SOME EQUALITIES FOR CONTINUED FRACTIONS OF
GENERALIZED ROGERS-RAMANUJAN TYPE

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Abstract. In this paper, we first discuss the convergence of the continued fractions of generalized Rogers-Ramanujan type in the modified sense. Then we prove several equalities concerning these continued fractions. The proofs of our main results are mainly based on the Bauer-Muir transformation.

1. Preliminary material

If the sequence \( \{S_n(0)\} \) of the approximants of the continued fraction \( b_0 + K(a_n/b_n) \) converges to a point \( f \) in the extended complex plane \( \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \), then we call that the continued fraction \( b_0 + K(a_n/b_n) \) converges to \( f \) in the classical sense, and write

\[
b_0 + K(a_n/b_n) = f,
\]

where

\[
S_n(0) = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \ldots + \frac{a_n}{b_n}.
\]

Two continued fractions \( b_0 + K(a_n/b_n) \) and \( d_0 + K(c_n/d_n) \) are equivalent if they have the same sequence of classical approximants. The following result is from [6] or [8].

Proposition 1.1. Continued fractions \( b_0 + K(a_n/b_n) \) and \( d_0 + K(c_n/d_n) \) are equivalent if and only if there exists a sequence of non-zero constants \( \{r_n\} \) with \( r_0 = 1 \) such that \( c_n = r_n r_{n-1} a_n \) \( (n = 1, 2, \ldots) \) and \( d_n = r_n b_n \) \( (n = 0, 1, \ldots) \).

As in [5] or [8], we introduce the following definition.

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Theorem A. If \( |q| < 1 \), then the continued fraction \( q + K(1/q^{2n+1}) \), which is divergent in the classical sense, is convergent in the modified sense.

In [7], Lee and Sohn continued to study this problem and obtained:

Theorem B (Modified Rogers-Ramanujan continued fraction). If \( |q| < 1 \), then the continued fraction \( q + K(1/q^{2n+1}) \) converges in the modified sense,
and
\[(1.2) \quad q + \frac{1}{q^3 + q^5 + q^7 + \ldots} = 1 + \frac{q^2}{1 + \frac{q^2}{1 + \frac{q^2}{1 + \ldots}}} \quad (m.c.) \]  

For a polynomial \(p(x)\) in \(\mathbb{C}[x]\), we introduce the following notation.
\[E(p) = \{x \in \mathbb{C} : p(x) = 0\} \cup \{0\} \]

The first aim of this paper is to discuss Theorem B further and get the following generalization.

**Theorem 1.5.** For any non-zero polynomial \(a_0(x)\) in \(\mathbb{C}[x]\), and for any fixed \(x \in \mathbb{C} - E(a_0)\), if \(|x| < 1\), then the continued fraction
\[k_1a_0 + \frac{k_2^2}{k_1a_1 + k_1a_2} + \frac{k_2^2}{k_1a_3} + \ldots,\]
which is divergent in the classical sense, converges in the modified sense, and
\[(1.3) \quad = k_2 + \frac{k_1a_0x}{k_2} + \frac{k_1a_0x^d}{k_2} + \frac{k_1a_0x^{2d}}{k_2} + \frac{k_1a_0x^{3d}}{k_2} + \ldots (m.c.),\]
where \(k_1\) and \(k_2\) are two non-zero constants, \(a_n = a_0x^n\) for \(n \geq 1\) and \(d\) is a positive constant.

**Remark 1.6.** We can get (1.2) by putting \(k_1 = k_2 = 1, a_0 = q\) and \(d = 2\) in (1.3).

In [4], Berndt and Yee studied the continued fractions of generalized Rogers-Ramanujan type and got the following equality.

**Theorem C.** For \(|q| < 1,\)
\[1 - \frac{q}{1 + 1} = 1 + \frac{q^2}{1 + \frac{q^2}{1 + \frac{q^2}{1 + \ldots}}} = 1 + \frac{q^3}{1 + 1 + \frac{q^3}{1 + 1 + \ldots}}.\]

As the second aim of this paper, we prove the following two theorems. The methods used in the proofs of these two theorems follow from [4].

