φ-FRAMES AND φ-RIESZ BASES ON LOCALLY COMPACT ABELIAN GROUPS

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Abstract. We introduce φ-frames in $L^2(G)$, as a generalization of $a$-frames defined in [8], where $G$ is a locally compact Abelian group and $φ$ is a topological automorphism on $G$. We give a characterization of φ-frames with regard to usual frames in $L^2(G)$ and show that φ-frames share several useful properties with frames. We define the associated φ-analysis and φ-preframe operators, with which we obtain criteria for a sequence to be a φ-frame or a φ-Bessel sequence. We also define φ-Riesz bases in $L^2(G)$ and establish equivalent conditions for a sequence in $L^2(G)$ to be a φ-Riesz basis.

1. Introduction and preliminaries

The theory of frames was introduced by Duffin and Schaeffer [10] in the early 1950s to deal with problems in nonharmonic Fourier series. There has been renewed interest in the subject related to its role in wavelet theory and a lot of new applications. Several kinds of frames have been introduced up to now; e.g. frames in Hilbert $C^*$-modules (modular frames) [14], frames of subspaces [7], $G$-frames [26], $p$-frames [1], frames for Banach spaces [6], $a$-frames [8], and many others for different purposes. In this paper we define and investigate φ-frames in $L^2(G)$, using the φ-bracket product, as a vector valued inner product on $L^2(G)$ introduced in [19], where $G$ is a locally compact Abelian (which will be abbreviated to “LCA”) group and $φ$ is a topological automorphism on $G$. One of the nice things about φ-frames is the fact that they are useful in studying Gabor systems in the way that there is a close relationship between these frames and Gabor frames in $L^2(G)$. Indeed, our results relate Gabor frames in $L^2(G)$, which have become a paradigm for the spectral analysis associated with time frequency methods [6], to φ-frames. Our construction is related to an extension of Casazza and Lammers’ definition of $a$-frames, $a > 0$, on $L^2(\mathbb{R})$ in [8], to the more general setting of $L^2(G)$, in a new and different approach. We characterize φ-frames in terms of the usual frames

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in \( L^2(G) \) (Theorem 2.1 below), which reveals the above mentioned relation, and we show that \( \varphi \)-frames have several useful properties in common with frames. We also define \( \varphi \)-Riesz bases in \( L^2(G) \) and establish equivalent conditions for a sequence to be a \( \varphi \)-Riesz basis, through which we establish a relation between \( \varphi \)-Riesz bases and usual Riesz bases in \( L^2(G) \).

Let \( G \) be a LCA group and \( \hat{G} \) denote the dual group of \( G \). We refer the reader to the usual text books about locally compact groups \([12, 16]\). Let the Fourier transform \( \hat{\cdot} : L^1(G) \rightarrow C_0(\hat{G}) \), \( f \mapsto \hat{f} \), be defined by \( \hat{f}(\xi) = \int_G f(x) \overline{\xi(x)} \, dx \). The Fourier transform can be extended to a unitary isomorphism from \( L^2(G) \) to \( L^2(\hat{G}) \) known as the Plancherel transform \([12, \text{ The Plancherel Theorem}]\). Let \( \varphi \) be a topological automorphism on \( G \). Let \( L \) be a uniform lattice in \( G \), that is, a discrete subgroup of \( G \) with compact quotient group \( G/L \). Then obviously \( \varphi(L) \) is also a uniform lattice in \( G \). Denote by \( \varphi(L) = \{ \gamma \in \hat{G}; \gamma(\varphi(L)) = \{1\} \} \), which is a uniform lattice in \( \hat{G} \) (see \([18, 21]\)). For a uniform lattice \( L \) in \( G \), a fundamental domain is a measurable set \( S_L \) in \( \hat{G} \) such that every \( x \in G \) can be uniquely written in the form \( x = ks, \) where \( k \in L \) and \( s \in S_L \). The existence of a fundamental domain for a uniform lattice in an LCA group is guaranteed by \([22, \text{ Lemma 2}]\).

Choosing the counting measure on \( L \), a relation between the Haar measures \( dx \) on \( G \) and \( d \hat{x} \) on \( G/\varphi(L) \) is given by the following special case of Weil’s formula \([12]\): For \( f \in L^1(G) \), we have \( \sum_{k \in \varphi(L)} f(x \varphi(k^{-1})) \in L^1(G/\varphi(L)) \) and
\[
\int_G f(x) \, dx = \int_{G/\varphi(L)} \sum_{k \in \varphi(L)} f(x \varphi(k^{-1})) \, d \hat{x},
\]
where \( \hat{x} = x \varphi(L) \).

