CONTINUITY OF $(\alpha, \beta)$-DERIVATIONS OF OPERATOR ALGEBRAS

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ABSTRACT. We investigate the continuity of $(\alpha, \beta)$-derivations on $B(X)$ or $C^*$-algebras. We give some sufficient conditions on which $(\alpha, \beta)$-derivations on $B(X)$ are continuous and show that each $(\alpha, \beta)$-derivation from a unital $C^*$-algebra into its a Banach module is continuous when $\alpha$ and $\beta$ are continuous at zero. As an application, we also study the ultraweak continuity of $(\alpha, \beta)$-derivations on von Neumann algebras.

1. Introduction

Let $B$ be a complex algebra with a subalgebra $A$, let $\alpha$ and $\beta$ be two mappings from $A$ into $B$, let $M$ be a $B$-module and hence an $A$-module. A linear mapping $\delta$ from $A$ into $M$ is called an $(\alpha, \beta)$-derivation, if $\delta(AB) = \delta(A)\alpha(B) + \beta(A)\delta(B)$ holds for all $A, B \in A$; moreover, $\delta$ is called inner, if there exists $M_0 \in M$ such that $\delta(A) = M_0\alpha(A) - \beta(A)M_0$ for each $A \in A$. An $(\alpha, \alpha)$-derivation is called briefly an $\alpha$-derivation. Clearly, an $id$-derivation is an ordinary linear derivation, where $id$ denotes the embedding map from $A$ into $B$, and every endomorphism $\alpha$ on $A$ is an $\frac{1}{2}$-derivation on $A$. Note that in our definition of an $(\alpha, \beta)$-derivation, no extra assumptions on $\alpha$ and $\beta$, such as linearity, are required. The purpose of this note is to investigate the continuity of $(\alpha, \beta)$-derivations from Banach algebras into their Banach modules.

In 1958, Kaplansky conjectured that every derivation on a $C^*$-algebra or a semisimple Banach algebra is continuous ([7, 8]). Sakai confirmed Kaplansky’s conjecture for the $C^*$-algebra case in [14], and from this, Kadison deduced the ultraweak continuity of derivations when the $C^*$-algebras are represented on Hilbert spaces ([5]). In [13], Ringrose generalized these results to the derivations from $C^*$-algebras into their Banach modules. The conjecture on the continuity of derivations on semisimple Banach algebras by Kaplansky was confirmed by Johnson and Sinclair in [4]. For the detail on automatic continuity of derivations of Banach algebras, we refer to [3, 15].

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On the continuity of \((\alpha, \beta)\)-derivations on \(C^\ast\)-algebras, M. Mirzavazibri and S. Moslehian proved that each \(+\,(\alpha, \beta)\)-derivation from a \(C^\ast\)-algebra \(A\) acting on a Hilbert space \(H\) into \(B(H)\) is continuous under the assumption that \(\alpha\) and \(\beta\) are \(+\)-linear continuous mappings from \(A\) into \(B(\beta)\) ([10, 11, 12]).

For an \((\alpha, \beta)\)-derivation \(\delta\) from a Banach algebra \(A\) into a Banach \(A\)-module \(M\), if both \(\alpha\) and \(\beta\) are bounded algebraic homomorphisms from \(A\) into itself, then \(M\), equipped with the \(A\)-module actions defined by \(A \cdot M = \beta(A)M\), \(M \cdot A = M\alpha(A)\), is also a Banach \(A\)-module, denoted by \(M_{\alpha, \beta}\), and \(\delta\) is indeed an ordinary derivation from \(A\) into \(M_{\alpha, \beta}\). Hence it is interesting to study the continuity of an \((\alpha, \beta)\)-derivation when at least one of \(\alpha\) and \(\beta\) is not an algebraic homomorphism.

This note is organized as follows. In Section 2 we give two examples of continuous \((\alpha, \beta)\)-derivations for two nonlinear and non-continuous mappings \(\alpha\) and \(\beta\). In Section 3 we give some sufficient conditions on which an \((\alpha, \beta)\)-derivation on \(B(\mathfrak{X})\), the algebra of all bounded linear operators on a complex Banach space \(\mathfrak{X}\), is continuous. In particular, we show that if \(\mathfrak{X}\) is simple and \(\alpha, \beta\) are surjective and continuous at zero, then each \((\alpha, \beta)\)-derivation on \(B(\mathfrak{X})\) is continuous. In Section 4, using a similar argument to the proof in [13], we show that every \((\alpha, \beta)\)-derivation of a unital \(C^\ast\)-algebra into its a Banach module is continuous if \(\alpha, \beta\) are continuous at zero, which generalizes the main results in [10]. As corollaries, we also get the ultraweak continuity of \((\alpha, \beta)\)-derivations of von Neumann algebras when the ultraweak continuity and linearity on \(\alpha\) and \(\beta\) are required.

For a complex Banach space \(\mathfrak{X}\), we denote by \(\mathfrak{X}^\ast\), \(B(\mathfrak{X})\) and \(K(\mathfrak{X})\), the Banach dual space of \(\mathfrak{X}\), the algebra of all bounded linear operators on \(\mathfrak{X}\) and the ideal of all compact operators in \(B(\mathfrak{X})\), respectively. For nonzero vectors \(\xi \in \mathfrak{X}\) and \(f \in \mathfrak{X}^\ast\), we denote by \(\xi \otimes f\) the rank operator defined by \((\xi \otimes f)(\eta) = f(\eta)\xi\) for each \(\eta \in \mathfrak{X}\). Sometimes we write \((\eta, f)\) in place of \(f(\eta)\). Let \(F_1(\mathfrak{X})\) denote the set of all rank one operators on \(\mathfrak{X}\). Obviously, \(A(\xi \otimes f)B = (A\xi) \otimes (Bf)\) for all \(A, B \in B(\mathfrak{X}), \xi \in \mathfrak{X}, f \in \mathfrak{X}^\ast\), where \(B^t\) denotes the transpose of the bounded linear operator \(B\), defined by \((\xi, B^t f) = (B\xi, f)\) for each \(\xi \in \mathfrak{X}\) and \(f \in \mathfrak{X}^\ast\).

2. Reduction and examples

Let \(A\) be a complex Banach algebra without identity. We take the direct sum \(A \oplus \mathbb{C}\) as a linear space \(A_I\). By a well-known fact, \(A_I\), endow with a Banach algebra structure, is a unital Banach algebra. In addition, if \(A\) is a \(C^\ast\)-algebra, then there exists a (unique) norm on \(A_I\) which makes \(A_I\) be a unital \(C^\ast\)-algebra. If we identify each \(A \in A\) with \((A, 0) \in A_I\), then \(A\) is a closed two-sided ideal of \(A_I\) ([6]). We write \((A, \lambda)\) as \(A + \lambda I\) for each \((A, \lambda) \in A_I\). If \(M\) is a Banach \(A\)-module, then it is a unital Banach \(A_I\)-module under the module action given by \((A + \lambda)M = AM + \lambda M\) and \(M(A + \lambda) = MA + \lambda M\) for every \(A + \lambda \in A_I\) and \(M \in M\).
For a given mapping \( \sigma : A \to A \), we can obtain its extension \( \sigma_I \) to \( A_I \) by \( \sigma_I(A + \lambda) = \sigma(A) + \lambda \) for each \( A + \lambda \in A_I \). Then \( \sigma_I \) is linear if and only if so is \( \sigma \), \( \sigma_I \) preserves the identity of \( A_I \) if and only if \( \sigma(0) = 0 \), and when \( A \) is a C*-algebra, \( \sigma_I \) is a *-mapping if and only if so is \( \sigma \). In both Banach algebra and C*-algebra cases, \( \sigma_I \) is continuous at 0 if and only if so is \( \sigma \).

