SOME NOTES ON EXTENSIONS
OF BASIC UNIVALENCE CRITERIA

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ABSTRACT. The object of the present paper is to obtain a more general condition for univalence of analytic functions in the open unit disk $U$. The significant relationships and relevance with other results are also given. A number of known univalent conditions would follow upon specializing the parameters involved in our main results.

1. Introduction

We denote by $U_r$ the disk $\{z \in \mathbb{C}: |z| < r\}$, where $0 < r \leq 1$, by $U = U_1$ the open unit disk of the complex plane and by $I$ the interval $[0, \infty)$. Let $A$ denote the class of analytic functions in the open unit disk $U$ which satisfy the usual normalization condition:

$$f(0) = f'(0) - 1 = 0.$$ 

Three of the most important and known univalence criteria for analytic functions defined in the open unit disk were obtained by Nehari [4], Ozaki-Nunokawa [7] and Becker [1]. Some extensions of these three criteria were given by (see [6, 9, 10, 11, 12, 13] and [14]). During the time, unlike there were obtained a lot of univalence criteria (see also [2], [3] and [5]).

Our univalence conditions contain as special cases, Tudor’s results and other results obtained by some of the authors cited in references.

**Theorem 1.1** (see [1]). Let $f \in A$. If for all $z \in U$

$$\frac{1}{(1 - |z|^2)} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1,$$ 

then the function $f$ is univalent in $U$.

**Theorem 1.2** (see [7]). Let $f \in A$. If for all $z \in U$

$$\left| \frac{z^2f'(z)}{f^2(z)} - 1 \right| < 1,$$
then the function \( f \) is univalent in \( U \).

**Theorem 1.3** (see [4]). Let \( f \in A \). If for all \( z \in U \)
\[
|\{f; z\}| \leq \frac{2}{(1 - |z|^2)^2},
\]
where
\[
\{f; z\} = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.
\]

Then the function \( f \) is univalent in \( U \).

In the present paper we consider the analyticity and univalence of functions \( f(z) \) belonging to the class \( A \). Our considerations are based on the theory of Loewner chains. A function \( L : U \times I \to \mathbb{C} \) is called a Loewner chain if it is analytic and univalent in \( U \) and \( L(z, s) \) is subordinate to \( L(z, t) \) for all \( 0 \leq s \leq t < \infty \). Consider \( f \) and \( g \) analytic functions in \( U \). We say that \( f \) is subordinate to \( g \), written \( f \prec g \), if there exists a function \( w \) analytic in \( U \) which satisfies \( w(0) = 0 \), \( |w(z)| < 1 \) and \( f(z) = g(w(z)) \) for all \( z \in U \).

### 2. Preliminaries

In proving our results, we will need the following theorem due to Ch. Pommerenke [8].

**Theorem 2.1.** Let \( L(z, t) = a_1(t)z + a_2(t)z^2 + \cdots, a_1(t) \neq 0 \) be analytic in \( U_r \) for all \( t \in I \), locally absolutely continuous in \( I \), and locally uniform with respect to \( U_r \). For almost all \( t \in I \), suppose that
\[
z \frac{\partial L(z, t)}{\partial z} = p(z, t) \frac{\partial L(z, t)}{\partial t}, \forall z \in U_r,
\]
where \( p(z, t) \) is analytic in \( U \) and satisfies the condition \( \Re\{p(z, t)\} > 0 \) for all \( z \in U, t \in I \). If \( |a_1(t)| \to \infty \) for \( t \to \infty \) and \( \{L(z, t)/a_1(t)\} \) forms a normal family in \( U_r \), then for each \( t \in I \), the function \( L(z, t) \) has an analytic and univalent extension to the whole disk \( U \).

### 3. Main results

Making use of Theorem 2.1 we can prove now, our main results.

