THE GROUP OF GRAPH AUTOMORPHISMS
OVER A MATRIX RING

SANGWON PARK AND JUNCHEOL HAN

Abstract. Let \( R = \text{Mat}_2(F) \) be the ring of all 2 by 2 matrices over a
finite field \( F \), \( X \) the set of all nonzero, nonunits of \( R \) and \( G \) the group
of all units of \( R \). After investigating some properties of orbits under
the left (and right) regular action on \( X \) by \( G \), we show that the graph
automorphisms group of \( \Gamma(R) \) (the zero-divisor graph of \( R \)) is isomorphic
to the symmetric group \( S_{|F|^2+1} \) of degree \( |F| + 1 \).

1. Introduction

The zero-divisor graph of a commutative ring has been studied extensitively
by Akbari, Anderson, Frazier, Lauve, Livinston and Mohammadian in \([1, 2, 3]\)
since its concept had been introduced by Beck in \([4]\). Recently, the zero-divisor
graph of a noncommutative ring (resp. a semigroup) has also been studied by
Redmond and Wu (resp. F. DeMeyer and L. DeMeyer) in \([10, 11, 12]\) (resp.
\([5]\)). The zero-divisor graph has been used to study the algebraic structures
of rings via their zero-divisors. In this paper, the group of the zero-divisor
graph automorphisms over a matrix ring over a finite field is investigated by
considering some group actions.

For a ring \( R \) with identity, let \( Z(R) \) be the set of all left or right zero-
divisors of \( R \), \( \Gamma(R) \) be the zero-divisor graph of \( R \) consisting of all vertices in
\( Z(R)^* = Z(R) \setminus \{0\} \), the set of all nonzero left or right zero-divisors of \( R \), and
edges \( x \rightarrow y \), which means that \( xy = 0 \) for \( x, y \in Z(R)^* \).

For a ring \( R \) with identity, let \( X(R) \) (simply, denoted by \( X \)) be the set of all
nonzero, nonunits of \( R \), \( G(R) \) (simply, denoted by \( G \)) be the group of all units
of \( R \). In this paper, we will consider some group actions on \( X \) by \( G \) given by
\( (g, x) \rightarrow gx \) (resp. \( (g, x) \rightarrow xg^{-1} \)) from \( G \times X \) to \( X \), called the left (resp.
right) regular action. If \( \phi : G \times X \rightarrow X \) is the left (resp. right) regular action,
then for each \( x \in X \), we define the orbit of \( x \) by \( o_\ell(x) = \{ \phi(g, x) = gx : g \in G \} \) (resp. \( o_r(x) = \{ \phi(g, x) = xg^{-1} : g \in G \} \)).

In Section 2, we will show that if \( R = \operatorname{Mat}_2(F) \) with \( F \) a finite field, then (1) the number of orbits under the left (resp. right) regular action on \( X \) by \( G \) is \(|F|^2 + 1\); (2) if \( N \) is the set of all nonzero nilpotents in \( R \), then \(|N| = |F|^2 - 1\) and \( o_\ell(x) \cap o_r(x) = o_\ell(x) \cap N = o_r(x) \cap N \) for all \( x \in N \); (3) \(|o_\ell(x) \cap o_r(y)| = |F| - 1\) for all \( x, y \in X \).

We recall that for all \( x \in X \) the set \( \operatorname{ann}_\ell(x) = \{ y \in X : xy = 0 \} \) (resp. \( \operatorname{ann}_r(x) = \{ z \in X : xz = 0 \} \) is called a left (resp. right) annihilator of \( x \). Let \( \operatorname{ann}_\ell^*(x) = \operatorname{ann}_\ell(x) \setminus \{ 0 \} \) (resp. \( \operatorname{ann}_r^*(x) = \operatorname{ann}_r(x) \setminus \{ 0 \} \)).