For an \( (\alpha, \beta) \)-derivation \( \delta \) from \( A \) into \( M \), since \( \delta \) is linear, we have \( \delta(0) = 0 \).

Using the equation \( 0 = \delta(0) = \delta(0A) = \delta(0A) \), we have \( \delta(A)\alpha(0) = 0 \) and \( \beta(0)\beta(A) = 0 \) for each \( A \in A \). Hence if let \( \alpha_0(A) = \alpha(A) - \alpha(0) \) and \( \beta_0(A) = \beta(A) - \beta(0) \) for each \( A \in A \), then \( \delta \) is a \( (\alpha_0, \beta_0) \)-derivation. So, for an \( (\alpha, \beta) \)-derivation \( \delta \), we sometimes can assume that \( \alpha(0) = 0 \) and \( \beta(0) = 0 \). Define the mapping \( \delta_I \) from \( A_I \) into \( M \) by \( \delta_I(A, \lambda) = \delta(A) \) for each \( (A, \lambda) \in A_I \).

Then \( \delta_I \) is an \( (\alpha_1, \beta_1) \)-derivation from \( A_I \) into \( M \), and \( \delta_I(0, 1) = \delta(0) = 0 \), \( \alpha_1(0, 1) = (0, 1) \) and \( \beta_1(0, 1) = (0, 1) \) by the assumptions that \( \alpha(0) = 0 \) and \( \beta(0) = 0 \). Obviously, \( \delta \) is bounded if and only if so is \( \delta_I \).

Hence, to obtain the continuity of an \( (\alpha, \beta) \)-derivation \( \delta \) of a Banach algebra \( A \), we can assume that A is unital, \( M \) is a unital \( A \)-module, \( \delta(1) = 0 \), \( \alpha(0) = \beta(0) = 0 \) and \( \alpha(1) = \beta(1) = 1 \), where 1 is the identity of \( A \).

The following examples yield continuous \( (\alpha, \beta) \)-derivations without the assumption of linearity and continuity of \( \alpha \) and \( \beta \).

**Example 2.1.** Let \( A \) be a von Neumann algebra acting on a separable Hilbert space \( \mathfrak{H} \), \( A \neq B(\mathfrak{H}) \). Let \( \alpha_0 \) and \( \beta_0 \) be bounded homomorphisms of \( A \) into itself, \( f_0, g_0 : A \to \mathbb{C} \) be two functionals without linearity and continuity, \( T_0 \) and \( S_0 \) be nonzero operators in \( A' \) with \( T_0S_0 = S_0T_0 = 0 \). Define the mappings \( \alpha, \beta \) and \( \delta \) from \( A \) into \( B(\mathfrak{H}) \) by

\[
\alpha(A) = \alpha_0(A) + f_0(A)S_0, \quad \beta(A) = \beta_0(A) + g_0(A)S_0, \quad \delta(A) = T_0\alpha(A) - \beta(A)T_0
\]

for each \( A \in A \). Then \( \alpha \) and \( \beta \) are neither continuous nor linear, and \( \delta \) is an \( (\alpha, \beta) \)-derivation from \( A \) into \( B(\mathfrak{H}) \). By calculation, we have \( \delta(A) = T_0\alpha_0(A) - \beta_0(A)T_0 \) for each \( A \in A \), hence \( \delta \) is continuous.

**Example 2.2.** Let \( \mathfrak{H} \) be a separable infinite dimensional Hilbert space, \( V \in B(\mathfrak{H}) \) a partial isometry with \( V^*V = I \), \( VV^* = P \neq I \). Let \( T_0 \in B(\mathfrak{H}) \) be a self-adjoint operator with \( T_0P = 0 \), let \( f_0 : B(\mathfrak{H}) \to \{ V^* \}' \) be a mapping without linearity and continuity; (e.g., \( f \) is a nonlinear and non-continuous functional on \( B(\mathfrak{H}) \)), where \( \{ V^* \}' \) is the commutant of \( \{ V^* \} \) in \( B(\mathfrak{H}) \). Define the mappings \( \alpha, \delta : B(\mathfrak{H}) \to B(\mathfrak{H}) \) by \( \alpha(A) = \frac{1}{2}(VAV^* + f_0(A)T_0) \) and \( \delta(A) = VAV^* \) for each \( A \in B(\mathfrak{H}) \). Then \( \alpha \) is neither linear nor continuous on \( B(\mathfrak{H}) \), but \( \delta \) is a continuous \( \alpha \)-derivation.

3. The \( B(\mathfrak{X}) \) case

In the following lemma, we list some properties given in [10] of an \( (\alpha, \beta) \)-derivation.
Lemma 3.1 ([10]). Let $\mathcal{B}$ be a complex algebra with a subalgebra $\mathcal{A}$ and let $\mathcal{M}$ be a $\mathcal{B}$-module. Let $\alpha, \beta : \mathcal{A} \to \mathcal{B}$ be two mappings. If $\delta$ is an $(\alpha, \beta)$-derivation from $\mathcal{A}$ into $\mathcal{M}$, then, for each $\lambda, \mu \in \mathbb{C}$ and $A, B, C \in \mathcal{A}$, the following equations hold:

(i) $\delta(A)\alpha(0) = \beta(0)\delta(A) = 0$;
(ii) $\delta(A)(\alpha(\lambda B + \mu C) - \lambda \alpha(B) - \mu \alpha(C)) = 0$;
(iii) $(\beta(\lambda A + \mu B) - \lambda \beta(A) - \mu \beta(B))\delta(C) = 0$;
(iv) $(\beta(AB) - \beta(A)\beta(B))\delta(C) = \delta(A)(\alpha(BC) - \alpha(B)\alpha(C))$. In particular, $(\delta(0) - \beta(0))\delta(C) = 0 = \delta(A)(\alpha(0) - \alpha(B)\alpha(0))$.