Let \( f, g \in L^2(G) \). The \( \varphi \)-bracket product of \( f, g \) is defined by
\[
[f, g]_{\varphi}(\hat{x}) = \sum_{k \in \varphi(L)} f(x \varphi(k^{-1})) \overline{g(x \varphi(k^{-1}))}
\]
for all \( x \in G \). We define the \( \varphi \)-norm of \( f \) as \( \|f\|_{\varphi} = ([f, f]_{\varphi}(\hat{x}))^{1/2} \). The \( \varphi \)-bracket product is in fact a vector valued inner product on \( L^2(G) \) (see \([19, \text{ Proposition 2.4}]\)). In particular, Cauchy Schwartz Inequality holds for it, i.e.,
\[
\|f, g\|_{\varphi} \leq \|f\|_{\varphi} \|g\|_{\varphi}
\]
for \( f, g \in L^2(G) \).

A sequence \( (g_n)_{n \in \mathbb{N}} \subseteq L^2(G) \) is called \( \varphi \)-orthonormal if \( [g_n, g_m]_{\varphi} = 0 \) for all \( n \neq m \in \mathbb{N} \) and \( \|g_n\|_{\varphi} = 1 \) for all \( n \in \mathbb{N} \). A \( \varphi \)-orthonormal sequence \( (g_n)_{n \in \mathbb{N}} \) is called a \( \varphi \)-orthonormal basis if \( [f, g_n]_{\varphi} = 0 \) a.e. for all \( n \in \mathbb{N} \), implies \( f = 0 \) a.e..

\([19, \text{ Proposition 14}]\) asserts that \( L^2(G) \) admits a \( \varphi \)-orthonormal basis.

One of the main tools in our studies is \( \varphi \)-factorable operators. For the sake of completeness, we recall some of our results on \( \varphi \)-factorable operators on
$L^2(G)$. For a detailed exposition of the $\varphi$-bracket product and $\varphi$-factorable operators confer [19, 20].

For $\gamma \in G$, denote by $M_\gamma$ the modulation operator on $L^2(G)$, i.e.,

$$M_\gamma f(x) = \gamma(x)f(x)$$

for all $f \in L^2(G)$. Let $U$ be a bounded operator from $L^2(G)$ to $L^2(E)$, where $E$ is a subgroup of $G$ or $G/\varphi(L)$. $U$ is called $\varphi$-factorable if

$$(1.4) \quad U(M_\gamma g) = M_\gamma U(g) \quad \text{for all } g \in L^2(G), \ \gamma \in \varphi(L)^{-1}.$$ 

It is easily verified that if $U : L^2(G) \to L^2(G)$ is a bounded $\varphi$-factorable operator, then its adjoint $U^*$ is also $\varphi$-factorable. Moreover,

$$(1.5) \quad [U(f), g]_\varphi = [f, U^*(g)]_\varphi, \quad \text{a.e. for all } f, g \in L^2(G).$$

We have the following Riesz Representation Theorem ([20, Theorem 2.4]), which characterizes all $\varphi$-factorable operators from $L^2(G)$ to $L^1(G/\varphi(L))$.

**Theorem 1.1.** A bounded operator $U : L^2(G) \to L^1(G/\varphi(L))$ is $\varphi$-factorable if and only if there exists $g \in L^2(G)$ such that $U(f) = [f, g]_\varphi$ a.e. for all $f \in L^2(G)$. Moreover $\|U\| = \|g\|_1$.

Let us now define a $\varphi$-frame and a $\varphi$-Bessel sequence.

**Definition 1.2.** A sequence $(f_n)_{n \in \mathbb{N}}$ in $L^2(G)$ is said to be a $\varphi$-frame if there exist $0 < A, B < \infty$, such that for every $f \in L^2(G)$,

$$(1.6) \quad A\|f\|_F^2(\hat{x}) \leq \sum_{n \in \mathbb{N}} |[f, f_n]_\varphi(\hat{x})|^2 \leq B\|f\|_F^2(\hat{x})$$

for a.e. $\hat{x} \in G/\varphi(L)$. $A, B$ are called $\varphi$-frame bounds. Those sequences which satisfy only the upper inequality in (1.6), are called $\varphi$-Bessel sequences. In this case $B$ is called $\varphi$-Bessel bound.