A graph automorphism \( f \) of a graph \( \Gamma(R) \) (where \( R \) denotes a ring) is defined to be a bijection \( f : \Gamma(R) \to \Gamma(R) \) which preserves adjacency. Note that the set \( \operatorname{Aut}(\Gamma(R)) \) of all graph automorphisms of \( \Gamma(R) \) forms a group under the usual composition of functions. In [3], Anderson and Livingston have shown that \( \operatorname{Aut}(\Gamma(\mathbb{Z}_n)) \) of all graph automorphisms of \( \Gamma(R) \) forms a group under \( \Gamma(R) \) a nonprime integer. For the case of noncommutative rings, it was shown by [8] that when \( R = \operatorname{Mat}_2(\mathbb{Z}_p) \) \((p \text{ is a prime}), \operatorname{Aut}(\Gamma(R)) \simeq S_{p+1}, \text{the symmetric group of degree } p+1. \text{ In Section 3, for the continuation of these investigation, we prove that } \operatorname{Aut}(\Gamma(R)) \simeq S_{|F|+1} \text{ when } R = \operatorname{Mat}_2(F) \text{ with } F \text{ a finite field.}

2. Orbits under the regular action in \( \operatorname{Mat}_2(F) \)

Recall that \( G \) is transitive on \( X \) (or \( G \) acts transitively on \( X \)) under the left (resp. right) regular action on \( X \) by \( G \) if there is an \( x \in X \) with \( o_\ell(x) = X \) (resp. \( o_r(x) = X \)) and the left (resp. right) regular action of \( G \) on \( X \) is said to be half-transitive if \( G \) is transitive on \( X \) or if \( o_\ell(x) \) (resp. \( o_r(x) \)) is a finite set with \(|o_\ell(x)| > 1\) (resp. \(|o_r(x)| > 1\)) and \(|o_\ell(x)| = |o_\ell(y)| \) (resp. \(|o_r(x)| = |o_r(y)|\)) for all \( x \) and \( y \in X \). In [7, Theorem 2.4 and Lemma 2.7], it was shown that if \( R = \operatorname{Mat}_2(F) \) with \( F \) a finite field, then \( G \) is half-transitive on \( X \) by the left (resp. right) regular action and \(|o_\ell(x)| = |F|^2 - 1\) (resp. \(|o_r(x)| = |F|^2 - 1\)) for all \( x \in X \).

**Lemma 2.1.** Let \( R = \operatorname{Mat}_2(F) \) with \( F \) a finite field. Then the number of orbits under the left (resp. right) regular action on \( X \) by \( G \) is \(|F|^2 + 1\).

**Proof.** Let \( \mu \) be the number of orbits under the left (resp. right) regular action on \( X \) by \( G \). Note that \(|G| = (|F|^2 - 1)(|F|^2 + 1)|. Thus \(|X| = |R| - |G| - 1 = |F|^4 - |F|^2 - 1 = (|F|^2 - 1) |F|^2 - 1 = (|F| + 1)(|F|^2 - 1)|. Since the cardinality of any orbit under the left (resp. right) regular action on \( X \) by \( G \) is \(|F|^2 - 1\) by [7, Lemma 2.7], \( \mu = \frac{|X|}{(|F|^2 - 1)} = |F| + 1 \). \( \square \)

The following theorem was shown in [6].

**Theorem 2.2.** The probability that \( n \) by \( n \) matrix over \( GF(p^n) \) be nilpotent is \( p^{n^2} \).

**Proof.** Refer [6, Theorem 1]. \( \square \)
By Theorem 2.2, we note that the number of all $2 \times 2$ nonzero nilpotent matrices over a finite field $F$ is equal to $|F|^2 - 1$.

**Theorem 2.3.** Let $R = \text{Mat}_2(F)$ with $F$ a finite field and $N$ be the set of all nonzero nilpotents in $R$. Then under the left (resp. right) regular action on $X$ by $G$, we have the following.

(i) $|\alpha r(x) \cap N| = |F| - 1$;

(ii) $\alpha r(x) \cap N = \alpha r(x) \cap N = \alpha x \cap \alpha r(x)$ for each $x \in N$.

**Proof.** (i) Consider the set $S_x = \{(\alpha I)x|\alpha \in F \setminus \{0\}\}$ for each $x \in N$ where $I$ is the identity matrix in $R$. Since $(\alpha I)x = x(\alpha I)$ for all $(\alpha I)x \in S$, $S_x \subseteq \alpha r(x) \cap N \cap \alpha r(x) \cap N$. Note that for all $\alpha, \beta \in F \setminus \{0\}(\alpha \neq \beta)$, $(\alpha I)x \neq (\beta I)x$, and so $|S_x| = |F| - 1$. Next, we will show that $\alpha x \cap N \subseteq S_x$. Let $y \in \alpha x \cap N$ be arbitrary. Let

$$x = \begin{bmatrix} -ab & b \\ -a^2b & ab \end{bmatrix} \in N \text{ for some } b(\neq 0), \alpha \in F.$$

