f, g : (0, 1) \times (0, \infty) \to [0, \infty) \) are continuous. Then \( T_n : K_n \to K_n \) is completely continuous.
3. Main results

Proof of Theorem 1.1. By (A2), there exist constants $C_1, C_2, N_1, N_2 > 0$ such that

\begin{equation}
4^\alpha a^\alpha_1 \mu_1^{\alpha_1+1} C_1 C_2^{\alpha_1} < 1,
\end{equation}

and

\begin{equation}
F(x) \leq C_1 x^{\alpha_1} + N_1, \quad G(x) \leq C_2 x^{\alpha_2} + N_2 \text{ for } x \geq \frac{1}{n}.
\end{equation}

Choose a constant $R > 0$ such that

\begin{equation}
R \geq \frac{1}{a} + \frac{2^{\alpha_1} a_1^{\alpha_1} C_1 + a \mu_1 N_1 + 4 \alpha_1 a_1^{\alpha_1} \mu_1^{\alpha_1+1} C_1 C_2^{\alpha_1}}{1 - 4 \alpha_1 a_1^{\alpha_1} \mu_1^{\alpha_1+1} C_1 C_2^{\alpha_1}}.
\end{equation}

For any $u \in \partial \Omega_R \cap K_n$, using (2.8) and (A1), we have

\begin{align*}
(T_n u)(t) &= (A_n (B_n u))(t) = \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t, s) f(s, (B_n u)(s)) ds \\
&= \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t, s) f(s, \frac{1}{n}) + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau)) d\tau ds \\
&\leq \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t, s) K(s) F(\frac{1}{n}) + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau)) d\tau ds.
\end{align*}

In view of (3.2) and (A2), it follows that

\begin{align*}
(T_n u)(t) &\leq \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t, s) K(s) (\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau)) d\tau)^{\alpha_1} + N_1) ds \\
&= \frac{1}{n} + C_1 \int_{1/n}^{1-1/n} H_n(t, s) K(s) (\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau)) d\tau)^{\alpha_1} ds \\
&\quad + N_1 \int_{1/n}^{1-1/n} H_n(t, s) K(s) ds \\
&\leq \frac{1}{n} + C_1 \int_{1/n}^{1-1/n} H_n(t, s) K(s) (\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) L(\tau) G(u(\tau)) d\tau)^{\alpha_1} ds \\
&\quad + N_1 \int_{1/n}^{1-1/n} H_n(t, s) K(s) ds \\
&\leq \frac{1}{n} + C_1 \int_{1/n}^{1-1/n} H_n(t, s) K(s) \\
&\quad \cdot \left( \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) L(\tau) (C_2(u(\tau))^{\alpha_2} + N_2) d\tau \right)^{\alpha_1} ds.
\end{align*}
\[ + N_1 \int_{1/n}^{1-1/n} H_n(t, s)K(s) ds. \]

Employing (i) of Lemma 2.3, we obtain

\[
(T_n u)(t) \leq \frac{1}{n} + C_1 \mu_n \int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)K(s) ds \\
\cdot \left( \frac{1}{n} + \mu_n \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau)L(\tau)(C_2(u(\tau))^\alpha_2 + N_2) d\tau \right)^{\alpha_1} \\
+ N_1 \mu_n \int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)K(s) ds
\]

\[
\leq \frac{1}{n} + C_1 \mu_n \int_{1/n}^{1-1/n} s(1 - s)K(s) ds \\
\cdot \left( \frac{1}{n} + \mu_n \int_{1/n}^{1-1/n} \tau(1 - \tau)L(\tau)(C_2(u(\tau))^\alpha_2 + N_2) d\tau \right)^{\alpha_1} \\
+ N_1 \mu_n \int_{1/n}^{1-1/n} s(1 - s)K(s) ds
\]