The rest of this paper is organized as follows: In Section 2 we investigate $\varphi$-frames and $\varphi$-Bessel sequences in $L^2(G)$, where $G$ is a second countable LCA group and $\varphi$ is a topological isomorphism on $G$. We characterize $\varphi$-frames in terms of frames in $L^2(G)$ (Theorem 2.1). We also define $\varphi$-pre-frame and $\varphi$-analysis operators. Then we study $\varphi$-frames and $\varphi$-Bessel sequences in terms of these operators. In Section 3 we introduce $\varphi$-Riesz bases and give equivalent conditions for a sequence in $L^2(G)$ to be a $\varphi$-Riesz basis (Theorem 3.4).

2. $\varphi$-Frames in $L^2(G)$

Throughout this paper we always assume that $G$ is a second countable LCA group, $L$ is a uniform lattice in $G$ and $\varphi$ is a topological isomorphism on $G$.

In this section we investigate $\varphi$-frames and characterize them with regard to standard frames in $L^2(G)$. We then define the associated $\varphi$-analysis and $\varphi$-preframe operators, with which we obtain criteria for a sequence to be a $\varphi$-frame or a $\varphi$-Bessel sequence.

Here is the characterization of $\varphi$-frames in terms of frames in $L^2(G)$.
Theorem 2.1. Let \((f_n)_{n \in \mathbb{N}}\) be a sequence in \(L^2(G)\). Then the following are equivalent.

1. \((f_n)_{n \in \mathbb{N}}\) is a \(\varphi\)-frame.
2. \((M, f_n)_{n \in \mathbb{N}, \gamma \in \varphi(L^1)}\) is a frame.

Proof. Let \((f_n)_{n \in \mathbb{N}}\) be a \(\varphi\)-frame with bounds \(A, B\) and \((g_n)_{n \in \mathbb{N}}\) be a \(\varphi\)-orthonormal basis for \(L^2(G)\). Define \(U : L^2(G) \to L^2(G)\) by \(Ug_n = M_n g_n\) for \(\gamma \in \varphi(L^1), n \in \mathbb{N}\). Note that \(M_n g_n\)'s form an orthonormal basis for \(L^2(G)\), which guarantees that \(U\) is well defined. Then \(U\) is \(\varphi\)-factorable and so we have

\[
[U^* f, g_n]_\varphi = [f, U(g_n)]_\varphi = [f, f_n]_\varphi,
\]

a.e. Since \((g_n)_{n \in \mathbb{N}}\) is a \(\varphi\)-orthonormal basis

\[
\|U^* f\|_\varphi^2(\hat{x}) = \sum_{n \in \mathbb{N}} |[U^* f, g_n]_\varphi(\hat{x})|^2 
= \sum_{n \in \mathbb{N}} |[f, f_n]_\varphi(\hat{x})|^2 
\leq B \|f\|_\varphi^2(\hat{x})
\]

for \(f \in L^2(G)\) and a.e. \(\hat{x} \in G/\varphi(L)\). Integrating (2.2) over \(G/\varphi(L)\) and using Weil's formula, we have \(\|U^* f\|_\varphi^2 \leq B \|f\|_\varphi^2\). That is, \(U^*\) is bounded. Also \(U^*\) is one-to-one. Indeed, if \(U^* f = 0\) for some \(f \in L^2(G)\), then \([U^* f, g_n]_\varphi = 0\). So by (2.1), \([f, f_n]_\varphi = 0\), which implies that \(f = 0\), since \((f_n)_{n \in \mathbb{N}}\) is a \(\varphi\)-frame. Similarly \(U^*^{-1}\) is bounded. Hence \(U^*\) is an isomorphism (note that \(U^*\) has dense range). Now by [3, Theorem 4.1], \((M, f_n)_{n \in \mathbb{N}, \gamma \in \varphi(L^1)}\) is a frame. This completes the proof of (1) \(\Rightarrow\) (2). Let \((M, f_n)_{n \in \mathbb{N}, \gamma \in \varphi(L^1)}\) be a frame. By [3, Theorem 4.1], \(U^*\) is an isomorphism. Thus using (2.2) we have

\[
A \|f\|_\varphi^2(\hat{x}) \leq \sum_{n \in \mathbb{N}} |[f, f_n]_\varphi(\hat{x})|^2 \leq B \|f\|_\varphi^2(\hat{x})
\]

for a.e. \(\hat{x} \in G/\varphi(L)\), in which \(A = \|U^*^{-1}\|^{-2}, B = \|U^*\|^2\). That is, (2) implies (1). \(\square\)