Since $y \in \alpha x$, $y = gx$ for some $g \in G$. Let $g = [\ell \; \ell'] \in G$. Then

$$y = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} -ab & b \\ -a^2b & ab \end{bmatrix} = \begin{bmatrix} -(p + qa)ab & (p + qa)b \\ -(r + sa)ab & (r + sa)b \end{bmatrix} \in \alpha x.$$

Since $y \in N$, we have $(p + qa)\alpha = (r + sa)(\neq 0)$ by the proof of Lemma 2.2, and so

$$y = \begin{bmatrix} p + qa & 0 \\ 0 & p + qa \end{bmatrix} \begin{bmatrix} -ab & b \\ -a^2b & ab \end{bmatrix} = ((p + qa)I)x \in S_x.$$

Therefore, $\alpha x \cap N \subseteq S_x$, and consequently we have $S_x = \alpha x \cap N$. By the similar argument, we have also $S_x = \alpha r(x) \cap N$. Hence $\alpha x \cap N = \alpha r(x) \cap N$ and $|\alpha x \cap N| = |\alpha r(x) \cap N| = |S_x| = |F| - 1$ for each $x \in N$.

(ii) By the proof of (i), we have that $\alpha x \cap N = \alpha r(x) \cap N$ for each $x \in N$. Note that $S_x = \alpha x \cap N = \alpha r(x) \cap N \subseteq \alpha x \cap \alpha r(x)$ for each $x \in N$ where $S_x$ is the set considered in the proof of (i). Let $y \in \alpha x \cap \alpha r(x)$ be arbitrary and let

$$x = \begin{bmatrix} -\alpha \beta & \beta \\ -\alpha^2 \beta & \alpha \beta \end{bmatrix} \in N \text{ (\forall} \alpha \in F, \forall \beta \in F \setminus \{0\})$$

be arbitrary. Then there exist $g = [a \; b]$, $h = [p \; q] \in G$ such that $y = gx = xh$.

Thus

$$\begin{align*}
(1) & \quad gx = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -\alpha \beta & \beta \\ -\alpha^2 \beta & \alpha \beta \end{bmatrix} = \begin{bmatrix} -\alpha (a + ba) & \beta (a + ba) \\ -\alpha (c + da) & \beta (c + da) \end{bmatrix}, \\
(2) & \quad xh = \begin{bmatrix} -\alpha \beta & \beta \\ -\alpha^2 \beta & \alpha \beta \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} -\beta (ap - r) & -\beta (aq - s) \\ -\alpha \beta (ap - r) & -\alpha \beta (aq - s) \end{bmatrix}.
\end{align*}$$
Let $\gamma = -\beta (a_q - s) \ (\equiv (1,2) - \text{entry of } y = xh)$. From (1) and (2), we have that

$$y = \begin{bmatrix} -\alpha \gamma & \gamma \\ -\alpha^2 \gamma & \alpha \gamma \end{bmatrix} \in N,$$

and so $o_\ell (x) \cap o_\ell (x) \subseteq o_\ell (x) \cap N$ for each $x \in N$. Hence $o_\ell (x) \cap o_\ell (x) = o_\ell (x) \cap N$ for each $x \in N$. Similarly, we have $o_\ell (x) \cap o_\ell (x) = o_\ell (x) \cap N$ for each $x \in N$. \hfill $\Box$

**Remark 1.** Let $R = \text{Mat}_2(R)$ with $F$ a finite field and $N$ be the set of all nonzero nilpotents in $R$. Choose $x_1 \in N$ so that $S_{x_1} = \{(\alpha I)x_1 | \alpha \in F \setminus \{0\} \} \subset N$. By Theorem 2.3, $o_\ell (x_1) \cap N = S_{x_1}$. Since $|N| = |F|^2 - 1$ by Theorem 2.2 and $|S_{x_1}| = |F| - 1$ by Theorem 2.3, we can choose $x_2 \in N \setminus S_{x_1}$. Then $S_{x_2} = o_\ell (x_1) \cap N$ and $S_{x_2} = o_\ell (x_2) \cap N$ are disjoint. Continuing in this way, we can choose $x_1, x_2, \ldots, x_{|F| + 1} \in N$ so that $x_{i+1} \in N(R) \setminus (S_{x_1} \cup S_{x_2} \cup \cdots \cup S_{x_i})$ for all $i = 1, \ldots, |F|$. Then we have