**Theorem 1.7.** For \(|q| < 1,\)
\[(1.4) \quad 1 - \frac{q}{1 + 1} = 1 + \frac{q^2}{1 + \frac{q^2}{1 + \frac{q^2}{1 + \ldots}}} = 1 + \frac{q^5}{1 + 1 + 1 + 1 + \ldots}.\]
Theorem 1.8. For $|q| < 1$,

\[
1 + \frac{q}{1 - \frac{q}{1 + 1 - 1 + 1 - 1 + \cdots}} = \frac{1}{1 - 1 + 1 + 1 + 1 + \cdots}.
\]

As an application of Theorems 1.7 and 1.8 and Corollaries 3.1 and 3.4, we can easily get the following four equalities.

Corollary 1.9.

\[
(1 + q + q^2 + q^3 + q^4) + (1 + q + q^2 + q^3 + q^4) = 2,
\]
\[
(1 + q + q^2 + q^3 + q^4) + (1 + q + q^2 + q^3 + q^4) = 2,
\]
\[
(1 + q + q^2 + q^3 + q^4) + (1 + q + q^2 + q^3 + q^4) = 2,
\]
\[
(1 + q + q^2 + q^3 + q^4) + (1 + q + q^2 + q^3 + q^4) = 2.
\]

On page 46 of Ramanujan’s lost notebook [9], there is a theorem which is stated as follows (see also [2]) and was proved in [3].

Theorem D. Let $k \geq 0$, $\alpha = (1 + \sqrt{1 + 4k})/2$ and $\beta = (-1 + \sqrt{1 + 4k})/2$. Then, for $|q| < 1$ and $\text{Re}(q) > 0$,

\[
1 + \frac{k + q}{1 + \frac{k + q^2}{1 + \cdots}} = \alpha + \frac{q}{\alpha + \beta q + \alpha + \beta q^2 + \cdots}.
\]

In [7], Lee and Sohn obtained the following generalization of (1.10).

Theorem E. For a fixed natural number $r$, suppose that $d \geq 1$ and $m_r \geq m_{r-1} \geq \cdots \geq m_1 \geq 1$. Then we have

\[
k_1 + \frac{k_2 + a_1}{k_1 + \frac{k_2 + a_2}{k_1 + \cdots}} = \alpha + \frac{a_1}{\alpha + \beta q + \alpha + \beta q^2 + \cdots},
\]

where

\[a_n = q^{m_1+(n-1)d} + q^{m_2+(n-1)d} + \cdots + q^{m_r+(n-1)d}\]

and

\[\beta = \frac{-k_1 + \sqrt{k_1^2 + 4k_2}}{2}, \quad \alpha = k_1 + \beta.\]
The last aim of this paper is to discuss Theorems D and E further. Our result is as follows.

**Theorem 1.10.** Let \( a_1(x) \) be a non-zero polynomial in \( \mathbb{C}[x] \). Then for any fixed \( x \) in \( \mathbb{C} - E(a_1) \), we have the following equality.

\[
(1.12) \quad k_1 + \frac{k_2 + a_1}{k_1} + \frac{k_2 + a_2}{k_1} + \cdots = \alpha + \frac{a_1}{\alpha + \beta x^d} + \frac{a_2}{\alpha + \beta x^{2d} + \cdots},
\]

where \( a_n = a_1 x^{(n-1)d} \) for \( n \geq 2 \), \( \beta = (-k_1 + \sqrt{k_1^2 + 4k_2})/2 \), \( \alpha = k_1 + \beta \) and \( d \) is a positive constant.

**Remark 1.11.** We can get (1.11) by putting \( a_1(q) = q^{m_1} + q^{m_2} + \cdots + q^{m_r} \) in (1.12), and (1.10) by putting \( k_1 = 1 \), \( k_2 = k \), \( a_1(q) = q \) and \( d = 1 \).