**Theorem 3.1.** Let \( m \) be a positive real number and let \( \alpha \) be a complex number such that \( \Re(\alpha) > \frac{1}{m+1} \) and \( f \in A \). Let \( g \) and \( h \) be two analytic functions in \( U \), \( g(z) = 1 + b_1z + \cdots, h(z) = c_0 + c_1z + \cdots \). If the following inequalities
\[
\left| \left( \frac{1}{\alpha} \frac{f'(z)}{g(z)} - 1 \right) - \frac{m - 1}{2} \right| < \frac{m + 1}{2}
\]

and
\begin{equation}
\left(\frac{1}{\alpha} \frac{f'(z)}{g(z)} - 1\right) |z|^{2(m+1)} + z^2 \left(1 - |z|^{m+1}\right)^2 \left[\frac{1}{\alpha} \frac{f'(z)h(z)}{f(z)} + \frac{1}{\alpha} \frac{f'(z)h^2(z)}{f(z)} + \frac{g'(z)h(z)}{g(z)} - h'(z)\right] + z |z|^{m+1} \left(1 - |z|^{m+1}\right) \left[\frac{1}{\alpha} \frac{f'(z)}{f(z)} + \frac{2}{\alpha} \frac{f'(z)h(z)}{g(z)} + \frac{g'(z)}{g(z)}\right] - \frac{m-1}{2} |z|^{m+1} \leq \frac{m+1}{2} |z|^{m+1}
\end{equation}
are satisfied for all \(z \in U\), then the function \(f\) is univalent in \(U\) where the principal branch is intended.

\textbf{Proof.} Let \(a\) and \(b\) be two positive real numbers such that \(m = \frac{b}{a}\). We prove that there exists a real number \(r \in (0, 1]\) such that the function \(L : U_r \times I \to \mathbb{C}\), defined formally by
\begin{equation}
L(z, t) = f^{1-\alpha}(e^{-at}z) \left[f(e^{-at}z) + \frac{(e^{bt} - e^{-at})zg(e^{-at}z)}{1 + (e^{bt} - e^{-at})zh(e^{-at}z)}\right]^{\alpha}
\end{equation}
is analytic in \(U_r\) for all \(t \in I\).

Let us consider the function \(\varphi_1(z, t)\) given by
\begin{equation}
\varphi_1(z, t) = 1 + (e^{bt} - e^{-at})zh(e^{-at}z).
\end{equation}
For all \(t \in I\) and \(z \in U\) we have \(e^{-at}z \in U\) and because \(h\) analytic, the function \(\varphi_1(z, t)\) is analytic in \(U\) and \(\varphi_1(0, t) = 1\). Then there exist a disc \(U_{r_1}\), \(0 < r_1 < 1\), in which \(\varphi_1(z, t) \neq 0\) for all \(t \in I\) and \(z \in U_{r_1}\).

For the function
\begin{equation}
\varphi_2(z, t) = \left[f(e^{-at}z) + \frac{(e^{bt} - e^{-at})zg(e^{-at}z)}{\varphi_1(z, t)}\right]^{\alpha},
\end{equation}
\(\varphi_2(z, t) = z^\alpha \varphi_3(z, t)\), it can be easily shown that \(\varphi_3(z, t)\) is analytic in \(U_{r_1}\) and \(\varphi_3(0, t) = e^{bt}\) for all \(t \in I\). From these considerations it follows that the function
\begin{equation}
L(z, t) = f^{1-\alpha}(z, t)\varphi_2(z, t)
\end{equation}
is analytic in \(U_{r_1}\) for all \(t \in I\) and has the following form
\[L(z, t) = a_1(t)z + \cdots .\]
We have
\begin{equation}
a_1(t) = e^{[\alpha(a+b)-a]t}
\end{equation}
Moreover, \( a_l(t) \neq 0 \) for all \( t \in I \).
From the analyticity of \( L(z, t) \) in \( U_{r_1} \), it follows that there exists a number \( r_2, 0 < r_2 < r_1 \), and a constant \( K = K(r_2) \) such that
\[
\left| \frac{L(z, t)}{a_l(t)} \right| < K, \quad \forall z \in U_{r_2}, \quad t \in I.
\]

Then, by Montel’s Theorem, \( \{ L(z, t) \} \) is a normal family in \( U_{r_2} \). From the analyticity of \( \frac{\partial L(z, t)}{\partial t} \), we obtain that for all fixed numbers \( T > 0 \) and \( r_3, 0 < r_3 < r_2 \), there exists a constant \( K_1 > 0 \) (that depends on \( T \) and \( r_3 \)) such that
\[
\left| \frac{\partial L(z, t)}{\partial t} \right| < K_1, \quad \forall z \in U_{r_3}, \quad t \in [0, T].
\]

Therefore, the function \( L(z, t) \) is locally absolutely continuous in \( I \), locally uniform with respect to \( U_{r_3} \).