Proof. By the equation $0 = \delta(0) = \delta(0A) = \delta(0A)$, we can obtain (i). Using the linearity and the multiplicative rule of $\delta$ to expand the left of the following equations: $\delta(A(\lambda B + \mu C)) - \lambda \delta(AB) - \mu \delta(AC) = 0, \delta((\lambda A + \mu B)C) - \lambda \delta(AC) - \mu \delta(BC) = 0, \delta((AB)C) - \delta(A(BC)) = 0$, we can get (ii), (iii) and (iv). \quad \Box

Theorem 3.2. Let $X$ be a complex Banach space, $\alpha$ and $\beta$ be mappings from $B(X)$ into itself. Let $\delta : B(X) \to B(X)$ be an $(\alpha, \beta)$-derivation. Suppose that $\alpha$ and $\beta$ satisfy one of the following conditions:

(i) $\alpha$ is an automorphism, $\beta$ is continuous at 0 and the set $\{\beta(T) : T \in F_1(X)\}$ separates the points of $X$ in the sense that, for each pair $\xi, \eta \in X$ with $\xi \neq \eta$, there is a rank one operator $T$ such that $\beta(T)\xi \neq \beta(T)\eta$, equivalently, the set $\{\beta(T) : T \in F_1(X)\}$ has no nonzero right annihilators in $B(X)$.

(ii) $\beta$ is an automorphism, $\alpha$ is continuous at 0 and the set $\{\alpha(T) : T \in F_1(X)\}$ has no nonzero left annihilators in $B(X)$.

(iii) $\alpha$ and $\beta$ are continuous at 0, span$\{\alpha(T)\xi : T \in F_1(X), \xi \in X\}$ is dense in $X$ and there is a rank one $S$ such that $\beta(S)$ is injective.

Then $\delta$ is continuous. Moreover, if (i), or when $X$ is reflexive and (ii), holds, $\delta$ is inner.

Proof. In order to obtain the continuity of $\delta$, we use the closed graph theorem. Let $A_n \in B(X), n = 1, 2, \ldots$, with $A_n \to 0$ and $\delta(A_n) \to A$. For every $\xi \otimes f, \eta \otimes g \in F_1(X)$ and $n = 1, 2, \ldots$, we have

$f(A_n \eta)\delta(\xi \otimes g) = \delta(\xi \otimes f \cdot A_n \cdot \eta \otimes g) = \delta(\xi \otimes f)\alpha(A_n \eta \otimes g) + \beta(\xi \otimes f)\delta(A_n)\alpha(\eta \otimes g) + \beta(\xi \otimes f)\beta(A_n)\delta(\eta \otimes g)$.

If $\alpha$ and $\beta$ are continuous at 0, letting $n \to \infty$, we have

$\delta(\xi \otimes f)\alpha(0) + \beta(\xi \otimes f)\alpha(\eta \otimes g) + \beta(\xi \otimes f)\beta(0)\delta(\eta \otimes g) = 0$.

Using Lemma 3.1, we have

$\beta(\xi \otimes f)\alpha(\eta \otimes g) = 0$

for every $\xi \otimes f, \eta \otimes g \in F_1(X)$.
If (i) holds, then for each \( \eta \otimes g \in F_1(\mathcal{X}) \), we have \( A\alpha(\eta \otimes g) = 0 \). Since \( \alpha \) is an automorphism, it is inner, i.e., there is an invertible bounded linear operator \( T_0 \in B(\mathcal{X}) \) such that \( \alpha(T) = T_0TT_0^{-1} \) for each \( T \in B(\mathcal{X}) \). So \( (AT_0)\eta \otimes g = 0 \) for all \( \eta \in \mathcal{X} \) and \( g \in \mathcal{X}^* \). Hence \( A = 0 \). Consequently, \( \delta \) is continuous.

In this case, we can show that \( \delta \) is inner. Choose \( \xi_0 \in \mathcal{X} \) and \( f_0 \in \mathcal{X}^* \) such that \( f_0(\xi_0) = 1 \), and define the mapping \( A_0 : \mathcal{X} \to \mathcal{X} \) by \( A_0\xi = \delta(T_0^{-1}\xi \otimes f_0)T_0\xi_0 \) for each \( \xi \in \mathcal{X} \). Obviously, \( A_0 \) is linear and bounded. For each \( T \in B(\mathcal{X}) \) and \( \xi \in \mathcal{X} \), we have

\[
\delta((T\xi) \otimes f_0) = \delta(T(\xi \otimes f_0)) = \delta(T)\alpha(\xi \otimes f_0) + \beta(T)\delta(\xi \otimes f_0)
\]

Multiplying by the operator \( T_0 \), we have

\[
\delta((T\xi) \otimes f_0)T_0 = \delta(T)T_0(\xi \otimes f_0) + \beta(T)\delta(\xi \otimes f_0)T_0.
\]

Applying mappings in two sides of the equation to \( \xi_0 \), we get \( A_0(T_0T\xi) = \delta(T)T_0\xi + \beta(T)A_0(T_0\xi) \). Since \( \xi \) is arbitrary, we have \( A_0T_0 = \delta(T)I + \beta(T)A_0I_0 \), and hence \( \delta(T) = A_0\alpha(T) - \beta(T)A_0 \) for each \( T \in B(\mathcal{X}) \). So \( \delta \) is inner.

If (ii) holds, then by the equation \( \beta(\xi \otimes f)A\alpha(\eta \otimes g) = 0 \), we have \( \beta(\xi \otimes f)A = 0 \). Since \( \beta \) is an automorphism, there is an invertible bounded linear operator \( S_0 \in B(\mathcal{X}) \) such that \( \beta(T) = S_0TS_0^{-1} \) for each \( T \in B(\mathcal{X}) \). So \( (\xi \otimes f)S_0^{-1}A = 0 \) for all \( \xi \in \mathcal{X} \) and \( f \in \mathcal{X}^* \). Hence \( A = 0 \), so \( \delta \) is continuous.

In this case, choose \( \xi_0 \in \mathcal{X} \) and \( f_0 \in \mathcal{X}^* \) such that \( f_0(\xi_0) = 1 \). We define the mapping \( B_0 : \mathcal{X} \to \mathcal{X} \) by

\[
\langle B_0\xi, f \rangle = \langle S_0^{-1}\delta((\xi_0 \otimes f)S_0)\xi, f_0 \rangle, \quad \xi \in \mathcal{X}, \; f \in \mathcal{X}^*.
\]

Suppose that \( \mathcal{X} \) is reflexive. Then \( B_0 \) is well-defined. The continuity and linearity of \( \delta \) imply the continuity and linearity of \( B_0 \). For each \( \xi \in \mathcal{X}, f \in \mathcal{X}^* \) and \( T \in B(\mathcal{X}) \), we have

\[
\langle (\beta(T)B_0 - B_0\alpha(T))\xi, f \rangle = \langle B_0\xi, \beta(T)f \rangle - \langle B_0\alpha(T)\xi, f \rangle = \langle S_0^{-1}\delta((\xi_0 \otimes \beta(T)f)S_0)\xi, f_0 \rangle - \langle S_0^{-1}\delta((\xi_0 \otimes f)S_0)\alpha(T)\xi, f_0 \rangle = \langle S_0^{-1}\delta((\xi_0 \otimes f)S_0T)\xi, f_0 \rangle - \langle S_0^{-1}\delta((\xi_0 \otimes f)S_0)\alpha(T)\xi, f_0 \rangle = \langle S_0^{-1}\beta((\xi_0 \otimes f)S_0)\delta(T)\xi, f_0 \rangle = \langle (\xi_0 \otimes f)\delta(T)\xi, f_0 \rangle = \langle \delta(T)\xi, f \rangle.
\]

Hence \( \delta(T) = \beta(T)B_0 - B_0\alpha(T) \) for each \( T \in B(\mathcal{X}) \), and so, \( \delta \) is inner.