Hence,

\[
(T_n u)(t) \leq \frac{1}{n} + C_1 \mu_n \int_{0}^{1} s(1 - s)K(s) ds \\
\cdot \left( \frac{1}{n} + \mu_n \int_{0}^{1} \tau(1 - \tau)L(\tau)(C_2(u)^{\alpha_2} + N_2) d\tau \right)^{\alpha_1} \\
+ \mu_n N_1 \int_{0}^{1} s(1 - s)K(s) ds \\
\leq \frac{1}{n} + a \mu_n N_1 + 2^{\alpha_1} a \mu_n C_1 \left( \frac{1}{n^{\alpha_1}} + b^{\alpha_1} \mu_n^{\alpha_1}(C_2(u)^{\alpha_2} + N_2)^{\alpha_1} \right) \\
\leq \frac{1}{n} + \frac{2^{\alpha_1} a \mu_n C_1}{n^{\alpha_1}} + a \mu_n N_1 + 2^{2\alpha_1} a b^{\alpha_1} \mu_n^{\alpha_1+1} C_1 (C_2^{\alpha_2} u^{\alpha_2} + N_2^{\alpha_2}).
Using (3.3), we obtain
\[(3.4)\quad \|T_n u\| \leq \|u\| \text{ for all } u \in \partial \Omega_R \cap K_n.\]

Now, by (A3), there exist constants \(C_3, C_4 > 0\) and \(\rho \in (0, R)\) such that
\[(3.5)\quad f(t, x) \geq C_3 x^{\beta_1}, g(t, x) \geq C_4 x^{\beta_2} \quad \text{for } x \in [0, \rho] \text{ and } t \in [\eta, 1].\]

Choose
\[(3.6)\quad r_n = \min \left\{ \rho, \frac{C_3 C_4^n \nu_n^{\beta_1+1}}{n^{\beta_2}} \left( \int_{\eta}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)ds \right)^{\beta_1+1} \right\}.\]

For any \(u \in \partial \Omega_{r_n} \cap K_n\), using (2.8), (3.5) and (ii) of Lemma 2.3, we have
\[(T_n u)(t) = (A_n(B_n u))(t)\]
\[= \frac{1}{n} + \int_{\eta}^{1-1/n} H_n(t, s)f(s, \frac{1}{n}) + \int_{1/n}^{1-1/n} H_n(s, \tau)g(\tau, u(\tau))d\tau ds\]
\[\geq \int_{1/n}^{1-1/n} H_n(t, s)f(s, \frac{1}{n}) + \int_{1/n}^{1-1/n} H_n(s, \tau)g(\tau, u(\tau))d\tau ds\]
\[\geq \int_{\eta}^{1-1/n} H_n(t, s)f(s, \frac{1}{n}) + \int_{1/n}^{1-1/n} H_n(s, \tau)g(\tau, u(\tau))d\tau ds\]
\[\geq C_3 \int_{\eta}^{1-1/n} H_n(t, s) \left( \int_{\eta}^{1-1/n} H_n(s, \tau)g(\tau, u(\tau))d\tau \right)^{\beta_1} ds\]
\[\geq C_3 \nu_n \int_{\eta}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)ds \cdot \left( \nu_n \int_{\eta}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau)g(\tau, u(\tau))d\tau \right)^{\beta_2} \]
\[\geq C_3 \nu_n^{\beta_1+1} \int_{\eta}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)ds \cdot \left( C_4 \int_{\eta}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau)g(\tau)^{\beta_2} d\tau \right)^{\beta_1} \]
\[\geq C_3 C_4^n \nu_n^{\beta_1+1} \left( \int_{\eta}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)ds \right)^{\beta_1+1}.\]

Thus, in view of (3.6), it follows that
\[(3.7)\quad \|T_n u\| \geq \|u\| \text{ for } u \in \partial \Omega_{r_n} \cap K_n.\]

By Lemma 2.1, \(T_n\) has a fixed point \(u_n \in K_n \cap (\overline{\Omega_R \setminus \Omega_{r_n}}).\)

Note that
\[(3.8)\quad r_n \leq u_n(t) \leq R \quad \text{for all } t \in [\frac{1}{n}, 1 - \frac{1}{n}]\]
and \( r_n \to 0 \) as \( n \to \infty \). Thus, we have exhibited a uniform bound for each \( u_n \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \) and for \( m \geq n \), \( \{u_m\} \) is uniformly bounded on \( \left[ \frac{1}{m}, 1 - \frac{1}{n} \right] \).