Let \( p : U_r \times I \to \mathbb{C} \) be the function in \( U_r \), \( 0 < r < r_3 \) for all \( t \in I \), defined by
\[
p(z, t) = \frac{z - \partial L(z, t)}{\partial z} - \frac{\partial L(z, t)}{\partial t}.
\]

If the function
\[
w(z, t) = \frac{p(z, t) - 1}{p(z, t) + 1} = \frac{z - \partial L(z, t)}{\partial z} - \frac{\partial L(z, t)}{\partial t} + \frac{\partial L(z, t)}{\partial z}
\]
is analytic in \( U \times I \) and \( |w(z, t)| < 1 \) for all \( z \in U \) and \( t \in I \), then \( p(z, t) \) has an analytic extension with positive real part in \( U \) for all \( t \in I \). From equality (3.8) we have
\[
w(z, t) = \frac{(1 + a)A_\alpha(z, t) + (1 - b)}{(1 - a)A_\alpha(z, t) + (1 + b)},
\]
where
\[
A_\alpha(z, t) = e^{-(a + b)t} \left\{ \frac{f'(e^{-at}z)}{\alpha f(e^{-at}z)} + (e^{at} - e^{-at})z^2 \left[ \frac{(1 - \alpha)f'(e^{-at}z)h(e^{-at}z)}{\alpha f(e^{-at}z)} \right. \right.
\[
+ \left. \left. \frac{f''(e^{-at}z)h^2(e^{-at}z)}{\alpha^2 g(e^{-at}z)} + \frac{g'(e^{-at}z)h(e^{-at}z)}{g(e^{-at}z)} - h'(e^{-at}z) \right] \right.
\[
+ \left. (e^{bt} - e^{-at})z \left[ \frac{(1 - \alpha)f'(e^{-at}z)h(e^{-at}z)}{\alpha f(e^{-at}z)} + 2 \frac{f'(e^{-at}z)h(e^{-at}z)}{\alpha g(e^{-at}z)} + \frac{g'(e^{-at}z)}{g(e^{-at}z)} \right) \right\} - 1
\]
for \( z \in U \) and \( t \in I \).
The inequality $|w(z, t)| < 1$ for all $z \in U$ and $t \in I$, where $w(z, t)$ is defined by (3.9), is equivalent to

$$
|A_\alpha(z, t) - \frac{b - a}{2a}| < \frac{b + a}{2a}, \quad \forall z \in U, \ t \in I.
$$

Define

$$
B_\alpha(z, t) = A_\alpha(z, t) - \frac{m - 1}{2}, \quad \forall z \in U, \ t \in I.
$$

From (3.1), (3.10) and $\Re(\alpha) > \frac{1}{m+1}$ we have

$$
|B_\alpha(z, t)| = \left| \left( \frac{f'(u)}{\alpha g(u)} - 1 \right) - \frac{m - 1}{2} \right| < \frac{m + 1}{2}
$$

and

$$
|B_\alpha(0, t)| = \left| \left( \frac{1}{\alpha} - 1 \right) - \frac{1}{\alpha} \right| < \frac{m + 1}{2}.
$$

Since $|e^{-at}| \leq |e^{-at}| = e^{-at} < 1$ for all $z \in \hat{U} = \{ z \in \mathbb{C} : |z| \leq 1 \}$ and $t > 0$, we find that $B_\alpha(z, t)$ is an analytic function in $\hat{U}$. Using the maximum modulus principle it follows that for all $z \in U - \{ 0 \}$ and each $t > 0$ arbitrarily fixed there exists $\theta = \theta(t) \in \mathbb{R}$ such that

$$
|B_\alpha(z, t)| < \max_{|z|=1} |B_\alpha(z, t)| = |B_\alpha(e^{i\theta}, t)|
$$

for all $z \in U$ and $t \in I$.