Obviously, if (iii) holds, then \( A = 0 \), which yields the continuity of \( \delta \). \( \Box \)

For a linear mapping \( T \) from a Banach space \( \mathcal{E} \) into a Banach space \( \mathcal{F} \), the separating space \( \mathcal{S}(T) \) is defined to be the set of elements \( \xi \) in \( \mathcal{F} \) such that
there is a sequence \( \{\xi_n\} \) in \( \mathcal{E} \) with \( \xi_n \to 0 \) in \( \mathcal{E} \) and \( T(\xi_n) \to \xi \) in \( \mathcal{F} \). Clearly, 
\[
\mathcal{S}(T) = \cap_{n=1}^{\infty} \{T(\eta) : \|\eta\| < \frac{1}{n}\},
\]
hence is a closed linear subspace of \( \mathcal{F} \), and by the closed graph theorem, \( T \) is continuous if and only if \( \mathcal{S}(T) = \{0\} \).

Recall that a Banach space \( X \) is called simple, if \( B(X) \) has a unique nontrivial norm-closed two-sided ideal. For example, \( L^p (1 \leq p < \infty) \), \( c_0 \) and a separable infinite dimensional Hilbert space \( \mathcal{H} \) are simple. In this case, the norm closure of all the finite rank operators is the ideal of compact operators, which is not-dense in \( B(X) \) and is the unique nontrivial norm-closed two-sided ideal of \( B(X) \).

**Proposition 3.3.** Suppose that \( X \) is a simple complex Banach space, \( A \) is a unital norm closed subalgebra of \( B(X) \), \( \alpha, \beta : A \to B(X) \) are surjective and continuous at 0. If at least one of \( \alpha \) and \( \beta \) is not an algebraic homomorphism, then every \( (\alpha, \beta) \)-derivation \( \delta \) from \( A \) into \( B(X) \) is automatically continuous.

**Proof.** We first show that \( \mathcal{S}(\delta) \) is a closed two-sided ideal of \( B(X) \). For arbitrary \( A \in \mathcal{S}(\delta) \) and \( B \in B(X) \), there is a sequence \( \{A_n\} \) in \( A \) with \( A_n \to 0 \) and \( \delta(A_n) \to A \). Since \( \alpha \) and \( \beta \) are surjective, there are \( S, T \in A \) such that \( B = \alpha(S) = \beta(T) \). Hence \( TA_n \to 0 \), \( A_n S \to 0 \). Also since \( \alpha \) and \( \beta \) are continuous at 0, using Lemma 3.1, we have 
\[
\delta(TA_n) = \delta(T)\alpha(A_n) + \beta(T)\delta(A_n) \to \delta(T)\alpha(0) + BA = BA \text{ and } \delta(A_n S) = \delta(A_n)\alpha(S) + \beta(A_n)\delta(S) \to AB + \beta(0)\delta(S) = AB. \]
Hence \( AB, BA \in \mathcal{S}(\delta) \). So \( \mathcal{S}(\delta) \) is a closed two-sided ideal of \( B(X) \). Since \( X \) is simple, we have \( \mathcal{S}(\delta) = 0, B(X), \) or \( K(X) \).

For an arbitrary \( A \in \mathcal{S}(\delta) \), let \( \{A_n\} \) be a sequence in \( A \) with \( A_n \to 0 \) and \( \delta(A_n) \to A \). For all pairs \( B, C \in A \), using (iv) of Lemma 3.1, we have 
\[
(\beta(A_n B) - \beta(A_n)\beta(B))\delta(C) = \delta(A_n)(\alpha(BC) - \alpha(B)\alpha(C)) \text{ and } (\beta(BC) - \beta(B)\beta(C))\delta(A_n) = \delta(B)(\alpha(BC) - \alpha(B)\alpha(C)).
\]
The continuity of \( \alpha \) and \( \beta \) at 0 implies that 
\[
(\beta(0) - \beta(0)\beta(B))\delta(C) = A(\alpha(BC) - \alpha(B)\alpha(C)) \text{ and } (\beta(BC) - \beta(B)\beta(C))A = \delta(B)(\alpha(0) - \alpha(C)\alpha(0)).
\]
Using (iv) of Lemma 3.1, we have
\[
A(\alpha(BC) - \alpha(B)\alpha(C)) = 0,
\]
(1)
\[
(\beta(BC) - \beta(B)\beta(C))A = 0
\]
for each \( A \in \mathcal{S}(\delta) \) and \( B, C \in A \). Similarly, using (ii) and (iii) of Lemma 3.1, for each \( A \in \mathcal{S}(\delta) \), \( B, C \in A \), \( \lambda, \mu \in \mathbb{C} \), we get that
\[
A(\alpha(\lambda B + \mu C) - \lambda \alpha(B) - \mu \alpha(C)) = 0,
\]
(3)
\[
(\beta(\lambda B + \mu C) - \lambda \beta(B) - \mu \beta(C))A = 0.
\]
Suppose that \( \mathcal{S}(\delta) = B(X) \) or \( \mathcal{S}(\delta) = K(X) \). Then it follows from (1), (2), (3) and (4) that both \( \alpha \) and \( \beta \) are algebraic homomorphisms, which is a contradiction. Hence \( \mathcal{S}(\delta) = 0 \), and so, \( \delta \) is continuous. \( \square \)
**Theorem 3.4.** Let $X$ be a simple Banach space, $\delta$ an $(\alpha, \beta)$-derivation from $B(X)$ into itself. Suppose that $\alpha, \beta : B(X) \to B(X)$ are surjective and continuous at $0$. Then $\delta$ is continuous.

*Proof.* If at least one of $\alpha$ and $\beta$ is not an algebraic homomorphism, then Proposition 3.3 yields the continuity of $\delta$. If both $\alpha$ and $\beta$ are algebraic homomorphisms, then they are bounded automorphisms of $B(X)$. Theorem 3.2 implies that $\delta$ is continuous.

Removing the continuity in above theorem, we have the following results.

**Theorem 3.5.** For a complex Banach space $X$ and two mappings $\alpha, \beta$ on $B(X)$, assume that $\alpha$ and $\beta$ are surjective and multiplicative and there are rank one operators $T_0$ and $S_0$ such that $\alpha(T_0) \neq 0$ and $\beta(S_0) \neq 0$. Then every $(\alpha, \beta)$-derivation from $B(X)$ into itself is continuous.

*Proof.* It suffices to show that $\alpha$ and $\beta$ are (bounded) automorphisms of $B(X)$. Assume that $\delta \neq 0$.

