$[r, s, t; f]$-COLORING OF GRAPHS

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Abstract. Let $f$ be a function which assigns a positive integer $f(v)$ to each vertex $v \in V(G)$, let $r$, $s$ and $t$ be non-negative integers. An $f$-coloring of $G$ is an edge-coloring of $G$ such that each vertex $v \in V(G)$ has at most $f(v)$ incident edges colored with the same color. The minimum number of colors needed to $f$-color $G$ is called the $f$-chromatic index of $G$ and denoted by $\chi'_f(G)$. An $[r, s, t; f]$-coloring of a graph $G$ is a mapping $c$ from $V(G) \cup E(G)$ to the color set $C = \{0, 1, \ldots, k - 1\}$ such that $|c(v_i) - c(v_j)| \geq r$ for every two adjacent vertices $v_i$ and $v_j$, $|c(e_i) - c(e_j)| \geq s$ and $a(v_i) \leq f(v_i)$ for all $v_i \in V(G)$, $a \in C$ where $a(v_i)$ denotes the number of $a$-edges incident with the vertex $v_i$ and $e_i$, $e_j$ are edges which are incident with $v_i$ but colored with different colors, $|c(e_i) - c(e_j)| \geq t$ for all pairs of incident vertices and edges. The minimum $k$ such that $G$ has an $[r, s, t; f]$-coloring with $k$ colors is defined as the $[r, s, t; f]$-chromatic number and denoted by $\chi_{r,s,t,f}(G)$. In this paper, we present some general bounds for $[r, s, t; f]$-coloring firstly. After that, we obtain some important properties under the restriction $\min\{r, s, t\} = 0$ or $\min\{r, s, t\} = 1$. Finally, we present some problems for further research.

1. Introduction

In this paper, the term graph is used to denote a simple connected graph $G$ with a finite vertex set $V(G)$ and a finite edge set $E(G)$. If multiple edges are allowed, $G$ is called a multigraph. The degree of a vertex $v$ in $G$ is the number of edges incident with $v$ and denoted by $d(v)$. We write $\delta(G) = \min\{d(v) : v \in V(G)\}$ and $\Delta(G) = \max\{d(v) : v \in V(G)\}$ to denote the minimum degree and maximum degree of $G$, respectively. Let $f$ be a function which assigns a positive integer $f(v)$ to each vertex $v \in V(G)$. We define $\Delta_f(G) = \max_{v \in V(G)}\{d(v)/f(v)\}$. Let $C$ denote the set of colors $\{0, 1, \ldots, k - 1\}$. A vertex (res. edge) coloring of a graph $G$ is a mapping $c$ from $V(G)$ (res. $E(G)$) to the color set $C$. A proper vertex (res. edge) coloring of
Then we focus on the case of values of will write let be a function which assigns a positive integer minimum and edges, respectively. The mapping non-negative integers defined as the two adjacent edges from got many interesting results. Let defined as above. An coloring of G is an edge-coloring of G such that each vertex v ∈ V(G) has at most f(v) edges colored with the same color. The minimum number of colors needed to f-color G is called the f-chromatic index of G and denoted by χ_f(G). Zhang and Liu [7, 8, 9] studied the f-coloring of graphs and got many interesting results.

Kemnitz and Marangio [6] studied the [r, s, t]-coloring of a graph G. Given non-negative integers r, s and t, an [r, s, t]-coloring of a graph G is a mapping c from V(G) ∪ E(G) to the color set C = {0, 1, . . . , k − 1} such that |c(v_i) − c(v_j)| ≥ r for every two adjacent vertices v_i and v_j, |c(e_i) − c(e_j)| ≥ s for every two adjacent edges e_i, e_j, and |c(e_i) − c(v_j)| ≥ t for all pairs of incident vertices and edges, respectively. The [r, s, t]-chromatic number χ_{r,s,t}(G) of G is the minimum k such that G has an [r, s, t]-coloring. Dekar, et al. [3] gave exact values of χ_{r,s,t}(G) of stars except one case.