2. The proofs of Theorem 1.5 and its corollaries

**Proof of Theorem 1.5.** We shall prove (1.3) from the left side to the right side by using the Bauer-Muir transformation. For convenience, we let

\[
R(x) = k_1 a_0 + \frac{k_2^2}{k_1 a_1 + k_1 a_2 + k_1 a_3 + \cdots}.
\]

We choose the modifying factors for \( R(x) \) as follows:

\[
\omega_i^{(0)} = k_2 - k_1 a_i = k_2 - k_1 a_0 x^d, \quad i = 0, 1, 2, \ldots.
\]

Obviously, \( \lambda_i^{(0)} = k_1 k_2 a_0 x^{(i-1)d} \neq 0 \) \((i = 1, 2, 3, \ldots)\). Then the Bauer-Muir transformation implies that

\[
R(x) = k_2 + \frac{k_1 k_2 a_0}{k_2} - \frac{k_2 x^d}{k_2 - k_2 x^d + k_1 a_0 x^d} + \frac{k_2 x^d}{k_2 - k_2 x^d + k_1 a_0 x^d} + \cdots = k_2 + \frac{k_1 k_2 a_0}{k_2} + R_1(x),
\]

where

\[
R_1(x) = k_2 - k_2 x^d + k_1 a_0 x^d + \frac{k_2 x^d}{k_2 - k_2 x^d + k_1 a_0 x^d} + \frac{k_2 x^d}{k_2 - k_2 x^d + k_1 a_0 x^d} + \cdots.
\]

Now we choose the modifying factors for \( R_1(x) \) as follows:

\[
\omega_i^{(1)} = k_2 x^d - k_1 a_0 x^{(i+1)d}, \quad i = 0, 1, 2, \ldots.
\]

We easily know that \( \lambda_i^{(1)} = k_1 k_2 a_0 x^{i d} \neq 0 \) \((i = 1, 2, 3, \ldots)\). By using the Bauer-Muir transformation once more, we have that

\[
R_1(x) = k_2 + \frac{k_1 k_2 a_0 x^d}{k_2} + \frac{k_2 x^d}{k_2 - k_2 x^d + k_1 a_0 x^d} + \frac{k_2 x^d}{k_2 - k_2 x^d + k_1 a_0 x^d} + \cdots = k_2 + \frac{k_1 k_2 a_0 x^d}{k_2} + R_2(x),
\]

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where
\[
R_2(x) = k_2 - k_2x^{2d} + k_1a_0x^{2d} + \frac{k_2^2x^{2d}}{k_2 - k_2x^{2d} + k_1a_0x^{2d}} + \frac{k_2^2x^{2d}}{k_2 - k_2x^{2d} + k_1a_0x^{2d}} + \ldots
\]
For \( j = 2, 3, 4, \ldots \), we let
\[
R_j(x) = k_2 - k_2x^{jd} + k_1a_0x^{jd} + \frac{k_2^2x^{jd}}{k_2 - k_2x^{jd} + k_1a_0x^{jd} + k_2 - k_2x^{jd} + k_1a_0x^{jd}} + \ldots
\]
By applying the Bauer-Muir transformation to \( R_j(x) \) \((j = 2, 3, 4, \ldots)\) and repeating the procedures as above, we find that, if we take \( \omega_{i}^{(j)} = k_2x^{jd} - k_1a_0x^{(j+i)d} \) \((i = 0, 1, 2, \ldots)\), then \( \lambda^{(j)} = k_1k_2a_0x^{(i-1+j)d} \neq 0 \) \((i = 1, 2, 3, \ldots)\). Similar discussions as above show that
\[
R(x) = k_2 + \frac{k_1k_2a_0}{k_2} + \frac{k_2^2x^{jd}}{k_2} - k_2x^{jd} + k_1a_0x^{jd} + \frac{k_2^2x^{jd}}{k_2 - k_2x^{jd} + k_1a_0x^{jd}} + \ldots
\]
By letting \( j \to \infty \) in (2.1), we get (1.3).

The left side of (1.3) and the continued fraction \( b_n^0 + K(a_n^0/b_n^0) \) are equivalent, where \( a_n^0 = r_nr_{n-1}k_2^2 \) for \( n > 0 \), \( b_n^0 = r_nk_1a_n \) for \( n \geq 0 \), and \( r_0 = 1, r_n = 1/k_2 \) for \( n \geq 1 \). But the continued fraction \( b_n^0 + K(a_n^0/b_n^0) \) diverges by Stern-Stolz’s theorem, cf. [6] and [8]. Hence the left side of (1.3) diverges in the classical sense.

Since \( x^n \to 0 \), we see that the right side of (1.3) converges by Proposition 1.1 and Worpitzky’s theorem (cf. Theorem I.3 in [8]). Hence the left side of (1.3) converges in the modified sense.