Since $\alpha$ and $\beta$ are surjective and multiplicative, it is not difficult to show that, for each $x \in X$, there are scales $f(x)$ and $g(x)$ such that $\alpha(xI) = f(x)I$, $\beta(xI) = g(x)I$ and $\alpha(I) = I$, $\beta(I) = I$. Note that $\delta(I) = \delta(I)\alpha(I) + \beta(I)\delta(I) = 2\delta(I)$, which yields $\delta(I) = 0$. Hence for $x \in X$ and $T \in B(X)$, $\lambda\delta(T) = \delta(T\cdot xI) = \delta(T)\alpha(xI) + \beta(T)\delta(xI) = \delta(T)\alpha(xI) = f(x)\delta(T)$, which implies that $f(x) = \lambda$, and thus, $\alpha(xI) = \lambda I$. Hence $\alpha$ is homogeneous. Similarly, using $\lambda\delta(T) = \delta(xI\cdot T)$, we can get that $\beta$ is also homogeneous.

Now we show that $\alpha$ and $\beta$ are injective.

Let $T_0 = x_0 \otimes f_0$ be the rank one operator such that $\alpha(T_0) \neq 0$. For each rank one operator $x \otimes f$, choose $g_0 \in X^*$ and $h_0 \in X$ such that $g_0(x) = f(h_0) = 1$. Then $\alpha(T_0)(x \otimes f) = (\alpha(x \otimes f))(\alpha(h_0 \otimes f))$, which implies $\alpha(x \otimes f) \neq 0$ for all rank one operators $x \otimes f$.

If $\alpha(T_0) = 0$, then $T = 0$. For, otherwise, there exists $x \in X$ with $TX \neq 0$. For $x \otimes f$, $f_0 \neq 0$, we have $T \otimes f_0$ is a rank one operator, but $\alpha(T \otimes f_0) = \alpha(T)(x \otimes f_0) = 0$, which is a contradiction with above argument.

Next we show $\alpha$ is injective on the set of all rank one operators. Let $R = \xi_0 \otimes f_0$ and $S = \eta_0 \otimes g_0$ be two arbitrary rank one operators with $\alpha(R) = \alpha(S)$. Since $R$ and $S$ are linearly independent, $\xi_0, \eta_0$ are linearly dependent, then $f_0$ and $g_0$ are linearly independent. Choosing $x \in X$ with $f_0(x) = 1$ and $g_0(x) = 0$, we have $R(x \otimes f_0) = \xi_0 \otimes f_0$ and $S(x \otimes f_0) = \eta_0 \otimes g_0 = 0$, which is impossible, for $0 \neq \alpha(R(x \otimes f_0)) = \alpha(S(x \otimes f_0)) = 0$. If $R$ and $S$ are linearly independent, and $\xi_0, \eta_0$ are linearly independent, then we can choose $h_0 \in X^*$ such that $h_0(x_0) = 0$ and $h_0(x_0) = 1$. Hence $(\xi_0 \otimes h_0)R = (\xi_0 \otimes h_0)(\xi_0 \otimes f_0) = 0$ and $(\xi_0 \otimes h_0)S = (\xi_0 \otimes h_0)((\eta_0 \otimes g_0) = \xi_0 \otimes g_0 = 0$. So $0 = \alpha((\xi_0 \otimes h_0)R) = \alpha((\xi_0 \otimes h_0)S) \neq 0$, which is a contradiction. Hence $R$ and $S$ are linearly independent. The homogeneity of $\alpha$ yields $R = S$.

If $\alpha(T) = \alpha(S)$ for $T, S \in B(X)$, then for each nonzero vectors $x \in X$ and $y \in X^*$, we have $\alpha(Sx \otimes y) = \alpha(Ty \otimes f)$. Obviously, $Sx = 0$ if and only if
$T \xi = 0$. If $S \xi \neq 0$, using the injectivity of $\alpha$ on the set of all rank one operators and the arbitrariness of $f$, we have $S \xi = T \xi$. Hence $S = T$. We have shown that $\alpha$ is injective. Similarly, we can show the injectivity of $\beta$.

Hence $\alpha$ and $\beta$ are multiplicative bijections on $B(\mathcal{X})$. By the celebrated result of Martindale in [9], $\alpha$ and $\beta$ are additive. Consequently, $\alpha$ and $\beta$ are surjective algebraic homomorphisms, hence are automorphisms on $B(\mathcal{X})$. By Theorem 3.2, $\delta$ is continuous. □

4. The $C^*$-algebra case

In this section we study the continuity of $(\alpha, \beta)$-derivations of $C^*$-algebras into their Banach modules. Inspiring the proof of the related results on the ordinary derivations in [13], we have the following Theorem 4.4. We start with some lemmas which can be found in Ex 4.6.39, Ex 4.6.13 and Ex 4.6.20 in [6] (see also Lemma 1 and the proof of Theorem 3 in [13]).

Lemma 4.1 ([6, 13]). Let $\mathcal{J}$ be a closed two-sided ideal in a unital $C^*$-algebra $\mathcal{A}$, $B \in \mathcal{J}$ a positive element with $\|B\| \leq 1$, $A \in \mathcal{J}$ with $AA^* \leq B^4$. Then $A = BC$ for some $C$ in $\mathcal{J}$ with $\|C\| \leq 1$.

Lemma 4.2 ([6, 13]). Suppose that $\mathcal{D}$ is an infinite dimensional unital $C^*$-algebra. Then there is an infinite sequence $\{A_1, A_2, \ldots\}$ of nonzero positive elements in $\mathcal{D}$ such that $A_jA_k = 0$ for $j \neq k$.

Lemma 4.3 ([6, 13]). Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital $C^*$-algebras and $\varphi$ is a $\ast$-homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. For each sequence $\{B_1, B_2, \ldots\}$ of positive elements of $\mathcal{B}$ such that $B_jB_k = 0$ when $j \neq k$, there is a sequence $\{A_1, A_2, \ldots\}$ of positive elements of $\mathcal{A}$ such that $A_jA_k = 0$ when $j \neq k$ and $\varphi(A_j) = B_j$ for each $j = 1, 2, \ldots$.

Theorem 4.4. Let $\mathcal{B}$ be a unital $C^*$-algebra, let $\mathcal{B}$ be a unital Banach algebra containing $\mathcal{A}$ as a unital Banach subalgebra, and let $\mathcal{M}$ be a Banach $\mathcal{B}$-module. Assume that $\alpha, \beta : \mathcal{A} \to \mathcal{B}$ are continuous at 0. Then every $(\alpha, \beta)$-derivation $\delta$ from $\mathcal{A}$ into $\mathcal{M}$ is continuous.