Denote $u = e^{-at}e^{i\theta}$. Then $|u| = e^{-at}$, $e^{-(a+b)t} = (e^{-at})^{m+1} = |u|^{m+1}$ and from (3.10) we have

$$
|B_\alpha(e^{i\theta}, t)|
$$

$$
= \left| \left( \frac{f'(u)}{\alpha g(u)} - 1 \right) u^{m+1} \right|
$$

$$
+ \frac{u^2 (1 - |u|^{m+1})^2}{|u|^{m+1}} \left[ \frac{(1 - \alpha)f'(u)h(u)}{\alpha f(u)} + \frac{f'(u)h^2(u)}{\alpha g(u)} + \frac{g'(u)h(u)}{g(u)} - h'(u) \right]
$$

$$
+ u \left( 1 - |u|^{m+1} \right) \left[ \frac{(1 - \alpha)f'(u)}{\alpha f(u)} + \frac{f'(u)h(u)}{\alpha g(u)} + \frac{g'(u)}{g(u)} \right] \leq \frac{m + 1}{2}.
$$

Because $u \in U$, the inequality (3.2) implies that

$$
|B_\alpha(e^{i\theta}, t)| \leq \frac{m + 1}{2},
$$

and from (3.13), (3.14) and (3.15), we conclude that

$$
|B_\alpha(z, t)| = \left| A_\alpha(z, t) - \frac{m - 1}{2} \right| \leq \frac{m + 1}{2}.$$
for all $z \in U$ and $t \in I$. Therefore $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$.

Since all the conditions of Theorem 2.1 are satisfied, we obtain that the function $L(z,t)$ has an analytic and univalent extension to the whole unit disk $U$ for all $t \in I$. For $t = 0$ we have $L(z,0) = f(z)$ for $z \in U$ and therefore the function $f$ is analytic and univalent in $U$.

**Remark 3.1.** (1) By putting $m = 1$ in Theorem 3.1 we obtain all Tudor’s results in [14].

(2) The univalence criteria which results from Theorem 3.1 when $m = 1$ and $\alpha = 1$ is due to Ovesea-Tudor and Owa in [6].

**Corollary 3.1.** Let $m$ be a positive real number and let $\alpha$ be a complex number such that $\Re(\alpha) > \frac{1}{m+1}$ and $f \in A$. Suppose that there exists an analytic function $h$ in $U$, $h(z) = c_0 + c_1 z + \cdots$. If the following inequality

\[ | \left( \frac{1}{\alpha} - 1 \right) | z |^{2(m+1)} + z^2 \left( 1 - |z|^{m+1} \right)^2 \left[ \frac{1}{\alpha} f'(z) h(z) + \frac{h^2(z)}{\alpha} + \frac{f''(z) h(z)}{f'(z)} - h'(z) \right] + z |z|^{m+1} \left( 1 - |z|^{m+1} \right) \left[ \frac{1}{\alpha} f'(z) + \frac{2h(z)}{\alpha} + \frac{f''(z)}{f'(z)} \right] - \frac{m+1}{2} |z|^{m+1} \]

holds true for all $z \in U$, then the function $f$ is univalent in $U$ where the principal branch is intended.

**Proof.** It results from Theorem 3.1 with $g = f'$.

If we choose $g = f'$ and $h = -\frac{1}{2} \frac{f''}{f'}$ in Theorem 3.1 we obtain the following univalence criterion.

**Corollary 3.2.** Let $m$ be a positive real number and let $\alpha$ be a complex number such that $\Re(\alpha) > \frac{1}{m+1}$ and $f \in A$. If the following inequality

\[ | \left( \frac{1}{\alpha} - 1 \right) | z |^{2(m+1)} + \left( 1 - |z|^{m+1} \right)^2 \left\{ \frac{1}{2} z^2 f; z \right\} + \frac{1}{2} \left( \frac{1}{\alpha} \right) \frac{z^2}{f(z)} \left( \frac{z f''(z)}{f'(z)} \right)^2 - \frac{z^2 f''(z)}{f'(z)} \right\} + |z|^{m+1} \left( 1 - |z|^{m+1} \right) \left( 1 - \frac{\alpha}{\alpha} \right) \left[ \frac{z f'(z)}{f(z)} - \frac{z f''(z)}{f'(z)} \right] - \frac{m-1}{2} |z|^{m+1} \]

\[ \leq \frac{m+1}{2} |z|^{m+1} \]
holds true for all \( z \in U \), then the function \( f \) is univalent in \( U \), where the principal branch is intended.