Here we present a new coloring which is defined as [r, s, t; f]-coloring. Let f be a function which assigns a positive integer f(v) to each vertex v ∈ V(G), let r, s and t be non-negative integers. An [r, s, t; f]-coloring of a graph G is a mapping c from V(G) ∪ E(G) to the color set C = {0, 1, . . . , k − 1} such that |c(v_i) − c(v_j)| ≥ r for every two adjacent vertices v_i and v_j, |c(e_i) − c(e_j)| ≥ s and α(v_i) ≤ f(v_i) for all v_i ∈ V(G), α ∈ C where α(v_i) denotes the number of α-edges incident with the vertex v_i and e_i, e_j are edges which are incident with v_i but colored with different colors, |c(e_i) − c(v_j)| ≥ t for all pairs of incident vertices and edges. The minimum k such that G has an [r, s, t; f]-coloring is defined as the [r, s, t; f]-chromatic number and denoted by χ_{r,s,t;f}(G). Clearly, if s = 1, r = t = 0, then c is an f-coloring; if f(v) = 1 for all v ∈ V(G) (we will write f ≡ 1 for short in the following), then c is an [r, s, t]-coloring; if f ≡ 1 and r = 1, s = t = 0, then c is a proper vertex coloring; if f ≡ 1 and s = 1, r = t = 0, then c is a proper edge coloring; if f ≡ 1 and r = s = t = 1, then c is a total coloring. Similarly, let r = s = t = 1, we get another new coloring which we define as f-total coloring.

In this paper, we at first discuss some interesting results for this new coloring. Then we focus on the case r = s = 1 which are not considered in the [r, s, t]-coloring.
2. Basic results

Lemma 2.1. If $H \subseteq G$, then $\chi_{r,s,t,f}(H) \leq \chi_{r,s,t,f}(G)$.

*Proof.* It is obvious that the restriction of an $[r, s, t; f]$-coloring of $G$ to the element of $H \subseteq G$ is still an $[r, s, t; f]$-coloring of $H$. □

Lemma 2.2. Let $f$ and $f'$ be two functions defined as in the definition of $[r, s, t; f]$-coloring. If $f'(v) \geq f(v)$ for all $v \in V(G)$, and $r' \leq r$, $s' \leq s$, $t' \leq t$, then $\chi_{r',s',t',f'}(G) \leq \chi_{r,s,t,f}(G)$.

*Proof.* The proof is trivial. We leave it to the readers. □

These two lemmas are obvious but useful to determine bounds and exact values of the $[r, s, t; f]$-chromatic number of graphs.

Theorem 2.3. If $a \geq 0$ is an integer, then $\chi_{ar,as,at,f}(G) = a(\chi_{r,s,t,f}(G) - 1) + 1$.

*Proof.* If $a = 0$ or $1$, then the assertion is obvious. Suppose $a \geq 2$ and $c$ is an $[r, s, t; f]$-coloring of $G$ with $\chi_{r,s,t,f}(G)$ colors. Then $|c(v_i) - c(v_j)| \geq r$ for every two adjacent vertices $v_i$ and $v_j$, $|c(e_i) - c(e_j)| \geq s$, $\alpha(v_i) \leq f(v_i)$ for all $v_i \in V(G)$, $\alpha \in C$ where $\alpha(v_i)$ denotes the number of $\alpha$-edges incident with the vertex $v_i$. Let $e_i$, $e_j$ are edges which are incident with $v_i$ but colored with different colors, $|c(e_i) - c(e_j)| \geq t$ for all pairs of incident vertices and edges. Let $d'(x) = a \cdot c(x)$ for all $x \in V(G) \cup E(G)$, and we use $\alpha'$, $C'$ denote the new color and the new color set, respectively. Then we have

$$|c'(v_i) - c'(v_j)| = a \cdot |c(v_i) - c(v_j)| \geq ar,$$
$$|c'(e_i) - c'(e_j)| = a \cdot |c(e_i) - c(e_j)| \geq as,$$
$$|c'(e_i) - c'(v_j)| = a \cdot |c(e_i) - c(v_j)| \geq at.$$

For $\alpha' \in C'$, if $\alpha'(v_i) \neq 0$, then there is color $\alpha \in C$ such that $\alpha' = ao$ and $\alpha'(v_i) = \alpha(v_i) \leq f(v_i)$; if $\alpha'(v_i) = 0$, obviously we have $\alpha'(v_i) \leq f(v_i)$. Anyway, $\alpha'(v_i) \leq f(v_i)$ for all $v_i \in V(G)$, $\alpha' \in C'$.

Therefore, $c'$ is an $[ar, as, at; f]$-coloring of $G$ with colors $\{0, 1, \ldots, a(\chi_{r,s,t,f}(G) - 1)\}$.