$$N = S_{x_1} \cup S_{x_2} \cup \cdots \cup S_{x_{|F| + 1}},$$

which is a disjoint union of $N$. Observe that $o_\ell (x_1), o_\ell (x_2), \ldots, o_\ell (x_{|F| + 1})$ are disjoint (equivalently, they are all distinct). Indeed, assume that there exist $o_\ell (x_i)$ and $o_\ell (x_j)$ for some $i, j (i < j, i \neq j)$ such that $o_\ell (x_i) = o_\ell (x_j)$. Then $x_j \in o_\ell (x_i) \cap N = S_{x_i}$, and so $S_{x_j} \subseteq S_{x_i}$, which is a contradiction. Since the number of orbits under the left regular action on $X$ by $G$ is $|F| + 1$ by Lemma 2.1, $X = o_\ell (x_1) \cup o_\ell (x_2) \cup \cdots \cup o_\ell (x_{|F| + 1})$. By the similar argument, we have $X = o_\ell (x_1) \cup o_\ell (x_2) \cup \cdots \cup o_\ell (x_{|F| + 1})$.

**Lemma 2.4.** Let $R = \text{Mat}_2(R)$ with $F$ a finite field and $N$ be the set of all nonzero nilpotents in $R$. Then for all $x, y \in N$, $y = gxg^{-1}$ for some $g \in G$.

**Proof.** Consider a group action on $X$ by $G$ given by $(g, x) \rightarrow gxg^{-1}$ from $G \times X$ to $X$, called conjugation.

Take $a = [\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}] \in N$. Let $o_\ell (a) = \{gag^{-1} | g \in G\}$ be the orbit of $a$ under conjugation, and $\text{stab}_G(a) = \{g \in G | ga = ag\}$ be the stabilizer of $a$ under conjugation. Then we have

$$\text{stab}_G(a) = \left\{ \begin{bmatrix} s & t \\ 0 & s \end{bmatrix} \in G | s \neq 0, t \in F \right\},$$

by easy computation, and so $|\text{stab}_G(a)| = |G|/[s \neq 0, t \in F]$. Hence

$$|o_\ell (a)| = \frac{|G|}{|\text{stab}_G(a)|} = \frac{|(F)^2 - |F||(|F|^2 - 1)}{|(F) - 1||F|} = |F|^2 - 1 = |N|,$$

Since $o_\ell (a) \subseteq N$, $o_\ell (a) = N$. Therefore we have the result. \hfill $\Box$

**Theorem 2.5.** Let $R = \text{Mat}_2(R)$ with $F$ a finite field and $N$ be the set of all nonzero nilpotents in $R$. Then under the left (resp. right) regular action on $X$ by $G$, $|o_\ell (x) \cap o_\ell (y)| = |F| - 1$ for each $x, y \in X$. 

Proof. First, we will show that \(\alpha_t(x) \cap \alpha_t(y) \neq \emptyset\) for each \(x, y \in X\). By Remark 1, we can choose \(x_1, \ldots, x_{|F|+1}\) (resp. \(y_1, \ldots, y_{|F|+1}\)) in \(N\) so that \(X = \alpha_t(x_1) \cup \cdots \cup \alpha_t(x_{|F|+1})\) (resp. \(X = \alpha_t(y_1) \cup \cdots \cup \alpha_t(y_{|F|+1})\)). Thus \(\alpha_t(x) = \alpha_t(x_i)\) and \(\alpha_t(y) = \alpha_t(y_j)\) for some \(x_i, y_j \in N\). Observe that \(\alpha_t(x_i) \cap \alpha_t(y_j) \neq \emptyset\). Indeed, since \(x_i\) and \(y_j\) are nonzero nilpotents in \(R\), \(y_j = gx_i g^{-1}\) for some \(g \in G\) by Lemma 2.4. Hence \(\alpha_t(x_i) \cap \alpha_t(y_j) = \alpha_t(x_i) \cap \alpha_t(gx_i g^{-1}) = \alpha_t(gx_i) \cap \alpha_t(gx_i)\) contains an element \(gx_i \in X\), and so \(\alpha_t(x) \cap \alpha_t(y) = \alpha_t(x_i) \cap \alpha_t(y_j) \neq \emptyset\).