**Remark 3.2.** (1) If we consider \( m = 1 \) and \( \alpha = 1 \) in Corollary 3.2, the inequality (3.18) becomes (1.3) and then we obtain the univalence criterion due to Nehari [4].

(2) Setting \( m = 1 \) in Corollary 3.2, we obtain the univalence criterion due to Raducanu [9].

**Corollary 3.3.** Let \( m \) be a positive real number and let \( \alpha \) be a complex number such that \( \Re(\alpha) > \frac{1}{m+1} \) and \( f \in A \). Suppose there exists an analytic function \( h(z) \) in \( U \), \( h(z) = c_0 + c_1 z + \cdots \). If the following inequalities

\[
\frac{z^2 f'(z)}{\alpha f^2(z)} - \frac{m+1}{2} < \frac{m+1}{2}
\]

and

\[
\frac{z^2 f'(z)}{\alpha f^2(z)} \left| z^{m+1} \right| - \frac{m-1}{2} \left| z^{m+1} \right| < \frac{m-1}{2} \left| z^{m+1} \right|
\]

are satisfied for all \( z \in U \), then the function \( f \) is univalent in \( U \) where the principal branch is intended.

**Proof.** It results from Theorem 3.1 with \( g(z) = \left( \frac{f(z)}{z} \right)^2 \). \( \Box \)

If we choose \( g(z) = \left( \frac{f(z)}{z} \right)^2 \) and \( h(z) = \frac{1}{z} - \frac{f(z)}{z^2} \) in Theorem 3.1 we obtain the following corollary.

**Corollary 3.4.** Let \( m \) be a positive real number and let \( \alpha \) be a complex number such that \( \Re(\alpha) > \frac{1}{m+1} \) and \( f \in A \). If the following inequalities

\[
\left| \frac{z^2 f'(z)}{\alpha f^2(z)} - 1 \right| - \frac{m-1}{2} \left| z^{m+1} \right| < \frac{m+1}{2}
\]

and

\[
\left| \frac{z^2 f'(z)}{\alpha f^2(z)} - 1 \right| + \left( \frac{\alpha - 1}{\alpha} \right) \left( 1 - |z|^{m+1} \right) \left| \frac{zf'(z)}{f(z)} \right| - \frac{m-1}{2} \left| z^{m+1} \right| \leq \frac{m+1}{2} \left| z^{m+1} \right|
\]

are satisfied for all \( z \in U \), then the function \( f \) is univalent in \( U \) where the principal branch is intended.
are satisfied for all \( z \in U \), then the function \( f \) is univalent in \( U \), where the principal branch is intended.

**Remark 3.3.** (1) If we consider \( \alpha = 1 \) in Corollary 3.4 we obtain the univalence criterion due to Raducanu et al. [13].

(2) Putting \( m = 1 \) in Corollary 3.4 we obtain the univalence criterion due to Raducanu [10].

(3) If we consider \( m = 1 \) and \( \alpha = 1 \) in Corollary 3.4, the inequalities (3.21) and (3.22) becomes (1.2) and then we obtain the univalence criterion due to Ozaki-Nunokawa [7].

**Corollary 3.5.** Let \( m \) be a positive real number and let \( \alpha \) be a complex number such that \( \Re(\alpha) > \frac{1}{m+1} \) and \( f \in A \). If the following inequality

\[
\left| \left( \frac{1}{\alpha} - 1 \right) - \frac{m-1}{2} \right| < \frac{m+1}{2},
\]

(3.23)

\[
\frac{1-\alpha}{\alpha} \left[ 1 - (1 - |z|^{m+1}) \right] \frac{zf'(z)}{f(z)} \quad + (1 - |z|^{m+1}) \frac{d}{dz} \left[ \log \frac{z^2 f'(z)}{f^2(z)} \right] - \frac{m-1}{2} |z|^{m+1}
\]

is satisfied for all \( z \in U \), then the function \( f \) is univalent in \( U \) where the principal branch is intended.