On the other hand, assume that $G$ has an $[ar, as, at; f]$-coloring $c$ with color set $\{0, 1, \ldots, a(\chi_{r,s,t,f}(G) - 1)\}$, $a \geq 2$. Then we have $|c(v_i) - c(v_j)| \geq ar$ for every two adjacent vertices $v_i$ and $v_j$, $|c(e_i) - c(e_j)| \geq as$, $\alpha(v_i) \leq f(v_i)$ for all $v_i \in V(G)$, $\alpha \in C$ where $\alpha(\alpha(v_i))$ denotes the number of $\alpha$-edges incident with the vertex $v_i$ and $e_i$, $e_j$ are edges which are incident with $v_i$ but colored with different colors, $|c(e_i) - c(v_j)| \geq at$ for all pairs of incident vertices and edges. We define a coloring $c'$ by $c'(x) = \lfloor c(x)/a \rfloor$ for all $x \in V(G) \cup E(G)$, in which
Let \( \lfloor c(x)/a \rfloor \) be the largest integer not larger than \( c(x)/a \). Let \( \alpha' = \lfloor \alpha/a \rfloor \in C', C' \) denote the color set of \( c' \). Clearly, \(|x| \geq \|x\|\) for any real number \( x \). So we have
\[
|c'(v_i) - c'(v_j)| \geq \|\frac{c(v_i) - c(v_j)}{a}\| \geq r,
\]
\[
|c'(e_i) - c'(e_j)| \geq \|\frac{c(e_i) - c(e_j)}{a}\| \geq s,
\]
\[
|c'(e_i) - c'(v_j)| \geq \|\frac{c(e_i) - c(v_j)}{a}\| \geq t.
\]

Let \( e_i, e_j \) are two edges incident with \( v_i \), if they are both \( \alpha \)-edges, then \( c'(e_i) = c'(e_j) = \alpha' \); if \( c(e_i) = \alpha \) and \( c(e_j) \neq \alpha \), then \( |c(e_i) - c(e_j)| \geq as \) for \( a \) is an \([ar, as, at; f]\)-coloring of \( G \). This implies \( |c'(e_i) - c'(e_j)| \geq \|\frac{c(e_i) - c(e_j)}{a}\| \geq s \geq 1. \)

Therefore, \( \alpha'(v_i) = \alpha(v_i) \leq f(v_i) \). So
\[
\alpha'(v_i) \leq f(v_i) \quad \text{for all} \quad v_i \in V(G), \alpha' \in C'.
\]

That is, \( c' \) is an \([r, s, t; f]\)-coloring of \( G \) with colors
\[
\{0, 1, \ldots, \left\lfloor \frac{a(xr, s, t, f(G) - 1) - 1}{a}\right\rfloor\},
\]
where \( \left\lfloor \frac{a(xr, s, t, f(G) - 1) - 1}{a}\right\rfloor \leq \chi_{r, s, t, f}(G) - 2 \). We get an \([r, s, t; f]\)-coloring of \( G \) with no more than \( \chi_{r, s, t, f}(G) - 1 \) colors, a contradiction. \( \square \)

**Corollary 2.4.** If \( r = s = t \) and \( f(v) \equiv 1 \), then
\[
\chi_{r, s, t, f}(G) = r(\chi''(G) - 1) + 1,
\]
where \( \chi''(G) \) is the total chromatic number of graph \( G \).

**Corollary 2.5.** Let \( G \) be a graph and let \( r, s, t, f \) be defined as in the definition of \([r, s, t; f]\)-coloring. Then
\[
\chi_{r, 0, 0, f}(G) = r(\chi(G) - 1) + 1,
\]
\[
\chi_{0, s, 0, f}(G) = s(\chi'(G) - 1) + 1,
\]
\[
\chi_{0, 0, t, f}(G) = t + 1.
\]

**Lemma 2.6 ([4]).** Let \( G \) be a graph. Then
\[
\Delta_f(G) \leq \chi'_{f}(G) \leq \max_{v \in V(G)} \left\{ \left\lfloor (1 + d(v))/f(v) \right\rfloor \right\} \leq \Delta_f(G) + 1.
\]

**Theorem 2.7.** Let \( G \) be a graph and let \( r, s, t, f \) be defined as in the definition of \([r, s, t; f]\)-coloring. Then
\[
\max\{r(\chi(G) - 1) + 1, s(\chi'(G) - 1) + 1, t + 1\}
\leq \chi_{r, s, t, f}(G) \leq r(\chi(G) - 1) + s(\chi'(G) - 1) + t + 1.
\]
Proof. (a) If \( f(v) = d(v) \) for all \( v \in V(G) \), then
\[
\Delta_f(G) = \max_{v \in V(G)} \{[d(v)/f(v)]\} = 1,
\]
and we can use one color to \( f \)-color \( G \). Therefore, \( \chi_f'(G) = \Delta_f(G) = 1 \). Let \( c \) be an \([r,0,0,f]\)-coloring of \( G \) with \( r(\chi(G)-1)+1 \) colors. Then we assign color \( r(\chi(G)-1)+t \) to all the edges of \( G \), we get an \([r, s, t, f]\)-coloring with \( r(\chi(G)-1)+t+1 \) colors. This is the upper bound, and the lower bound is obvious by Lemma 2.2 and Corollary 2.5.