To show that \( \{u_m\} \) for \( m \geq n \), is equicontinuous on \( \left[ \frac{1}{m}, 1 - \frac{1}{n} \right] \), consider for \( t \in \left[ \frac{1}{m}, 1 - \frac{1}{n} \right] \), the integral equation

\[
u_m(t) = u_m\left( \frac{1}{m} \right) + \int_{1/m}^{1-1/m} H_m(t, s) f(s, (B_m u_m)(s)) \, ds.
\]

Employ Lemma 2.2 to obtain

\[
u_m(t) = u_m\left( \frac{1}{m} \right) + \int_{1/m}^{1-1/m} \left[ G_m(t, s) + \frac{\alpha(t - \frac{1}{m})}{1 - \frac{2}{m} + \frac{2}{m} - \alpha \eta} G_m(\eta, s) \right] f(s, (B_m u_m)(s)) \, ds
\]

\[
= u_m\left( \frac{1}{m} \right) + \frac{m}{m - 2} \int_{1/m}^{1-1/m} \frac{(s - \frac{1}{m})(1 - \frac{1}{m} - t)}{m - 2} f(s, (B_m u_m)(s)) \, ds
\]

\[
+ \frac{1}{1 - \frac{2}{m} + \frac{2}{m} - \alpha \eta} \int_{1/m}^{1-1/m} G_m(\eta, s) f(s, (B_m u_m)(s)) \, ds.
\]

Differentiate with respect to \( t \) to obtain

\[
u'_m(t) = - \frac{m}{m - 2} \int_{1/m}^{t} (s - \frac{1}{m}) f(s, (B_m u_m)(s)) \, ds
\]

\[
+ \frac{m}{m - 2} \int_{1/m}^{1-1/m} (1 - \frac{1}{m} - s) f(s, (B_m u_m)(s)) \, ds
\]

\[
+ \frac{\alpha(t - \frac{1}{m})}{1 - \frac{2}{m} + \frac{2}{m} - \alpha \eta} \int_{1/m}^{1-1/m} G_m(\eta, s) f(s, (B_m u_m)(s)) \, ds,
\]

which implies that for \( t \in \left[ \frac{1}{m}, 1 - \frac{1}{n} \right] \)

\[
|\nu'_m(t)| \leq \int_{1/m}^{1-1/m} f(s, (B_m u_m)(s)) \, ds
\]

\[
+ \frac{\alpha}{1 - \frac{2}{m} + \frac{2}{m} - \alpha \eta} \int_{1/m}^{1-1/m} G_m(\eta, s) f(s, (B_m u_m)(s)) \, ds.
\]

Hence, for \( m \geq n \), \( \{u_m\} \) is equicontinuous on \( \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \).

For \( m \geq n \), define

\[
v_m = \begin{cases} 
  u_m\left( \frac{1}{m} \right), & \text{if } 0 \leq t \leq \frac{1}{n}, \\
  u_m(t), & \text{if } \frac{1}{n} \leq t \leq 1 - \frac{1}{n}, \\
  \alpha u_m(\eta), & \text{if } 1 - \frac{1}{n} \leq t \leq 1.
\end{cases}
\]
Since $v_m$ is a constant extension of $u_m$ to $[0, 1]$, the sequence $\{v_m\}$ is uniformly bounded and equicontinuous on $[0, 1]$. Thus, there exists a subsequence $\{v_{m_k}\}$ of $\{v_m\}$ converging uniformly on $[0, 1]$ to $v \in P \cap (\overline{H_R} \setminus \Omega _{r})$.