**Proof.** It results from Theorem 3.1 with \( g(z) = f'(z) \) and \( h(z) = \frac{1}{z} - \frac{f'(z)}{f(z)} \). □

**Remark 3.4.** (1) If we consider \( m = 1 \) in Corollary 3.5 we obtain the univalence criterion due to Raducanu [11].

(2) For \( m = 1 \) and \( \alpha = 1 \) in Corollary 3.5 we obtain Goluzin’s criterion for univalence [3].

**Corollary 3.6.** Let \( m \) be a positive real number and let \( \alpha \) be a complex number such that \( \Re(\alpha) > \frac{1}{m+1} \) and \( f \in A \). If the following inequality

\[
\left| \left( \frac{1}{\alpha} - 1 \right) |z|^{m+1} + z(1 - |z|^{m+1}) \left( \frac{1-\alpha}{\alpha} \frac{f'(z)}{f(z)} + \frac{f''(z)}{f'(z)} \right) - \frac{m-1}{2} \right| \leq \frac{m+1}{2}
\]

(3.24)

is satisfied for all \( z \in U \), then the function \( f \) is univalent in \( U \) where the principal branch is intended.

**Proof.** It results from Theorem 3.1 with \( g(z) = f'(z) \) and \( h(z) = 0 \). □

**Remark 3.5.** If we consider \( \alpha = m = 1 \) in Corollary 3.6, the inequality (3.24) becomes (1.1) and then we obtain the univalence criterion due to Becker [1].

Finally, if we take \( \alpha \to \infty \) in Corollary 3.6 (\( z \in U \)) we obtain another univalence criterion as follows.
Corollary 3.7. Let $m$ be a positive real number and $f \in A$. If the following inequality
\begin{equation}
(3.25) \quad \left| z(1 - |z|^{m+1}) \left[ \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right] - |z|^{m+1} - \frac{m - 1}{2} \right| \leq \frac{m + 1}{2}
\end{equation}
holds true for all $z \in U$, then the function $f$ is univalent in $U$ where the principal branch is intended.

Remark 3.6. If we consider $\alpha \to \infty$, $z \in U$ in the Corollaries 3.1 and 3.2 we can obtain other new univalence criteria.

Remark 3.7. The famous univalence criteria obtained by Nehari, Ozaki-Nunokawa and Becker contain $|z|^2$ in their expressions. From Theorem 3.1 we obtain new and more general results with $|z|^{m+1}$ ($m > 0$) instead of $|z|^2$.

Example 3.1. The function
\begin{equation}
(3.26) \quad f(z) = \frac{z}{1 - \frac{m}{m+1}} (m \geq 1)
\end{equation}
is analytic and univalent in $U$.

Proof. From equality (3.26) we have
\begin{equation}
(3.27) \quad \frac{z^2 f'(z)}{f^2(z)} - 1 = \frac{m}{m + 1} z^{m+1}.
\end{equation}
Taking into account (3.27), $\alpha = 1$ and $m \geq 1$, the conditions (3.21) and (3.22) in Corollary 3.4 becomes, respectively,
\begin{align*}
\left| \left( \frac{z^2 f'(z)}{f^2(z)} - 1 \right) - \frac{m - 1}{2} \right| &= \left| \frac{m}{m + 1} z^{m+1} - \frac{m - 1}{2} \right| < \frac{m + 1}{2(m + 1)} < \frac{m + 1}{2}
\end{align*}
and
\begin{align*}
\frac{1}{|z|^{m+1}} \left| \left( \frac{z^2 f'(z)}{f^2(z)} - 1 \right) - \frac{m - 1}{2} |z|^{m+1} \right| &= \frac{1}{|z|^{m+1}} \left( \frac{m}{m + 1} z^{m+1} - \frac{m - 1}{2} |z|^{m+1} \right) < \frac{m^2 + 2m - 1}{2(m + 1)} < \frac{m + 1}{2},
\end{align*}
which are satisfied the conditions (3.21) and (3.22) of Corollary 3.4. It follows that the function $f$ defined by (3.26) is analytic and univalent in $U$. By using the Mathematica 7.0 program, for $m = 5$, we can obtain the graphic of $f(z) = \frac{z}{1 - \frac{m}{m+1}}$ (see Figure 1). \hfill \square

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