It follows from Theorem 1.5 that the following results are obvious.

\[ \square \]
Corollary 2.1. For \(|q| < 1\), the continued fraction \(k_1q + K(k^2_2/k^2_1q^{2n+1})\) converges in the modified sense, and

\[(2.2) \quad k_1q + \frac{k^2_2}{k^3_1q^2 + k^2_1q^3 + \ldots} = k_2 + \frac{k_1k_2q}{k_2} + \frac{k^2_2q^2}{k_2} + \frac{k_1k_2q^3}{k_2} + \frac{k^2_2q^4}{k_2} + \ldots \quad (m.c.),\]

where \(k_1\) and \(k_2\) are two non-zero constants if \(q \neq 0\).

**Proof.** We can get (2.2) from Theorem 1.5 by putting \(a_0 = q\) and \(d = 2\). \(\square\)

Remark 2.2. We can get (1.2) from (2.2) by putting \(k_1 = 1\) and \(k_2 = 1\).

Corollary 2.3. For \(|q| < 1\), the continued fraction \(kq + K(1/kq^{2n+1})\) converges in the modified sense, and

\[(2.3) \quad kq + \frac{1}{kq^2 + k^2q^3 + k^3q^4 + \ldots} = 1 + \frac{kq}{1} + \frac{q^2}{1} + \frac{k^2q^3}{1} + \frac{q^4}{1} + \ldots \quad (m.c.),\]

where \(k\) is a non-zero constant.

**Proof.** We can get (2.3) from Theorem 1.5 by putting \(a_0 = q\), \(d = 2\), \(k_1 = k\) and \(k_2 = 1\). \(\square\)

Corollary 2.4. For \(|q| < 1\), the continued fraction \(q + K(k^2/q^{2n+1})\) converges in the modified sense, and

\[(2.4) \quad q + \frac{k^2}{q^2 + q^3 + q^4 + \ldots} = k + \frac{kq}{k} + \frac{k^2q^2}{k} + \frac{kq^3}{k} + \frac{k^2q^4}{k} + \ldots \quad (m.c.),\]

where \(k\) is a non-zero constant.

**Proof.** We can get (2.4) from Theorem 1.5 by putting \(a_0 = q\), \(d = 2\), \(k_1 = 1\) and \(k_2 = k\). \(\square\)

### 3. The proofs of Theorems 1.7, 1.8 and their corollaries

**Proof of Theorem 1.7.** For convenience, we denote the left side of (1.4) by \(r(q)\). That is

\[r(q) = 1 - \frac{q}{1} - \frac{q^2}{1} - \frac{q^3}{1} - \ldots.\]

Set

\[(3.1) \quad f(q, a) = 1 - \frac{q}{1 + a}.\]

Then

\[(3.2) \quad f(q, a) = \frac{a + q - aq}{a + q} \quad \text{and} \quad 1 - f(q, a) = \frac{aq}{a + q}.\]

Hence

\[(3.3) \quad \frac{1 - f(q, a)}{f(q, a)} = \frac{aq}{a + q - aq} = \frac{q}{1 + q(1-a)} = \frac{q(1-a)}{a}.\]
Let \( F(q, A) = r(q) \) with
\[
F(q, A) = 1 - \frac{q}{1 + A}.
\]

Then (3.1), (3.2) and (3.3) imply that
\[
F(q, A) = \frac{A + q - Aq}{A + q} = \frac{1}{1 + \frac{q(1 - A)}{A}}.
\]

By replacing \( A \) on the right side of (3.4) by \( f(q^2, a) \) and using (3.3), we get that
\[
F(q, A) = \frac{1}{1 + \frac{q^3}{1} + \frac{q^2(1 - a)}{a}}.
\]

We replace \( a \) in (3.5) by \( f(q^3, a), \ldots \). By repeating this procedure we see that the 2\( n \)th approximant of the left hand side of (1) is equal to the \( n \)th approximant of the right hand side. Since \( q^n \to 0 \), we know that two sides of (1.4) converge by Worpitzky’s theorem (cf. Theorem I.3 in [8]). Hence the limit of the sequence of the 2\( n \)th approximants of the left hand side of (1) is equal to the one of the sequence of the \( n \)th approximants of the left hand side. Therefore (1.4) is true.