We introduce the notation

$$x_{n_k}(t) = v_{n_k}(t), \quad y_{n_k}(t) = \frac{1}{n_k} + \int_{1/n_k}^{1} H_{n_k}(t, s) g(s, v_{n_k}(s)) ds,$$

$$\xi(t) = \lim_{n_k \to \infty} x_{n_k}(t), \quad \eta(t) = \lim_{n_k \to \infty} y_{n_k}(t),$$

and for $t \in [0, 1]$ consider the integral equation

$$x_{n_k}(t) = x_{n_k}\left(\frac{1}{n_k}\right) + \int_{1/n_k}^{1} H_{n_k}(t, s) f(t, y_{n_k}(s)) ds.$$

Letting $n_k \to \infty$, we have

$$\xi(t) = \xi(0) + \int_{0}^{1} H(t, s) f(t, \eta(s)) ds,$$

and

$$\eta(t) = \int_{0}^{1} H(t, s) g(s, \xi(s)) ds, \quad t \in [0, 1].$$

Moreover,

$$\xi(0) = 0, \quad x(1) = \alpha \xi(\eta), \quad \eta(0) = 0, \quad \eta(1) = \alpha \eta(\eta).$$

Hence, $(\xi(t), \eta(t))$ is a solution of the system (1.2).

Since

$$f, g : (0, 1) \times (0, \infty) \to (0, \infty),$$

$f(t, 0), g(t, 0)$ are not identically 0, and $H$ is of fixed sign on $(0, 1) \times (0, 1)$, it follows that $\xi, \eta > 0$ on $(0, 1)$. \hfill \Box

Example 3.1. Let

$$f(t, y) = \frac{1}{t(1-t)} \left( \frac{1}{y} + 3y^{1/3} \right), \quad g(t, x) = \frac{1}{t(1-t)} \left( \frac{1}{x} + 4x \right)$$

and $\alpha = 2, \eta = \frac{1}{7}$. Choose

$$K(t) = L(t) = \frac{1}{t(1-t)}, \quad F(y) = \frac{1}{y} + 3y^{1/3}, \quad G(x) = \frac{1}{x} + 4x,$$

and $\alpha_1 = \frac{1}{2}, \alpha_2 = 2, \beta_1 = \beta_2 = 1$. Then $(A_1) - (A_3)$ are satisfied. Hence, by

Theorem 1.1, system (1.2) has a positive solution.
Example 3.2. Let

\[ f(t, y) = \frac{e^{\frac{x}{t}}}{t(1-t)}, \quad g(t, x) = \frac{e^{\frac{x}{t}}}{t(1-t)} \]

and \( \alpha = 2, \eta = \frac{1}{2} \). Choose

\[ K(t) = L(t) = \frac{1}{t(1-t)}, \quad F(y) = e^{\frac{y}{t}}, \quad G(x) = e^{\frac{x}{t}}. \]

Choose constant \( M_1 \) such that

\[ M_1 \leq \frac{4(n-3)}{n} e^{-\frac{\alpha}{3}} \int_{1/3}^{1-1/n} (s-\frac{1}{n})(1-\frac{1}{n}-s) ds. \]

Then (A1), (A2) and (A4) are satisfied. Hence, by Theorem 1.2, system (1.2) has a positive solution.
Proof of Theorem 1.3. For \( u \in \partial \Omega_{M_2} \cap K_n \), using (2.8), we have

\[
(T_n u)(t) = \frac{1}{n} \int_{1/n}^{1-1/n} H_n(t, s) f(s, (B_n u)(s)) ds
\]

\[
= \frac{1}{n} \int_{1/n}^{1-1/n} H_n(t, s) f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau)) d\tau) ds.
\]