Proof. For each $A$ in $\mathcal{A}$, we define the mappings $L_A, S_A, \gamma_A, \sigma_A : \mathcal{A} \to \mathcal{M}$ by $L_A(T) = \delta(\alpha(T))$, $S_A(T) = \beta(\alpha(T)) = \delta(\beta(T))$, $\gamma_A(T) = \sigma_A(T) = \beta(T)\delta(\alpha(T))$ for each $T$ in $\mathcal{A}$. Since $\delta$ is linear, we have $L_A$ and $S_A$ are linear, hence $\gamma_A = L_A - S_A$ is linear. It follows from (iii) of Lemma 3.1 that $\sigma_A$ is linear. The continuity of $\alpha$ and $\beta$ at 0 implies the continuity of $\gamma_A$ and $\sigma_A$ at 0, hence at every $T \in \mathcal{A}$. Hence $\gamma_A$ and $\sigma_A$ are bounded. Let $J = \{A \in \mathcal{A} : L_A \text{ is bounded}\}$. Obviously, $0 \in J$, $J$ is a subspace of $\mathcal{A}$, and $\mathcal{J} = \{A \in \mathcal{A} : S_A \text{ is bounded}\}$. Firstly, we claim that $J$ is a norm closed two-sided ideal of $\mathcal{A}$.

For each $J$ in $J$ and $A$ in $\mathcal{A}$, since $L_JA$ is the composition of the bounded mapping $L_J$ and the (bounded) multiplication on the left by $A$: $T \to AT$ from
\(A\) into itself, we have that \(JA\) is in \(J\); on the other hand, by Lemma 3.1(iv), we have
\[
S_{AJ}(T) = \beta(AJ)\delta(T) = \beta(A)S_J(T) + \gamma_A(JT) - \gamma_A(J)(T)
\]
for each \(T \in A\). Hence \(S_{AJ}\) is continuous at 0, which yields that \(S_{AJ}\) is continuous. Hence \(AJ\) is in \(J\), so \(J\) is a two-sided ideal of \(A\).

Let \(\{A_n\}\) be a sequence in \(J\) with \(A_n \to A \in A\). For each \(T\) in \(A\) noting that \(\sigma_T\) is bounded, we have \(\lim_{n \to \infty} S_{A_n}(T) = \lim_{n \to \infty} \beta(A_n)\delta(T) = \lim_{n \to \infty} \sigma_T(A_n) = \sigma_T(A) = \beta(A)\delta(T) = S_A(T)\). Since \(\{S_{A_n}\}\) is a sequence in \(B(A, M)\), the set of all the bounded linear mappings from \(A\) into \(M\), using the principle of uniform boundedness, we have \(S_A\) is also bounded, hence \(A\) belongs to \(J\). We have established the claim.

Now we show that the restriction \(\delta|_J\) to \(J\) of \(\delta\) is bounded. For otherwise, choose a sequence \(A_1, A_2, \ldots\) in \(J\) such that for each \(n\), \(\|A_n\|^2 \leq \frac{1}{n^2}\) and \(\|\delta(A_n)\| \geq n\). Let \(B = (\sum_{n=1}^{\infty} A_n A_n^*)^\frac{1}{2}\). Then \(B\) is a positive element in \(J\) with \(\|B\| \leq 1\) and \(A_n A_n^* \leq B^2\) for each \(n\). By Lemma 4.1, for each \(n\), there exists \(C_n\) in \(J\) such that \(\|C_n\| \leq 1\) and \(A_n = BC_n\). Hence \(\|L_B(C_n)\| = \|\delta(A_n)\| \geq n\) for each \(n\), which contradicts the continuity of \(L_B\). This proves that \(\delta|_J\) is bounded.

Let \(\pi: A \to A/J\) be the canonical quotient mapping which is a surjective *-homomorphism. We claim that \(A/J\) is finite dimensional. On the contrary, using Lemma 4.2, we choose an infinite sequence \(\{\tilde{A}_1, \tilde{A}_2, \ldots\}\) of nonzero positive elements in \(A/J\) such that \(\tilde{A}_j A_k = 0\) when \(j \neq k\). By Lemma 4.3, there is an infinite sequence \(\{A_1, A_2, \ldots\}\) of nonzero positive elements in \(A\) such that \(A_j A_k = 0\) when \(j \neq k\). Since \(\tilde{A}_j = \pi(A_j)\), we have that \(A_j\) and hence, \(A_j^2\) is not in \(J\), which implies that \(L_{A_j^2}\) is unbounded. Consequently, we have constructed a sequence \(A_1, A_2, \ldots\) of positive elements in \(A\) such that \(A_j^2 \notin J\) and \(A_j A_k \neq 0\) when \(j \neq k\). Replacing \(A_j\) by an appropriate scalar multiple, we may assume also that \(\|A_j\| \leq 1\) for each \(j\). Since \(L_{A_j^2}\) is unbounded, there is \(T_j\) in \(A\) such that \(\|T_j\| \leq 2^{-j}\) and \(\|L_{A_j^2}(T_j)\| = \|\delta(A_j^2 T_j)\| \geq \|\gamma_{A_j}\| + j\). Let \(A = \sum_{j=1}^{\infty} A_j T_j\). Then \(A \in A\), \(\|A\| \leq 1\) and \(A_j A = A_j^2 T_j\). Hence, for each \(j = 1, 2, \ldots\),
\[
\|\sigma_A\| \geq \|\sigma_{A_j}(A_j)\| = \|\beta(A_j)\delta(A)\| = \|\delta(A_j A) - \delta(A_j)\alpha(A)\|
\geq \|\delta(A_j^2 T_j)\| - \|\gamma_{A_j}\| \geq j,
\]
which is impossible. Hence \(A/J\) is finite dimensional.

Since \(\delta|_J\) is norm continuous and \(J\) has finite codimension in \(A\), it follows that \(\delta\) is norm continuous.

\[\square\]

Remark. When \(A\) in above theorem is only a Banach algebra, we can also get the closed two-sided ideal \(J\) of \(A\). Using the closed graph theorem, we can show that if \(J\) has a bounded left approximate identity, then the restriction
δ|_J of δ to J is bounded. Firstly, we recall the Cohen’s factorization theorem ([11], Corollary 12 in Chapter 1), which tells us if B is a Banach algebra with a bounded left approximate identity, then for each sequence \( \{A_n\} \) in B with \( A_n \to 0 \), there exist \( A, B_n \in B \) with \( A_n = AB_n \), \( (n = 1, 2, \ldots) \) and \( B_n \to 0 \). Now we show the boundedness of \( \delta|_J \). Let \( A_n \in J \) \( (n = 1, 2, \ldots) \) with \( A_n \to 0 \) and \( \delta(A_n) \to J \). It follows from the Cohen’s factorization theorem that there exist \( A, B_n \in J \) with \( A_n = AB_n \), \( (n = 1, 2, \ldots) \) and \( B_n \to 0 \). Since \( \delta(A_n) = \delta(A)\alpha(B_n) + \beta(A)\delta(B_n) \) for each \( n \) and \( A \in J \), by Lemma 3.1(i), the boundedness of \( S_A \) and the continuity of \( \alpha \) at 0 yield \( J = 0 \). Hence \( \delta|_J \) is continuous.