Next, we will show that \(|\alpha_t(x) \cap \alpha_t(y)| = |F| - 1\) for each \(x, y \in X\). Since \(\alpha_t(x) \cap \alpha_t(y) \neq \emptyset\) for each \(x, y \in X\), we choose \(z \in \alpha_t(x) \cap \alpha_t(y)\), and then \(\alpha_t(z) = \alpha_t(x) \cap \alpha_t(y)\). Consider a set \(S_z = \{\alpha I \mid \alpha \in F \setminus \{0\}\}\) where \(I\) is the identity matrix in \(R\). Since \((\alpha I)z = z(\alpha I)\) for all \((\alpha I)z \in S_z\), \(S_z \subseteq \alpha_t(x) \cap \alpha_t(y)\). Note that for all \(\alpha, \beta \in F \setminus \{0\}\) (\(\alpha \neq \beta\)), \((\alpha I)z \neq (\beta I)z\), and so \(|S_z| = |F| - 1\). Thus \(|\alpha_t(x) \cap \alpha_t(y)| = |\alpha_t(z) \cap \alpha_t(z)| \geq |S_z| = |F| - 1\). Since \(X = \alpha_t(x_1) \cup \cdots \cup \alpha_t(x_{|F|+1})\), we have \(\alpha_t(y) = X \cap \alpha_t(y) = \alpha_t(x_1) \cap \alpha_t(y) \cup \cdots \cup \alpha_t(x_{|F|+1}) \cap \alpha_t(y)\). Clearly, \(\alpha_t(x_1) \cap \alpha_t(y), \ldots, \alpha_t(x_{|F|+1}) \cap \alpha_t(y)\) are disjoint, and thus \(|\alpha_t(y)| = |F|^2 - 1 = |\alpha_t(x_1) \cap \alpha_t(y)| + \cdots + |\alpha_t(x_{|F|+1}) \cap \alpha_t(y)| \geq (|F| - 1)(|F| + 1) = |F|^2 - 1\), which implies that \(|\alpha_t(x_1) \cap \alpha_t(y)| = \cdots = |\alpha_t(x_{|F|+1}) \cap \alpha_t(y)| = |F| + 1\). Since \(\alpha_t(x) = \alpha_t(x_i)\) for some \(x_i \in N\), we have that \(|\alpha_t(x) \cap \alpha_t(y)| = |\alpha_t(x) \cap \alpha_t(y)| = |F| - 1\) for each \(x, y \in X\). \(\square\)

The following example illustrates Theorem 2.3 and Theorem 2.5 for a certain finite field.

**Example 1.** Consider \(F = \mathbb{Z}_2[x]/(1 + x + x^2)\), a field of order 4 where \(\mathbb{Z}_2\) is the Galois field of order 2. To simplify notation, we denote \(f(x) + (1 + x + x^2) \in F\) by \(f(x)\) for all \(f(x) \in \mathbb{Z}_2[x]\). Thus \(F = \{0, 1, x, 1 + x\}\). Let \(R = \text{Mat}_2(F)\) and let \(N\) be the set of all nonzero nilpotents of \(R\). Then \(|X| = (|F| + 1)(|F|^2 - 1) = 75\) and \(|N| = |F|^2 - 1 = 15\). Note that under the left (resp. right) regular action on \(X\) by \(G\), there are \(z_1, z_2, z_3, z_4, z_5 \in N\) such that \(X = \alpha_t(z_1) \cup \alpha_t(z_2) \cup \alpha_t(z_3) \cup \alpha_t(z_4) \cup \alpha_t(z_5)\) (resp. \(X = \alpha_t(z_1) \cup \alpha_t(z_2) \cup \alpha_t(z_3) \cup \alpha_t(z_4) \cup \alpha_t(z_5)\)), where \(z_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\), \(z_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\), \(z_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\), \(z_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\), and \(z_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\).

We compute the followings by a computer programming (using Mathematica Ver. 6):

\[
\begin{align*}
\alpha_t(z_1) \cap N &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
\alpha_t(z_2) \cap N &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
\alpha_t(z_3) \cap N &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \\
\alpha_t(z_4) \cap N &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \\
\alpha_t(z_5) \cap N &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\end{align*}
\]
with \( o_t(z_i) \cap N = o_r(z_i) \cap N \) for all \( i = 1, \ldots , 5 \).