(b) If there is a vertex \( u \in V(G) \) such that \( f(u) < d(u) \), then \( \chi_f'(G) \geq 2 \). In this case, consider \( c \) mentioned in part (a). We use colors \( r(\chi(G)-1)+t, r(\chi(G)-1)+t+s, \ldots, r(\chi(G)-1)+t+s(\chi_f'(G)-1) \) to color the edges. Then we get an \([r, s, t, f]\)-coloring with \( r(\chi(G)-1)+s(\chi_f'(G)-1)+t+1 \) colors. The lower bound can be got by Lemma 2.2 and Corollary 2.5. \( \square \)

**Lemma 2.8.** Let \( G \) be a graph and let \( r, s, t, f \) be defined as in the definition of \([r, s, t, f]\)-coloring. If \( t > r(\chi(G)-1)+s(\chi_f'(G)-1) \), then
\[
\chi_{r,s,t,f}(G) \geq r(\chi(G)-1)+s(\delta_f(G)-1)+t+1,
\]
where \( \delta_f(G) = \min_{v \in V(G)} \{[d(v)/f(v)]\} \).

**Proof.** Let \( c \) be an \([r, s, t, f]\)-coloring of \( G \) with \( \chi_{r,s,t,f}(G) \) colors. By Theorem 2.7 and the assumption on \( t \) we obtain \( 2t+1 > r(\chi(G)-1)+s(\chi_f'(G)-1)+t+1 \geq \chi_{r,s,t,f}(G) \). So \( \chi_{r,s,t,f}(G) \leq 2t \). If there is a vertex \( v \) and incident edges \( e_1, e_2 \) such that \( c(e_1) < c(v) < c(e_2) \) or an edge \( e = v_1v_2 \) such that \( c(v_1) < c(e) < c(v_2) \), then at least \( 2t+1 \) colors are needed which contradicts with the conclusion \( \chi_{r,s,t,f}(G) \leq 2t \). Therefore, if \( x \) is an arbitrary element of \( G \), then \( c(x) < c(y) \) for all elements \( y \) that are incident to \( x \) or \( c(x) > c(y) \) for all \( y \). By induction, we obtain either \( c(v) < c(e) \) for all vertices \( v \) and all edges \( e \) incident to \( v \) or always \( c(v) > c(e) \). Without loss of generality, we assume \( c(v) < c(e) \).

Consider the vertex \( u \) which obtains the greatest color \( c(u) \). In order to proper coloring the vertex set of graph \( G \), at least \( \chi_{r,0,0,f}(G) \) colors are needed. By Corollary 2.5 we have \( \chi_{r,0,0,f}(G) = r(\chi(G)-1)+1 \). Therefore, \( c(u) \geq r(\chi(G)-1) \). In the \( f \)-coloring, denote by \( r(u) \) the color numbers appeared on the edges which are incident with \( u \). Obviously, we have \( r(u)f(u) \geq d(u) \), which implies \( r(u) \geq \min_{v \in V(G)} \{[d(v)/f(v)]\} = \delta_f(G) \). That is to say, there are at least \( \delta_f(G) \) different colors which are greater than \( c(u) \) by our assumption appeared on \( u \). Then we get \( \chi_{r,s,t,f}(G) \geq c(u)+t+s(\delta_f(G)-1) \geq r(\chi(G)-1)+s(\delta_f(G)-1)+t+1 \). \( \square \)

By Lemma 2.6, all graphs are partitioned into two classes. One is graphs with \( \chi_f'(G) = \Delta_f(G) \), called \( C_f \), 1, or \( f \)-class 1, and the other with \( \chi_f'(G) = \Delta_f(G)+1 \), called \( C_f \) 2, or \( f \)-class 2.
Just as the case we discussed in Theorem 2.7, $\chi_f'(G) = \Delta_f(G) = 1$ when $f(v) = d(v)$ for all $v \in V(G)$. This also implies that $\delta_f(G) = 1$. So by Theorem 2.7 and Lemma 2.8 we have the following result.