Let \( S \) be a von Neumann algebra acting on a separable Hilbert space \( H \), let \( M \) be a dual normal \( S \)-module. If \( M_* \) is the predual of \( M \), we write \( \langle M, \omega \rangle \) in place of \( M(\omega) \) for each \( M \in M \) and \( \omega \in M_* \). Then \( M_* \) is a Banach \( S \)-module under the following module actions determined by

\[
\langle M, \omega A \rangle = \langle AM, \omega \rangle, \quad \langle M, A \omega \rangle = \langle MA, \omega \rangle
\]

for \( \omega \in M_* \), \( A \in S, M \in M \). In [13], using the properties of the Mackey topologies on \( M_* \) and \( S \), Ringrose proved that the mappings \( A \to \omega A, A \to A \omega \) are continuous from the unit ball of \( S \) (with strong\(^*\) topology) into \( M_* \) (with norm topology). Hence, for a \( C^* \)-subalgebra \( A \) of \( S \) and a pair of ultraweakly and strong\(^*\) continuous linear mappings \( \alpha, \beta \) from \( A \) into \( S \), the mappings \( A \to \alpha(A)\omega, A \to \omega \beta(A) \) are strong\(^*\)-norm continuous from the unit ball of \( A \) into \( M_* \). We have the following corollary.

**Corollary 4.5.** Let \( S \) be a von Neumann algebra acting on a separable Hilbert space \( H \), and let \( A \) be a unital \( C^* \)-subalgebra of \( S \), with the weak closure \( R \). Suppose that \( M \) is a dual normal \( S \)-module and \( \alpha, \beta \) are two ultraweakly and strong\(^*\) continuous linear mappings from \( A \) into \( S \). Then every \( (\alpha, \beta) \)-derivation \( \delta \) from \( A \) into \( M \) is ultraweakly-weak\(^*\) continuous, and extends to an ultraweakly-weak\(^*\) continuous \( (\overline{\alpha}, \overline{\beta}) \)-derivation of \( R \), where \( \overline{\alpha} \) and \( \overline{\beta} \) are the extension of \( \alpha \) and \( \beta \) to \( R \), respectively.

**Proof.** By Theorem 4.4, \( \delta \) is norm continuous. To establish the ultraweak-weak\(^*\) continuity of \( \delta \), it suffices to show that, for each \( \omega \) in \( M_* \), the linear functional \( \varphi : A \to C \), defined by \( \varphi(A) = \langle \delta(A), \omega \rangle \) for each \( A \in A \), is ultraweakly continuous (equivalently, show that \( \varphi \) is continuous on the unit ball of \( A \) under the weak operator topology). By Lemma 7.1.3 in [6], we only need to prove that the restriction of \( \varphi \) to \( A^+_1 \), the set of all positive elements in the unit ball of \( A \), is strongly continuous at 0.

Let \( \{T_i\} \) be a net converging strongly to 0 in \( A^+_1 \). Then \( \{T_i^{1/2}\} \) converges strongly, and hence under the strong\(^*\) topology, to 0. Since \( \alpha \) and \( \beta \) are ultraweakly and strong\(^*\) continuous, by the previous argument of Corollary 4.5,
both \( \{\|\alpha(T^{1/2}_i)\omega\|\} \) and \( \{\|\omega\beta(T^{1/2}_i)\|\} \) converge to 0. It follows that

\[
|\varphi(T_i)| = \|\delta(T^{1/2}_i T^{1/2}_j)\omega\| = \|\delta(T^{1/2}_i)\alpha(T^{1/2}_j) + \beta(T^{1/2}_i)\delta(T^{1/2}_j)\omega\| = \|\delta(T^{1/2}_i)\alpha(T^{1/2}_j)\omega + \omega\beta(T^{1/2}_i)\| \leq \|\delta\|\|\alpha(T^{1/2}_i)\omega\| + \|\omega\beta(T^{1/2}_i)\| \to 0.
\]

Hence we have proved that \( \delta \) is ultraweakly-weak\(^*\) continuous. Since by Kaplansky density theorem, the unit ball of \( \mathcal{A} \) is weakly dense in the unit ball of \( \mathcal{R} \), and the unit ball \( \mathcal{M} \) is weak\(^*\) compact, we have that \( \delta \) can extend without increase in norm to an ultraweak-weak\(^*\) continuous linear mapping, denoted by \( \tilde{\delta} \), from \( \mathcal{R} \) into \( \mathcal{M} \).

Now, we claim that \( \tilde{\delta} \) is an \((\pi, \bar{\delta})\)-derivation. For a given arbitrary element \( \omega \in \mathcal{M}_\pi \), define a bilinear form \( F_\omega : \mathcal{R} \times \mathcal{R} \to \mathbb{C} \) by \( F_\omega(A, B) = \langle \tilde{\delta}(AB) - \bar{\delta}(A)\pi(B) - \bar{\delta}(A)\bar{\delta}(B), \omega \rangle \) for each pair \( A, B \in \mathcal{R} \). Clearly, \( F_\omega(A, B) = 0 \) when \( A \) and \( B \) are in \( \mathcal{A} \). For self-adjoint operators \( A, B \in \mathcal{R} \), by Kaplansky density theorem, we choose self-adjoint element \( \{A_i\} \) and \( \{B_i\} \) in \( \mathcal{A} \) which converges strongly to \( A \) and \( B \), respectively and \( \|A_i\| \leq |A|, \|B_i\| \leq \|B\| \) for each \( i \). Also since the joint multiplication is strongly continuous on the bounded sets of self-adjoint elements, we have \( \{A_iB_i\} \) converges strongly to \( AB \), and hence \( F_\omega(A, B) = \lim F_\omega(A_i, B_i) = 0 \). Since \( \omega \) is arbitrary, we have \( \bar{\delta}(AB) - \tilde{\delta}(A)\pi(B) - \bar{\delta}(A)\bar{\delta}(B) = 0 \) for arbitrary self-adjoint operators, and hence for any elements, in \( \mathcal{R} \). Consequently, \( \delta \) is an \((\pi, \bar{\delta})\) derivation. \( \square \)

The following corollary is a direct result of Corollary 4.5.

**Corollary 4.6.** Let \( \mathcal{R} \) and \( \mathcal{S} \) be von Neumann algebras acting on a separable Hilbert space \( \mathfrak{h} \), \( \mathcal{R} \subseteq \mathcal{S} \) and let \( \mathcal{M} \) be a dual normal \( \mathcal{S}\)-module. For two given ultraweakly and strong\(^*\) continuous linear mappings \( \alpha, \beta : \mathcal{R} \to \mathcal{S} \), every \((\alpha, \beta)\)-derivation \( \delta : \mathcal{R} \to \mathcal{M} \) is ultraweakly-weak\(^*\) continuous.

**Corollary 4.7.** Let \( \mathcal{S} \) be a von Neumann algebra acting on a separable Hilbert space \( \mathfrak{h} \), \( \mathcal{A} \) be an ultraweakly closed unital subalgebra of \( \mathcal{S} \). Suppose that \( \mathcal{M} \) is a dual normal \( \mathcal{S}\)-module, \( \alpha, \beta : \mathcal{A} \to \mathcal{S} \) are ultraweakly and strong\(^*\) continuous linear mappings. Then for each \((\alpha, \beta)\)-derivation \( \delta : \mathcal{A} \to \mathcal{M} \), there is a central projection \( P \) in \( \mathcal{A} \cap \mathcal{A}^* \) such that \( (\mathcal{A} \cap \mathcal{A}^*)(1 - P) \) is finite dimensional and the mapping \( \delta : \mathcal{A} \to \delta(PA) \) from \( \mathcal{A} \) into \( \mathcal{M} \) is norm continuous.