In view of (A_1), (A_5) and Lemma 2.3, we obtain

\[
\leq \frac{1}{n} \int_{1/n}^{1-1/n} H_n(t, s) K(s) F\left(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau)) d\tau\right) ds
\]

\[
\leq \frac{1}{n} \int_{1/n}^{1-1/n} H_n(t, s) K(s) F\left(\int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau)) d\tau\right) ds
\]

\[
\leq \frac{1}{n} \int_{1/n}^{1-1/n} H_n(t, s) K(s) F\left(\int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, M_2) d\tau\right) ds
\]

\[
\leq \frac{1}{n} \int_{1/n}^{1-1/n} H_n(t, s) K(s) F\left(\int_{1/n}^{1-1/n} \left(\tau - \frac{1}{n}\right)g(\tau, M_2) d\tau\right) ds
\]

\[
= \frac{1}{n} + F(\nu_n) \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - 1 - \tau)g(\tau, M_2) d\tau \int_{1/n}^{1-1/n} H_n(t, s) K(s) ds
\]

\[
\leq \frac{1}{n} + \mu_n F(\nu_n) \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - 1 - \tau)g(\tau, M_2) d\tau
\]

\[
\cdot \int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - 1 - s)K(s) ds
\]

\[
\leq \frac{1}{n} + a \mu_n F(\nu_n) \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - 1 - \tau)g(\tau, M_2) d\tau \leq M_2,
\]

which implies that

\[
\|T_n u\| \leq \|u\| \quad \text{for all } u \in \partial \Omega_{M_2} \cap K_n.
\]

By (A_5), we can choose \( \rho \in (0, M_2) \) such that (3.7) holds. Hence, \( T_n \) has a fixed point \( u_n \in K_n \cap (\Omega_{M_2} \setminus \Omega_{\rho}) \). By the same process as done in Theorem 1.1, the system (1.2) has a positive solution. \( \square \)

Example 3.3. Let

\[
f(t, y) = \begin{cases} \frac{y^{1/2}}{\rho(1-\tau)}, & y \leq 1, \\ \frac{y^{1/2}}{\rho(1-\tau)}, & y > 1, \end{cases}
\]

\[
g(t, x) = \begin{cases} \frac{x^{1/2}}{\rho(1-\tau)}, & x \leq 1, \\ \frac{x^{1/2}}{\rho(1-\tau)}, & x > 1, \end{cases}
\]
and $\alpha = 2$, $\eta = \frac{1}{3}$. Choose

$$K(t) = L(t) = \frac{1}{t(1-t)}$$

and $\beta_1 = \beta_2 = 1$. Choose constant $M_2$ such that

$$M_2 \geq \max\left\{ \frac{1}{n} + 6F(e(1-3/n)) \int_{1/3}^{1-1/n} \frac{(s-1/n)(1-1/n-s)}{s(1-s)} ds \right\}.$$ 

Then (A1), (A3) and (A5) are satisfied. Hence, by Theorem 1.3, system (1.2) has a positive solution.

**Proof of Theorem 1.4.** By (A1) and (A4), we obtain (3.10). By (A5) we can choose a constant $M_2 > M_1$ such that (3.11) holds. Then $T_n$ has a fixed point $u_n \in K_n \cap (\Omega_{M_2} \setminus \Omega_{M_1})$. By the same process as done in Theorem 1.1, the system (1.2) has a positive solution. $\square$

**Example 3.4.** Let

$$f(t, y) = \frac{1}{t(1-t)} \frac{1}{\sqrt{y}}, \quad g(t, x) = \frac{1}{t(1-t)} \frac{1}{x^2}$$

and $\alpha = 2$, $\eta = \frac{1}{3}$. Choose

$$K(t) = L(t) = \frac{1}{t(1-t)}$$

and $\beta_1 = \beta_2 = 1$. Choose $M_1$ and $M_2$ such that $M_1 \leq \frac{4(n-3)}{\sqrt{n(n^2+1)}} \int_{1/3}^{1-1/n} \frac{1}{s-1/n} (1-1/n-s) ds$ and $M_2 \geq \frac{1}{6n^3} \left( \frac{1}{n} - \sqrt{\frac{n-3}{n^2-1}} \int_{1/3}^{1-1/n} \frac{1}{s-1/n} (1-1/n-s) ds \right)^{-1/2}$. Then (A1), (A4) and (A5) are satisfied. Hence, by Theorem 1.4, system (1.2) has a positive solution.

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