**Corollary 2.9.** Suppose that $t > r(\chi(G) - 1) + s(\chi_f'(G) - 1)$.

1. If $f(v) = d(v)$ for all $v \in V(G)$, then
   $$\chi_{r,s,t,f}(G) = r(\chi(G) - 1) + t + 1;$$

2. If (1) is not satisfied, but $G$ is a $C_f$ 1 graph with $\Delta_f(G) = \delta_f(G)$, then
   $$\chi_{r,s,t,f}(G) = r(\chi(G) - 1) + s(\chi_f'(G) - 1) + t + 1.$$

Corollary 2.9 provides a subclass of graphs that can reach the upper bound of Theorem 2.7.

In Section 3 and Section 4, we will give some restriction to the parameters $r, s, t, f$ in order to obtain some new results.

### 3. $\min\{r, s, t\} = 0$

We consider the case only one of $r$, $s$, $t$ equals 0. The case where two of $r$, $s$, $t$ equal 0 is discussed in Corollary 2.5.

**Theorem 3.1.** Let $G$ be a graph. Then

$$\chi_{r,s,0,t,f}(G) = \max\{r(\chi(G) - 1) + 1, s(\chi_f'(G) - 1) + 1\}.$$

*Proof.* This equation can be obtained by Theorem 2.7 and the fact that vertices and edges can be colored independently. □

**Lemma 3.2 ([6]).** Let $G$ be a graph. Then

1. If $\chi(G) = 2$, then
   $$\chi_{r,0,t}(G) = \begin{cases} 
   r + 1 & \text{if } r \geq 2t; \\
   2t + 1 & \text{if } t \leq r < 2t; \\
   r + t + 1 & \text{if } r < t.
   \end{cases}$$

2. If $\chi(G) \geq 3$ and $r \geq t$, then
   $$\chi_{r,0,t}(G) = r(\chi(G) - 1) + 1;$$

3. If $\chi(G) \geq 3$ and $r < t$, then
   $$\max\{r(\chi(G) - 1) + 1, t + 1\} \leq \chi_{r,0,t}(G) \leq r(\chi(G) - 3) + t + 1 + \min\{t, 2r\}.$$

**Theorem 3.3.** Let $G$ be a graph. If $f(v) = d(v)$ for all $v \in V(G)$, then

$$\chi_{r,0,t,f}(G) = \chi_{r,0,t}(G),$$

where $\chi_{r,0,t}(G)$ is the same as that in Lemma 3.2.

*Proof.* If $f(v) = d(v)$ for all $v \in V(G)$, then $\chi_f'(G) = \Delta_f(G) = 1$. That is, we can color all the edges of $G$ with one color and the condition $\alpha(v) \leq f(v)$ for all $v \in V(G), \alpha \in C$ in the definition of $[r, s, t; f]$-coloring has no influence. Therefore, we have $\chi_{r,0,t,f}(G) = \chi_{r,0,t}(G)$. □
Note that if there is a vertex \( u \in V(G) \) such that \( f(u) < d(u) \), then at least 2 colors are needed for the edges of \( G \). Therefore, \( s = 0 \) is impossible in this case.

**Lemma 3.4** ([6]). Let \( G \) be a graph. Then

(1) If \( \Delta(G) \geq 2 \) and \( G \) is of class 1, then

\[
\chi_{0,s,t}(G) = \begin{cases} 
  s(\Delta(G) - 1) + 1 & \text{if } s \geq 2t; \\
  s(\Delta(G) - 1) + 2t - s + 1 & \text{if } t \leq s < 2t; \\
  s(\Delta(G) - 1) + t + 1 & \text{if } s < t.
\end{cases}
\]

(2) If \( \Delta(G) \geq 2 \), \( G \) is of class 2 and \( s \geq t \), then

\[
\chi_{0,s,t}(G) = s(\chi'(G) - 1) + 1;
\]

(3) If \( \Delta(G) \geq 2 \), \( G \) is of class 2 and \( s < t \), then

\[
s(\Delta(G) - 1) + t + 1 \leq \chi_{0,s,t}(G) \leq \min\{s\Delta(G) + t + 1, t\Delta(G) + 1\}.
\]

**Theorem 3.5.** Let \( G \) be a graph. Then

(a) if \( f(v) = d(v) \) for all \( v \in V(G) \), then \( \chi_{0,s,t,f}(G) = t + 1 \);

(b) otherwise,

(1) \( \Delta_f(G) \geq 2 \) and \( G \) is of \( C_f \) 1, then

\[
\chi_{0,s,t,f}(G) = \begin{cases} 
  s(\Delta_f(G) - 1) + 1 & \text{if } s \geq 2t; \\
  s(\Delta_f(G) - 1) + 2t - s + 1 & \text{if } t \leq s < 2t; \\
  s(\Delta_f(G) - 1) + t + 1 & \text{if } s < t.
\end{cases}
\]