Also we compute the followings by a computer programming (using Mathematica Ver. 6):

\[
\begin{align*}
o_t(z_1) \cap o_r(z_1) &= \left\{ \begin{array}{ccc} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right\}, \\
o_t(z_1) \cap o_r(z_2) &= \left\{ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{array} \right\}, \\
o_t(z_1) \cap o_r(z_3) &= \left\{ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right\}, \\
o_t(z_1) \cap o_r(z_4) &= \left\{ \begin{array}{ccc} 1 + x & 1 + x & 1 + x \\ 1 + x & 1 + x & 1 + x \\ 1 + x & 1 + x & 1 + x \end{array} \right\}, \\
o_t(z_1) \cap o_r(z_5) &= \left\{ \begin{array}{ccc} 1 + x & 1 + x & 1 + x \\ 1 + x & 1 + x & 1 + x \\ 1 + x & 1 + x & 1 + x \end{array} \right\}.
\end{align*}
\]

3. Automorphism of graph over \( \text{Mat}_2(F) \)

**Lemma 3.1.** Let \( R \) be a ring with identity and \( f : \Gamma(R) \to \Gamma(R) \) be a graph automorphism of \( \Gamma(R) \). Then for all \( x \in X \), \( f(\text{ann}_x^r(x)) = \text{ann}_x^r(f(x)) \) (and \( f(\text{ann}_x^l(x)) = \text{ann}_x^l(f(x)) \)).

**Proof.** Let \( y \in f(\text{ann}_x^r(x)) \) be arbitrary. Then \( y = f(z) \) for some \( z \in \text{ann}_x^r(x) \). Since \( xx = 0 \) and \( f \) preserves adjacency, \( 0 = f(z)f(x) = yf(x) \) and so \( y \in \text{ann}_x^r(f(x)) \). Hence \( f(\text{ann}_x^r(x)) \subseteq \text{ann}_x^r(f(x)) \). Let \( z \in \text{ann}_x^r(f(x)) \) be arbitrary. Then \( zf(x) = 0 \). Since \( f \) is one-to-one, there exists \( z_1 \in X \) such that \( f(z_1) = z \). Then \( 0 = zf(x) = f(z_1)f(x) \). Since \( f \) preserves adjacency, \( z_1x = 0 \). Since \( z_1 \in \text{ann}_x^r(x) \) and \( z = f(z_1) \in f(\text{ann}_x^r(x)) \), \( \text{ann}_x^r(f(x)) \subseteq f(\text{ann}_x^r(x)) \).

By the similar argument, we have \( f(\text{ann}_x^l(x)) = \text{ann}_x^l(f(x)) \). \( \square \)

**Lemma 3.2.** Let \( R \) be a ring with identity. If \( \text{ann}_x^r(x) \neq \emptyset \) (resp. \( \text{ann}_x^l(x) \neq \emptyset \)) for some \( x \in X \), then \( \text{ann}_x^r(x) \) (resp. \( \text{ann}_x^l(x) \)) is a union of orbits under the left (resp. right) regular action on \( X \) by \( G \).

**Proof.** Let \( y \in \text{ann}_x^r(x) \) be arbitrary. Then we have \( o_t(y) \subseteq \text{ann}_x^r(x) \), and so \( \bigcup_{y \in \text{ann}_x^r(x)} o_t(y) \subseteq \text{ann}_x^r(x) \). Since \( \text{ann}_x^r(x) \) is not empty, it is clear that \( \text{ann}_x^r(x) \subseteq \bigcup_{y \in \text{ann}_x^r(x)} o_t(y) \). Hence \( \text{ann}_x^r(x) = \bigcup_{y \in \text{ann}_x^r(x)} o_t(y) \), i.e., \( \text{ann}_x^r(x) \) is a union of orbits under the left regular action on \( X \) by \( G \). By the similar argument, \( \text{ann}_x^l(x) \) is a union of orbits under the right regular action on \( X \) by \( G \). \( \square \)

**Corollary 3.3.** Let \( R \) be a finite ring with identity. Then for all \( x \in X \), \( \text{ann}_x^r(x) \) (resp. \( \text{ann}_x^l(x) \)) is a union of finite number of orbits under the left (resp. right) regular action on \( X \) by \( G \).

**Proof.** By [8, Proposition 1.2], all \( x \in X \) are zero-divisors, and so \( \text{ann}_x^r(x) \neq \emptyset \) (resp. \( \text{ann}_x^l(x) \neq \emptyset \)) for all \( x \in X \). Hence for all \( x \in X \text{ann}_x^r(x) \) (resp. \( \text{ann}_x^l(x) \))
is a union of finite number of orbits under the left (resp. right) regular action on $X$ by $G$ by Lemma 3.2. □

The following lemma is well-known in [9].