We now intend to define \(\varphi\)-pre-frame and \(\varphi\)-analysis operators. First, we need to introduce a vector space which plays the role of \(L^2(\mathbb{N})\) in the standard case. To this end, define \(l_2^G(G/\varphi(L))\) as the space of the sequences in \(L^2(G/\varphi(L))\) convergent in \(L^1(G/\varphi(L))\), i.e.,

\[
l_2^G(G/\varphi(L)) = \{\{g_i\}_{i \in \mathbb{N}} \subseteq L^2(G/\varphi(L)) : \int_{G/\varphi(L)} \sum_{i \in \mathbb{N}} |g_i(\hat{x})|^2 d\hat{x} < \infty\}.
\]

\(l_2^G(G/\varphi(L))\) is an inner-product space with the inner product defined as follows:

\[
\langle \{g_i\}, \{h_i\} \rangle_{l_2^G(G/\varphi(L))} = \sum_{i \in \mathbb{N}} g_i \overline{h_i}
\]
for \( \{g_i\}_{i \in \mathbb{N}}, \{h_i\}_{i \in \mathbb{N}} \in l_1^2(G/\varphi(L)) \). Note that \( \sum_{i \in \mathbb{N}} g_i h_i \in L^1(G/\varphi(L)) \). Indeed,

\[
\left\| \sum_{i \in \mathbb{N}} g_i h_i \right\|_{L^1(G/\varphi(L))} \leq \int_{G/\varphi(L)} \left| \sum_{i \in \mathbb{N}} g_i(\hat{x}) |h_i(\hat{x})| \right| d\hat{x} \\
\leq \left( \int_{G/\varphi(L)} \left| \sum_{i \in \mathbb{N}} |g_i(\hat{x})|^2 d\hat{x} \right| \right)^{1/2} \left( \int_{G/\varphi(L)} \left| \sum_{i \in \mathbb{N}} |h_i(\hat{x})|^2 d\hat{x} \right| \right)^{1/2} < \infty.
\]

For \( \{g_i\}_{i \in \mathbb{N}} \in l_1^2(G/\varphi(L)) \), define the pointwise norm by

\[
\left\| \{g_i\}_{i \in \mathbb{N}} \right\|_{l_1^2(G/\varphi(L))}(\hat{x}) = \left( \sum_{i \in \mathbb{N}} |g_i(\hat{x})|^2 \right)^{1/2},
\]

and the uniform norm by

\[
\left\| \{g_i\}_{i \in \mathbb{N}} \right\|_{l_1^2(G/\varphi(L))} = \left( \int_{G/\varphi(L)} \left| \sum_{i \in \mathbb{N}} |g_i(\hat{x})|^2 d\hat{x} \right| \right)^{1/2}.
\]

Let \( \{f_n\}_{n \in \mathbb{N}} \) be a \( \varphi \)-bounded \( \varphi \)-Bessel sequence in \( L^2(G) \). Define the \( \varphi \)-analysis operator as the mapping \( T_\varphi : L^2(G) \rightarrow l_1^2(G/\varphi(L)) \) given by

\[
T_\varphi f = \{\langle f, f_n \rangle \varphi \}_{n \in \mathbb{N}}.
\]

Define \( \theta : L^2(G) \rightarrow L^1(G/\varphi(L)) \) by \( \theta(f) = [T_\varphi f, \{g_n\}_{n \in \mathbb{N}}]_{l_1^2(G/\varphi(L))} \) for some sequence \( \{g_n\}_{n \in \mathbb{N}} \in l_1^2(G/\varphi(L)) \). Note that if \( T_\varphi \) is bounded, then \( \theta \) is a bounded \( \varphi \)-factorable operator. So by Riesz Representation Theorem for \( \varphi \)-factorable operators (Theorem 1.1), there exists \( T_\varphi^* \{\{g_n\}_{n \in \mathbb{N}}\} \in L^2(G) \) with

\[
\| T_\varphi^* \{\{g_n\}_{n \in \mathbb{N}}\} \|_2 = \| \theta \|
\]