(2) \( \Delta_f(G) \geq 2 \), \( G \) is of \( C_f \) 2 and \( s \geq t \), then

\[
\chi_{0,s,t,f}(G) = s(\chi'_f(G) - 1) + 1;
\]

(3) \( \Delta_f(G) \geq 2 \), \( G \) is of \( C_f \) 2 and \( s < t \), then

\[
s(\Delta_f(G) - 1) + t + 1 \leq \chi_{0,s,t,f}(G) \leq \min\{s\Delta_f(G) + t + 1, t\Delta_f(G) + 1\}.
\]

**Proof.** (a) If \( f(v) = d(v) \) for all \( v \in V(G) \), then we can color all the vertices with color 0 and all the edges with color \( t \). Then we obtain an \([0, s, t; f]\)-coloring of \( G \) with \( t + 1 \) colors. On the other hand, by Theorem 2.7 we get \( \chi_{0,s,t,f}(G) \geq t + 1 \). Therefore, \( \chi_{0,s,t,f}(G) = t + 1 \).

(b) If there is a vertex \( u \in V(G) \) such that \( f(u) < d(u) \), then the proof is similar to the proof in [4] (see A. Kemnitz, M. Marangio [4] Lemmas 7, 8, 9) just using \( \Delta_f(G) \) instead of \( \Delta(G) \). We don’t mention it here. \( \square \)

4. \( \min\{r, s, t\} = 1 \)

In this section we will consider the three parameters \( \chi_{r,1,1,f}(G) \), \( \chi_{1,s,1,f}(G) \), \( \chi_{1,1,t,f}(G) \), especially the last one.

**Theorem 4.1.** If \( r \geq \frac{\chi'_f(G)}{\chi(G) - 1} + 1 \), then \( \chi_{r,1,1,f}(G) = r(\chi(G) - 1) + 1 \).
Lemma 4.3. Let \( G \) be a graph and let \( t \) and \( f \) be defined as in the definition of \([r, s, t; f]\)-coloring. Then we have

\[
\Delta_f(G) + t \leq \chi_{1,1,t,f}(G) \leq \chi(G) + \chi_f(G) + t - 1.
\]

Proof. The upper bound can be obtained by Theorem 2.7. On the other hand, by Lemma 2.2 we get \( \chi_{1,1,t,f}(G) \geq \chi_{0,1,t,f}(G) \). Then by Theorem 3.5 we obtain the lower bound. \( \square \)

When we investigate the \([r, s, t; f]\)-chromatic number under the special case \( r = s = 1 \), we can improve the result in Lemma 4.3 as Theorem 4.6.

Lemma 4.4 ([7]). Let \( G \) be a complete graph \( K_n \). If \( k \) and \( n \) are odd integers, \( f(v) = k \) and \( k|d(v)| \) for all \( v \in V(G) \), then \( G \) is of \( C_f \) 2. Otherwise, \( G \) is of \( C_f \) 1.

Lemma 4.5 ([2], Brook’s Theorem). \( \chi(G) \leq \Delta(G) + 1 \) holds for every graph \( G \). Moreover, \( \chi(G) = \Delta(G) + 1 \) if and only if either \( \Delta(G) \neq 2 \) and \( G \) has a complete graph \( K_{\Delta(G)+1} \) as a connected component, or \( \Delta(G) = 2 \) and \( G \) has an odd cycle as a connected component.

Theorem 4.6. Let \( G \) be a graph and let \( t \), \( f \) be defined as in the definition of \([r, s, t; f]\)-coloring. Then we have

\[
\chi_{1,1,t,f}(G) \leq \Delta(G) + \Delta_f(G) + t.
\]
Proof. We now consider three cases depending on $G$.

**Case 1.** If $G$ is neither a complete graph nor an odd cycle, then $\chi(G) \leq \Delta(G)$ by Lemma 4.5 and $\chi'_f(G) \leq \Delta_f(G) + 1$ by Lemma 2.6. Hence, the inequality is true.

**Case 2.** $G$ is the complete graph $K_n$ on $n$ vertices. By Lemma 4.4 we know that $K_n$ is of $C_f$ 1 except one case. Then we have $\chi'_f(G) = \Delta_f(G)$. By Lemma 4.3, we have $\chi_{1,1,t;f}(G) \leq (\Delta(G) + 1) + \Delta_f(G) + t - 1 = \Delta(G) + \Delta_f(G) + t$. Now we assume that $k$ and $n$ are odd integers, $f(v) = k$ and $k|d(v)$ for all $v \in V(G)$. Then Lemma 4.4 implies that $G$ is of $C_f$ 2.