The following equality follows from the similar arguments as in the proof of Theorem 1.7.

**Corollary 3.1.** For \( |q| < 1 \),
\[
1 - \frac{aq}{1 + \frac{aq}{1 + \frac{aq}{1 + \frac{aq}{1 + \frac{aq}{1 + \cdots}}}}} = \frac{1}{1 + \frac{a^2/2}{1 + \frac{a^2/4}{1 + \frac{a^2/8}{1 + \frac{a^2/16}{1 + \cdots}}}}}
\]
where \( k = 1, 2, 3, \ldots \) and \( a \) is a constant.

By Corollary 3.1, we can easily get the following two equalities.

**Corollary 3.2.** For \( |q| < 1 \),
\[
1 - \frac{q}{1 + \frac{q}{1 + \frac{q}{1 + \frac{q}{1 + \cdots}}}} = \frac{1}{1 + \frac{q^2}{1 + \frac{q^4}{1 + \frac{q^8}{1 + \frac{q^{12}}{1 + \cdots}}}}}
\]

**Corollary 3.3.** For \( |q| < 1 \),
\[
1 - \frac{q}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^4}{1 + \cdots}}}} = \frac{1}{1 + \frac{q^2}{1 + \frac{q^2}{1 + \frac{q^2}{1 + \cdots}}}}
\]
where \( k = 1, 2, 3, \ldots \).
Proof of Theorem 1.8. For convenience, we denote the left side of the equality (1.5) by $B(q)$. That is

$$B(q) = 1 + \frac{q}{1 - \frac{q^2}{1 + \frac{q^3}{1 - \frac{q^3}{1 + \cdots}}}}.$$ 

Set

$$g(q, a) = 1 + \frac{q}{1 - \frac{a}{a - q}}.$$ 

Then

$$g(q, a) = \frac{a - q + aq}{a - q}$$ and $g(q, a) - 1 = \frac{aq}{a - q}$. 

Hence

$$g(q, a) - 1 = \frac{aq}{a - q + aq} = \frac{q}{\frac{a}{a - q} + a} = \frac{q}{1 + \frac{q(a - 1)}{a}}.$$ 

Let $G(q, A) = B(q)$ with

$$G(q, A) = 1 + \frac{q}{1 - A}.$$ 

Then (3.6), (3.7) and (3.8) imply that

$$G(q, A) = \frac{A - q + Aq}{A - q} = \frac{1}{1 - \frac{q(A - 1)}{A}}.$$ 

By replacing $A$ on the right side of (3.9) by $g(q^2, a)$ and using (3.8), we get that

$$G(q, A) = 1 + \frac{q}{1 + \frac{q^3}{1 + \frac{q^2(a - 1)}{a}}}.$$ 

We replace $a$ in (3.10) by $f(q^3, a), \ldots$. By repeating the procedure as above we see that the $2n$th approximant of the left hand side of (1.5) is equal to the $n$th approximant of the right hand side. Since $q^n \to 0$, we see that two sides of (1.5) converge by Worpitzky’s theorem (cf. Theorem I.3 in [8]). Hence the limit of the sequence of the $2n$th approximants of the left hand side of (1.5) is equal to the one of the sequence of the $n$th approximants of the right hand side. Therefore (1.5) is true. 

It follows from similar arguments in Theorem 1.8 that we can get the following.

**Corollary 3.4.** For $|q| < 1$,

$$1 + \frac{aq}{1 - \frac{aq}{1 + \frac{aq^{k+1}}{1 - \frac{aq^{2k+1}}{1 + \frac{aq^{3k+1}}{1 + \cdots}}}}} = \frac{1}{1 - \frac{aq}{1 + \frac{aq^{k+2}}{1 + \frac{aq^{3k+2}}{1 + \frac{aq^{5k+2}}{1 + \cdots}}}},$$

where $k = 1, 2, 3, \ldots$ and $a$ is a constant.
By Corollary 3.4, we can easily get the following two equalities.