**Lemma 3.4.** Let $p$ be a prime number and $\alpha, \beta$ be positive integers. Then $p^{\alpha} - 1$ is a divisor of $p^{\beta} - 1$ if and only if $\alpha$ is a divisor of $\beta$.

**Proof.** Refer [9, Lemma 3.3, p 32]. □

By using the preceding lemma, we describe $ann_\ast(x)$ (and $ann_\ast(x)$) for all $x \in X$ effectively as follows:

**Theorem 3.5.** Let $R = \text{Mat}_2(F)$ with $F$ a finite field. Then $ann_\ast(x) = \alpha(x)$ for all $x \in ann_\ast(x)$ (and $ann_\ast(x) = \alpha(x)$ for all $z \in ann_\ast(x)$).

**Proof.** By [7, Lemma 2.7], we have $|\alpha(x)| = |F|^2 - 1$ for all $x \in X$. Hence we observe that

1. since $ann_\ast(x)$ is a union of a finite number of orbits under the left regular action of $G$ on $X$ by Corollary 3.3 and the left regular action of $G$ on $X$ is half-transitive by [7, Theorem 2.4], $|\alpha(x)|$ is a divisor of $|ann_\ast(x)|$ for all $y \in ann_\ast(x)$;
2. $|\alpha(x)|$ is a divisor of $|F|$ since $ann_\ast(x)$ is an additive subgroup of $F$ for all $x \in X$.

Let $|F| = p^k$ for some prime $p$ and some positive integer $\alpha$. Then $|\alpha(x)| = p^{2\alpha} - 1$ and $|F| = p^{2\alpha}$. Since $ann_\ast(x) \neq R$, we have $|\alpha(x)| = p^k$ for some positive integer $k$ $(2\alpha \leq k < 4\alpha)$ by (2). By (1) and Lemma 3.4, $|ann_\ast(x)| = p^{2\alpha} - 1$, and so $|ann_\ast(x)| = |\alpha(x)|$. Since $\alpha(x) \subseteq ann_\ast(x), ann_\ast(x) = \alpha(x)$ for all $y \in ann_\ast(x)$. Similarly, we can show that $ann_\ast(x) = \alpha(x)$ for all $z \in ann_\ast(x)$. □

**Theorem 3.6.** Let $R = \text{Mat}_2(F)$ with $F$ a finite field. Then $\text{Aut}(\Gamma(R)) \neq \{1\}$.

**Proof.** If $|F| = 2$, then $F$ is isomorphic to $Z_2$, and so $\text{Aut}(\Gamma(R)) \neq \{1\}$ by [8, Theorem 3.5]. Suppose that $|F| \geq 3$ and let $N(R)$ be the set of all nonzero nilpotents in $R$. By Theorem 2.3, $|\alpha(x) \cap N(R)| = |F| - 1 \geq 2$ for each $x \in X$. Take $x_1, x_2 \in \alpha(x) \cap N(R)$ so that $x_1 \neq x_2$. Since $x_1$ and $x_2$ are nilpotents, we have $ann_\ast(x_1) = \alpha(x_1) = \alpha(x_2) = ann_\ast(x_2)$ by Theorem 3.5. Observe that $ann_\ast(x_1) = ann_\ast(x_2)$. Indeed, if $a \in ann_\ast(x_1)$, then $0 = x_1a = gx_1a = 0$ for some $g \in G$ since $x_2 \in \alpha(x_1)$, which implies that $a \in ann_\ast(x_2)$, and so $ann_\ast(x_1) \subseteq ann_\ast(x_2)$. Similarly, we have $ann_\ast(x_2) \subseteq ann_\ast(x_1)$.

By a similar argument, we have $ann_\ast(x_1) = \alpha(x_1) = \alpha(x_2) = ann_\ast(x_2)$ by Theorem 3.5. Since $\alpha(x_1) = \alpha(x_2), x_2 = gx_1$ for some $g \in G$. Let $f = (x_1, x_2)$ be a transposition in $S_X$, the symmetric group on $X$. Since $x_1 \neq x_2, f \neq 1$. We will show that $f \in \text{Aut}(\Gamma(R))$. Let $yz = 0$ for some $y, z \in X$. Then we consider the following cases.

**Case 1.** $y = z = x_1$. 

Then \( f(y)f(z) = x_2x_2 = 0 \) since \( x_2 \in N(R) \).