**Case 2.1.** If $f(v) = d(v)$, then we have $\chi'_f(G) = \Delta_f(G) = 1$. We can assign all the edges with one color $n + t - 1$ and assign the vertices differently with colors $0, 1, \ldots, n - 1$. Therefore, we obtain an $[1,1,t;f]$-coloring of $K_n$ with $n + t = \Delta(K_n) + \Delta_f(K_n) + t$ colors.

**Case 2.2.** If $f(v) \equiv 1$, then it becomes an $[1,1,t]$-coloring of $K_n$. Let $c$ be a proper edge coloring of $K_n$ with $n$ colors and $M_i$ $(1 \leq i \leq n)$ be the matchings corresponding to the color classes. Further more, each $M_i$ contains all vertices but one $v_i$ (We know that it is true for $K_n$ when $n$ is odd, because $\chi'(K_n) = n = \Delta + 1, |M_i| \leq \frac{n-1}{2}, 1 \leq i \leq n$, and if there is an integer $j$ such that $|M_j| < \frac{n-1}{2}$, then $\chi'(K_n) > \Theta(K_n) = \frac{n(n-1)}{2}$, a contradiction). For $1 \leq i \leq n$, color the vertex $v_i$ with color $n - i$ and the edges in $M_i$ with $n + t - 3 + i$. Since $v_i$ is not incident to $M_i$, then we obtain an $[1,1,t;1]$-coloring of $K_n$ with $2n + t - 3 = \Delta(K_n) + \Delta_1(K_n) + t$ colors.

**Case 2.3.** If $1 < f(v) < d(v)$, then $f(v) = k \geq 3$ and

$$\Delta_f(G) = \max_{v \in V(G)} \{[d(v)/f(v)]\} = \frac{n-1}{k} \overset{\text{def}}{=} 2\alpha.$$ 

Let $M_i$ be defined as in Case 2.2 and let $M'_i = M_i$, $M''_i = \bigcup_{j=2}^{k+1} M_{i-2k+j}$, $2 \leq i \leq 2\alpha + 1$. Color the vertex $v_i$ with color $n - i$ and the edges in $M'_i$ with color $n + t - 3 + i$, $2 \leq i \leq 2\alpha + 1$. We obtain an $[1,1,t;f]$-coloring of $K_n$ with $n + t - 3 + (2\alpha + 1) + 1 = \Delta(K_n) + \Delta_f(K_n) + t$ colors.

**Case 3.** $G$ is an odd cycle. Then $\Delta = 2, \Delta_f(G) = \max_{v \in V(G)} \{[d(v)/f(v)]\} \leq 2$.

**Case 3.1.** If $f(v) = d(v)$ for all $v \in V(G)$, then $\chi'_f(G) = \Delta_f(G) = 1$. We assign colors 0 and 1 to the vertices along the odd cycle alternately and assign color 2 to the final vertex. Then we color all the edges of $G$ with color $\Delta(G) + \Delta_f(G) + t - 1 = t + 2$. We obtain an $[1,1,t;f]$-coloring of $G$ with $\Delta(G) + \Delta_f(G) + t$ colors.

**Case 3.2.** If there is a vertex $u \in V(G)$ such that $f(u) < d(u) = 2$, which implies $f(u) = 1$, $\Delta_f(G) = 2$. We color $u$ with color 2 and the other vertices with 0 and 1 alternately. Denoted by $e_1, e_2$ the edges incident with $u$. Next, we color edge $e_1$ with color $t + 2$, color the edge adjacent with $e_1$ but not $e_2$
with color \( t + 1 \). In this order, we color the edges along the cycle with colors \( t + 2, t + 1 \) alternately except for coloring \( c_2 \) with color \( t + 3 \). Then we obtain an \([1, 1, t; f]\)-coloring of \( G \) with \( t + 4 = \Delta(G) + \Delta_f(G) + t \) colors.

In any case, we all prove that \( \chi_{1,1,t;f}(G) \leq \Delta(G) + \Delta_f(G) + t \).

\[ \Box \]

**Lemma 4.7.** Let \( t \geq 2 \) be an integer. Then

1. If \( \delta_f(G) = \Delta_f(G) \), then \( \chi_{1,1,t;f}(G) \geq \Delta_f(G) + t + 1 \);
2. If \( t \geq \Delta_f(G) \), then \( \chi_{1,1,t;f}(G) \geq \Delta_f(G) + t + 1 \).