such that \( \theta(f) = \langle f, T_\varphi^* \{\{g_n\}_{n \in \mathbb{N}}\} \rangle \varphi \). Note that \( \| T_\varphi \| = \| T_\varphi^* \| \). Indeed,

\[
\| T_\varphi f, \{g_n\}_{n \in \mathbb{N}} \|_{l_1^2(G/\varphi(L))} \|_{L^1(G/\varphi(L))} = \int_{G/\varphi(L)} \| T_\varphi f, \{g_n\}_{n \in \mathbb{N}} \|_{l_1^2(G/\varphi(L))}(\hat{x}) d\hat{x} = \int_{G/\varphi(L)} \left( \sum_{n \in \mathbb{N}} |\langle f, f_n \rangle \varphi(\hat{x})| \right)^2 d\hat{x} \leq \left( \int_{G/\varphi(L)} \left( \sum_{n \in \mathbb{N}} |\langle f, f_n \rangle \varphi(\hat{x})|^2 d\hat{x} \right) \right)^{1/2} \left( \int_{G/\varphi(L)} \left( \sum_{n \in \mathbb{N}} |g_n(\hat{x})|^2 d\hat{x} \right) \right)^{1/2} = \| T_\varphi f \|_{l_1^2(G/\varphi(L))} \| \{g_n\}_{n \in \mathbb{N}} \|_{l_1^2(G/\varphi(L))}.
\]

Hence

\[
\| T_\varphi^* \{\{g_n\}_{n \in \mathbb{N}}\} \|_2 = \| \theta \| = \sup_{\| f \|_2 \leq 1} \| [T_\varphi f, \{g_n\}_{n \in \mathbb{N}}]_{l_1^2(G/\varphi(L))} \|_{L^1(G/\varphi(L))}\]
\[ \leq \| T_\varphi \| \{ g_n \} \|_{l^1(G/\varphi(L))}. \]

That is, \( \| T_\varphi^* \| \leq \| T_\varphi \| \). Also obviously, \( T_\varphi = T_\varphi^{**} \). So \( \| T_\varphi \| = \| T_\varphi^* \| \).

To obtain the \( \varphi \)-preframe operator \( T_\varphi^* \) explicitly, we calculate as follows. Let \( f \in L^2(G) \), \( \{ g_n \}_{n \in \mathbb{N}} \in l^2(G/\varphi(L)) \). Then we have

\[
\begin{align*}
[f, T_\varphi^* \{ g_n \}]_\varphi(\hat{x}) &= [T_\varphi f, \{ g_n \}_{n \in \mathbb{N}}]_{l^2(G/\varphi(L))}(\hat{x}) \\
&= \sum_{n \in \mathbb{N}} T_\varphi f(\hat{x}) g_n(\hat{x}) \\
&= \sum_{n \in \mathbb{N}} [f, g_n]_\varphi(\hat{x}) g_n(\hat{x}) \\
&= [f, \sum_{n \in \mathbb{N}} g_n]_\varphi(\hat{x}).
\end{align*}
\]

Thus

\[
\int_{G/\varphi(L)} [f, T_\varphi^* \{ g_n \}]_\varphi(\hat{x}) d\hat{x} = \int_{G/\varphi(L)} [f, \sum_{n \in \mathbb{N}} g_n]_\varphi(\hat{x}) d\hat{x}.
\]

That is,

\[
\langle f, T_\varphi^* \{ g_n \} \rangle_{L^2(G)} = \langle f, \sum_{n \in \mathbb{N}} g_n \rangle_{L^2(G)}.
\]

Hence

\[
(2.4) \quad T_\varphi^* \{ g_n \} = \sum_{n \in \mathbb{N}} g_n.
\]

\( T_\varphi^* \) is called the \( \varphi \)-preframe operator.

In the following proposition we characterize \( \varphi \)-Bessel sequences in terms of the \( \varphi \)-preframe operator. To be more precise, we show that a \( \varphi \)-bounded sequence is \( \varphi \)-Bessel if and only if the \( \varphi \)-preframe operator is bounded.