**Case 2.** \( y = z = x_2 \).

Then \( f(y)f(z) = x_1x_1 = 0 \) since \( x_1 \in N(R) \).

**Case 3.** \( y = x_1, z = x_2 \).

Then \( f(y)f(z) = x_2x_1 = gx_1x_1 = 0 \) since \( x_1 \in N(R) \).

**Case 4.** \( y = x_2, z = x_1 \).

Then \( f(y)f(z) = x_1x_2 = g^{-1}x_2x_2 = 0 \) since \( x_2 \in N(R) \).

**Case 5.** \( y, z \neq x_1, x_2 \).

Then \( f(y)f(z) = yz = 0 \).

Consequently, if \( yz = 0 \) for some \( y, z \in X \), then \( f(y)f(z) = 0 \), which implies that \( f \in \text{Aut}(\Gamma(R)) \), and so \( \text{Aut}(\Gamma(R)) \neq \{1\} \).

**Corollary 3.7.** Let \( R = \text{Mat}_2(F) \) with \( F \) a finite field and \( N(R) \) be the set of all nonzero nilpotents in \( R \). Consider \( X = o_\ell(a_1) \cup \cdots \cup o_\ell(a_{|F|+1}) \) as mentioned in Remark 1. For all \( j = 1, \ldots, |F|+1 \), let \( s_j = (1, j) \) be a transposition in \( S_{|F|+1} \), the symmetric group of degree \( |F|+1 \). If \( f_{s_j} = (x_1, x_j) \) is a transposition in \( S_X \), the symmetric group on \( X \), then \( f_{s_j} \) is a graph automorphism in \( \Gamma(R) \).

**Proof.** By Lemma 3.1 and Theorem 3.5, \( f_{s_j}(o_\ell(x_1)) = o_\ell(f_{s_j}(x_1)) = o_\ell(x_j) \). Then \( f_{s_j} \) is a graph automorphism in \( \Gamma(R) \) by the similar argument as given in the proof in Theorem 3.6. \( \square \)

**Theorem 3.8.** Let \( R = \text{Mat}_2(F) \) with \( F \) a finite field. Then \( \text{Aut}(\Gamma(R)) \simeq S_{|F|+1} \).

**Proof.** Let \( N(R) \) be the set of all nonzero nilpotents in \( R \). We choose \( x_1, \ldots, x_{|F|+1} \in N(R) \) so that \( X = o_\ell(x_1) \cup \cdots \cup o_\ell(x_{|F|+1}) \) by Remark 1. Let \( f \in \text{Aut}(\Gamma(R)) \) be arbitrary. By Lemma 3.1 and Theorem 3.5, for each \( j = 1, \ldots, |F|+1 \), \( f(o_\ell(x_j)) = o_\ell(f(x_j)) = o_\ell(x_{i_j}) \) for some \( i_j \) \( (1 \leq i_j \leq |F|+1) \). Thus \( f \) is determined by the permutation

\[
\begin{pmatrix}
\ell & \cdots & |F|+1 \\
i_1 & \cdots & i_{|F|+1}
\end{pmatrix} \in S_{|F|+1}.
\]

Since \( S_{|F|+1} \) is generated by the transpositions \( s_2 = (1, 2), \ldots, s_{|F|+1} = (1, |F|+1) \), and each \( f_{s_j} = (x_1, x_j) \), a transposition in \( S_X \), is a graph automorphism in \( \Gamma(R) \) by Corollary 3.7, \( f \) is generated by \( f_{s_1}, \ldots, f_{s_{|F|+1}} \). Hence the map \( \sigma : \text{Aut}(\Gamma(R)) \rightarrow S_{|F|+1} \) by \( \sigma(f) = f_\ell \) is bijective. Also \( \sigma \) is a group homomorphism by observing that for all \( s_i, s_j \in S_{|F|+1} (i, j = 2, \ldots, |F|+1) \), \( (f_{s_i} \circ f_{s_j}) = f_{s_is_j} \). Therefore, \( \text{Aut}(\Gamma(R)) \simeq S_{|F|+1} \). \( \square \)

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References


Sangwon Park
Department of Mathematics
Dong-A University
Pusan 609-714, Korea
E-mail address: swpark@donga.ac.kr

Juncheol Han
Department of Mathematics Education
Pusan National University
Pusan 609-735, Korea
E-mail address: jchan@pusan.ac.kr