**Proof.** Assume that we have an \([1, 1, t; f]\)-coloring of \( G \) with colors \( \{0, 1, \ldots, \Delta_f(G) + t - 1 \} \). We at first prove that the vertex \( u \) with \( [d(u)/f(u)] = \Delta_f(G) \) must be assigned color 0 or \( \Delta_f(G) + t - 1 \). Consider \( u \) and all the edges which are incident to it. We denote the subgraph by \( H \). Then at least \( \Delta_f(G) \) colors are needed for \([1, 1, t; f]\)-coloring the edges of \( H \). Without loss of generality, we denote the colors by \( C_1 < C_2 < \cdots < C_{\Delta_f} \). If there is an integer \( i \), such that \( C_i < c(u) < C_{i+1} \), then \( C_{\Delta_f} \geq 2t + \Delta_f(G) - 2 > \Delta_f(G) + t - 1 \), a contradiction. If \( c(u) < C_1 \), then \( C_1 \geq t \) which implies that \( c(u) = 0 \) and \( C_1 = t + 1, C_2 = t + 2, \ldots, C_{\Delta_f} = \Delta_f(G) + t - 1 \); If \( c(u) > C_{\Delta_f} \), then we can get \( c(u) = \Delta_f(G) + t - 1 \) and \( C_1 = t + 1, C_2 = t + 2, \ldots, C_{\Delta_f} = \Delta_f(G) + t - 1 \) by the same way. Without loss of generality, we assume that \( c(u) = 0 \).

1. If \( \delta_f(G) = \Delta_f(G) \), then every vertex must be assigned color 0 or \( \Delta_f(G) + t - 1 \). Let \( uv \) be an edge colored with color \( \Delta_f(G) + t - 1 \). We see that \( v \) can be labeled by neither 0 nor \( \Delta_f(G) + t - 1 \), a contradiction.
2. If \( t \geq \Delta_f(G) \), let \( uv \) be an edge colored with color \( t \), then \( c(v) \geq 2t \geq \Delta_f(G) + t \) by the assumption \( t \geq \Delta_f(G) \), a contradiction.

\[ \Box \]

**Lemma 4.8** ([7]). Let \( G(V, E) \) be a bipartite graph and \( \Delta_f(G) = \max_{v \in V(G)} \{[d(v)/f(v)]\} \).

Then \( \chi'_f(G) = \Delta_f(G) \).

**Theorem 4.9.** Let \( G(V, E) \) be a bipartite graph. Then

1. \( \Delta_f(G) + t \leq \chi_{1,1,t;f}(G) \leq \Delta_f(G) + t + 1 \);
2. If \( t \geq \Delta_f(G) \) or \( \delta_f(G) = \Delta_f(G) \), then \( \chi_{1,1,t;f}(G) = \Delta_f(G) + t + 1 \).

**Proof.** If \( G \) is a bipartite graph, then \( \chi(G) = 2 \) and \( \chi'_f(G) = \Delta_f(G) \) by Lemma 4.8. Together with Lemma 4.3 we obtain (1).

(2) can be obtained by Lemma 4.7 and (1) of Theorem 4.9.

Note that for a bipartite graph \( G \), \( \chi(G) = 2 \) and \( \chi'_f(G) = \Delta_f(G) \). If \( t \geq \Delta_f(G) \), by (2) of Theorem 4.9 we get \( \chi_{1,1,t;f}(G) = \Delta_f(G) + t + 1 \); If \( t = 0 \), by Theorem 3.1 we get \( \chi_{1,1,t;f}(G) = \max\{2, \Delta_f(G)\} \); If \( 1 \leq t < \Delta_f(G) \), by (1) of Theorem 4.9 we have \( \chi_{1,1,t;f}(G) = \Delta_f(G) + t + 1 \) or \( \Delta_f(G) + t \). We may ask what conditions are needed for a bipartite graph \( G \) with \( 1 \leq t < \Delta_f(G) \) to satisfy \( \chi_{1,1,t;f}(G) = \Delta_f(G) + t + 1 \)?
5. Problems for further research

In this paper, we present a new coloring of a graph $G$ for the first time. We named it an $[r, s, t; f]$-coloring of $G$ and investigate some interesting properties on the $[r, s, t; f]$-chromatic number. Some are the generalization of the results about the $[r, s, t]$-coloring and the other are new. However, all the results in our paper are correct for $[r, s, t]-$coloring just let $f(v) = 1$ for all $v \in V(G)$.

Finally, we present the following problems for further research.

Problem 1. Find the properties of the $f$-total coloring as we defined in Section 1. Is there a conjecture like the TCC for it?

Problem 2. Find the other results on the chromatic number $\chi_{1,1,t,f}(G)$.

Problem 3. Find the exact values of $\chi_{r,s,t,f}(G)$ for some special graphs.

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