**Corollary 3.5.** For $|q| < 1$,

\[
1 + \frac{q}{1} - \frac{q^3}{1} + \frac{q^5}{1} - \frac{q^7}{1} + \cdots = \frac{1 - q}{1 - q} \frac{q^4}{1} - \frac{q^8}{1} + \frac{q^{12}}{1} + \cdots.
\]

**Corollary 3.6.** For $|q| < 1$,

\[
1 + \frac{q}{1} - \frac{q^{k+1}}{1} + \frac{q^{2k+1}}{1} - \frac{q^{2k+1}}{1} + \cdots = \frac{1 - q}{1 - q} \frac{q^{k+2}}{1} + \frac{q^{3k+2}}{1} - \frac{q^{3k+2}}{1} + \cdots,
\]

where $k = 1, 2, 3, \ldots$.

The proofs of Corollary 1.9. We need only to prove the equality (1.6). The proofs of the equalities (1.7), (1.8) and (1.9) follow from similar reasoning. By Theorem 1.7, we have

\[
(3.11) \quad \frac{q}{1} - \frac{q^3}{1} + \frac{q^5}{1} - \frac{q^7}{1} + \cdots = -1 + \frac{1}{1 - 1} - \frac{1}{1 - 1} - \frac{1}{1 - 1} + \frac{1}{1 - 1} + \cdots.
\]

By Theorem 1.8, we have

\[
(3.12) \quad \frac{q}{1} - \frac{q^3}{1} + \frac{q^5}{1} - \frac{q^7}{1} + \cdots = 1 - \frac{1}{1 - 1} - \frac{1}{1 - 1} - \frac{1}{1 - 1} + \frac{1}{1 - 1} + \cdots.
\]

The equality (1.6) follows from (3.11) and (3.12). \(\square\)

### 4. The proof of Theorem 1.10

**Proof of Theorem 1.10.** We shall prove the equality (1.12) from the left side to the right side by using the Bauer-Muir transformation. We denote the left side of (1.12) by $B(x)$. That is

\[
B(x) = k_1 + \frac{k_2 + a_1}{k_1} \frac{k_2 + a_2}{k_1} + \cdots.
\]

We choose the modifying factors for $B(x)$ as follows:

\[
\omega_i^{(0)} = \beta, \quad i = 0, 1, 2, \ldots.
\]

Since

\[
\lambda_i^{(0)} = a_1 x^{(i-1)d} = a_i \neq 0, \quad i = 1, 2, 3, \ldots,
\]

it follows from the Bauer-Muir transformation that

\[
B(x) = k_1 + \frac{k_2 + a_1}{k_1} \frac{k_2 + a_2}{k_1} + \cdots
\]

\[
= k_1 + \beta + \frac{a_1}{k_1 + \beta} \frac{a_1}{k_1 + \beta} \cdots
\]

\[
= \beta + \frac{a_1}{B_1(x)},
\]
where
\[ B_1(x) = \alpha + \frac{k_2x^d + a_2}{\alpha - \beta x^d} + \frac{k_3x^d + a_3}{\alpha - \beta x^d} + \ldots. \]

We choose the modifying factors for \( B_1(x) \) as follows:
\[ \omega_i^{(1)} = \beta x^d, \quad i = 0, 1, 2, \ldots. \]

Then \( \lambda_i^{(1)} = a_{i+1} \neq 0 \) (\( i = 1, 2, 3, \ldots \)). The Bauer-Muir transformation yields that
\[
B_1(x) = \alpha + \beta x^d + \frac{a_2}{\alpha - \beta x^d + \beta x^d} \left( \frac{k_2x^d + a_2}{\alpha - \beta x^{2d}} + \frac{(k_2x^d + a_3)x^d}{\alpha - \beta x^{2d}} + \ldots \right)
\]
\[ = \alpha + \beta x^d + \frac{a_2}{B_2(x)}, \]
where
\[ B_2(x) = \alpha + \frac{k_2x^{2d} + a_3}{\alpha - \beta x^{2d}} + \frac{k_2x^{2d} + a_4}{\alpha - \beta x^{2d}} + \ldots. \]

By taking \( \omega_i^{(k)} = \beta x^{kd} (i \geq 0) \) we see that \( \lambda_i^{(k)} = a_{i+k} \neq 0 \) (\( i \geq 1 \)) for any \( k \geq 0 \). The equality (1.12) follows. \( \square \)

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