**Proof.** Let \( \mathcal{R} = \mathcal{A} \cap \mathcal{A}^* \). Then \( \mathcal{R} \) is a von Neumann algebra. As in the proof in Theorem 4.4, set \( \mathcal{J} = \{A \in \mathcal{R} : L_A \) is bounded from \( \mathcal{A} \) into \( \mathcal{M} \} \). By the same argument, one can see that \( \mathcal{J} \) is a two-sided ideal of \( \mathcal{R} \). Now we show that \( \mathcal{J} \) is ultraweakly closed. Let \( \{A_i\} \) be a net of elements in \( \mathcal{J} \) converging ultraweakly to \( A \). Since \( \mathcal{J} \) is a two-sided ideal of a von Neumann algebra, it is selfadjoint, for let \( J \in \mathcal{J} \) and \( J = W|J| \) be its polar decomposition, we have \( W \in \mathcal{R} \) and \( J^* = |J|W^* = WJW^* \in \mathcal{J} \). Using Kaplansky density theorem, we assume that \( \|A_i\| \leq \|A\| \) for each \( i \). By Corollary 4.6, the restriction of \( \delta \)
\[ \delta \] to \( \mathcal{R} \) is bounded and ultraweakly-weak* continuous. Hence, for each \( T \in \mathcal{A} \), we have 
\[ L_A(T) = \delta(\mathcal{A}) \alpha(T) + \beta(\mathcal{A}) \delta(T) = \text{weak}^*-\lim, \delta(\mathcal{A}) \alpha(T) + \beta(\mathcal{A}) \delta(T) = \text{weak}^*-\lim, \delta(\mathcal{A}, T) = \text{weak}^*-\lim, L_A(T); \] and moreover, for each \( \iota \), we have 
\[ \| L_A(T) \| = \| \delta(\mathcal{A}) \alpha(T) + \beta(\mathcal{A}) \delta(T) \| 
\leq \| \delta|_{\mathcal{R}} \| \| \mathcal{A} \| \| \alpha \| \| T \| + \| \beta \| \| \mathcal{A} \| \| \delta(T) \| 
\leq \| \delta|_{\mathcal{R}} \| \| \mathcal{A} \| \| T \| + \| \beta \| \| \mathcal{A} \| \| \delta(T) \|. \]

Using the principle of uniform boundedness, we have \( \{ \| L_{A_n} \| \} \) is bounded. So 
\( L_A \), as the pointwise limit of the net \( \{ L_{A_n} \} \) of continuous mappings from \( \mathcal{A} \)
into \( \mathcal{M} \), is continuous, and thus \( A \in \mathcal{J} \). Hence \( \mathcal{J} \) is an ultraweakly-two-sided ideal of \( \mathcal{R} \), so there is a unique central projection \( P \) in \( \mathcal{R} \) such that \( \mathcal{J} = \mathcal{R}P \).

Now we claim that \( \mathcal{R}(I - P) \) is finite dimensional. For, otherwise, there is a sequence of nontrivial pairwise orthogonal projections \( \{ Q_n \} \) in \( \mathcal{R} \) with sum \( I - P \). Since for each \( n \), the mapping \( L_{Q_n} \) is unbounded, there exists \( A_n \) in \( \mathcal{A} \) such that \( \| A_n \| \leq 2^{-n} \) and \( \| \delta(Q_n A_n) \| > 2^n \). Let \( A = \sum_{n=1}^{\infty} Q_n A_n \). Then \( \| A \| \leq 1 \) and \( Q_n A = Q_n A_n \) for each \( n \). Consequently, \( 2^n \leq \| \delta(Q_n A_n) \| = \| \delta(Q_n A) \| \leq \| \delta|_{\mathcal{R}} \| \| \alpha(A) \| + \| \beta \| \| \delta(A) \| \) for each \( n \), which is impossible. Hence \( \mathcal{R}(I - P) \) is finite dimensional. \( \square \)

**Remark.** Applying Corollary 4.7 to \( \delta^*(A) = \delta(A^*)^* \) on \( \mathcal{A}^* \), we have that there is a central projection \( Q \) in \( \mathcal{A} \cap \mathcal{A}^* \) such that \( (\mathcal{A} \cap \mathcal{A}^*)(I - Q) \) is finite dimensional and the mapping \( A \rightarrow \delta(AQ) \) from \( \mathcal{A} \) into \( \mathcal{M} \) is norm continuous.

**Corollary 4.8.** Suppose that \( \mathcal{A} \) is a CSL algebra acting on a separable Hilbert space \( \mathcal{H} \), i.e., \( \mathcal{A} \) is a reflexive algebra whose lattice \( \text{Lat}(\mathcal{A}) \) of invariant projections is commutative. If \( \alpha, \beta : \mathcal{A} \rightarrow B(\mathcal{H}) \) are ultraweakly and strong* continuous linear mappings, then every \( (\alpha, \beta) \)-derivation from \( \mathcal{A} \) into \( B(\mathcal{H}) \) is bounded.

**Proof.** The proof is the same as that of Corollary 2.3 in [2], we describe it briefly. Let \( \mathcal{L} = \text{Lat}(\mathcal{A}) \) and \( \mathcal{R} = \mathcal{A} \cap \mathcal{A}^* \). Then \( \mathcal{R} = \mathcal{L}' \) with center \( \mathcal{L}'' \). By Corollary 4.7 and its remark, there are projections \( P \) and \( Q \) in \( \mathcal{L}'' \) such that \( \mathcal{R}P^\perp \) and \( \mathcal{R}Q^\perp \) are finite dimensional, and the mappings \( L_P : A \in \mathcal{A} \rightarrow \delta(PA) \in B(\mathcal{H}) \) and \( R_Q : A \in \mathcal{A} \rightarrow \delta(AQ) \in B(\mathcal{H}) \) are continuous. Let \( P^\perp = \sum_{i=1}^{k} P_i \) and \( Q^\perp = \sum_{j=1}^{l} Q_j \) be the sum of minimal projections in \( \mathcal{L}'' \), each of which is finite rank, for \( \mathcal{R}P^\perp = \sum_{i=1}^{k} \oplus B(P_i \mathcal{H}) \) and \( \mathcal{R}Q^\perp = \sum_{j=1}^{l} \oplus B(Q_j \mathcal{H}) \) are finite dimensional. Hence for each \( A \in \mathcal{A} \), we have \( \delta(A) = \delta(PA) + \delta(P^\perp AQ_j + \sum_{j=1}^{l} \delta(P_i AQ_j) \). Since \( P_i AQ_j \) is finite dimensional and \( L_P, R_Q \) are continuous, we have \( \delta \) is continuous. \( \square \)

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CONTINUITY OF \((\alpha, \beta)\)-DERIVATIONS OF OPERATOR ALGEBRAS

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