**Remark 2.2.** (i) For \( f \in L^2(G) \) we have

\[
\| f \|_\varphi(\hat{x}) = \sup \{ \| [f, g]_\varphi(\hat{x}) \|; \| g \|_\varphi(\hat{x}) \leq 1 \}
\]

for a.e. \( \hat{x} \in G/\varphi(L) \). Indeed, by Cauchy Schwartz Inequality (1.3) we have

\[
\sup \{ \| [f, g]_\varphi(\hat{x}) \|; \| g \|_\varphi(\hat{x}) \leq 1 \} \leq \| f \|_\varphi(\hat{x})
\]

for a.e. \( \hat{x} \in G/\varphi(L) \). Also

\[
\sup \{ \| [f, g]_\varphi(\hat{x}) \|; \| g \|_\varphi(\hat{x}) \leq 1 \} \geq \| f, f \|_\varphi(\hat{x}) = \| f \|_\varphi(\hat{x})
\]

for a.e. \( \hat{x} \in G/\varphi(L) \).

(ii) By a similar argument as in the standard \( L^2 \)-space theory it is verified that \( (L^2(G), \| \cdot \|_\varphi) \) is a Banach space.

We say \( g \in L^2(G) \) is \( \varphi \)-bounded if there exists \( M > 0 \) so that \( \| g \|_\varphi \leq M \) a.e.. Note that for \( f, g \in L^2(G) \) the function \( [f, g]_\varphi g \) need not generally be in \( L^2(G) \). But if \( f, g, h \in L^2(G) \) and \( g, h \) are \( \varphi \)-bounded, then \( [f, g]_\varphi h \in L^2(G) \) (see [19]).
Proposition 2.3. Let \((f_n)_{n \in \mathbb{N}}\) be a \(\varphi\)-bounded sequence in \(L^2(G)\). Then \((f_n)_{n \in \mathbb{N}}\) is \(\varphi\)-Bessel with bound \(B\) if and only if \(T^*_\varphi\) is a well defined bounded operator from \(L^2(G/\varphi(L))\) into \(L^2(G)\) and \(\|T^*_\varphi\| \leq \sqrt{B}\).

Proof. Let \((f_n)_{n \in \mathbb{N}}\) be a \(\varphi\)-Bessel sequence with bound \(B\) in \(L^2(G)\). Assume that \((g_n)_{n \in \mathbb{N}} \in L^2_2(G/\varphi(L)), n \in \mathbb{N}\). Then for \(m, n \in \mathbb{N}, n \geq m,\) we have

\[
\| \sum_{i=1}^{n} g_i f_i - \sum_{i=1}^{m} g_i f_i \|_\varphi(x) \\
= \| \sum_{i=m+1}^{n} g_i f_i \|_\varphi(x) \\
= \sup_{\|g\|_\varphi \leq 1} \left| \sum_{i=m+1}^{n} g_i f_i \right| \\
= \sup_{\|g\|_\varphi \leq 1} \left( \sum_{i=m+1}^{n} |g_i f_i| \right) \\
\leq \left( \sum_{i=m+1}^{n} |g_i|^2 \right)^{1/2} \sup_{\|g\|_\varphi \leq 1} \left( \sum_{i=m+1}^{n} |f_i|^2 \right)^{1/2} \\
\leq \sqrt{B} \left( \sum_{i=m+1}^{n} |g_i|^2 \right)^{1/2}.
\]

So \(\sum_{i=1}^{n} g_i f_i\) is Cauchy in \((L^2(G), \|\cdot\|_\varphi)\) and therefore convergent. Thus \(T^*_\varphi\) is well defined. Also obviously \(\|T^*_\varphi\| \leq B\). For the converse assume \(T^*_\varphi\) and so \(T^*_\varphi\) is bounded. Then \(\|T^*_\varphi(hf)\|_{L^2(G/\varphi(L))} \leq \|T^*_\varphi\| \|hf\|_2\) for every \(h \in L^\infty(G/\varphi(L))\).

Therefore,

\[
\int_{G/\varphi(L)} \sum_{n \in \mathbb{N}} |hf_n|^2 d\hat{x} \leq \int_{G/\varphi(L)} \|hf\|_\varphi^2(\hat{x}) \|T^*_\varphi\|^2 d\hat{x}.
\]

That is,

\[
\int_{G/\varphi(L)} |h(\hat{x})|^2 \sum_{n \in \mathbb{N}} |f_n|^2 d\hat{x} \leq \int_{G/\varphi(L)} |h(\hat{x})|^2 \|f\|_\varphi^2(\hat{x}) \|T^*_\varphi\|^2 d\hat{x}
\]

for every \(h \in L^\infty(G/\varphi(L))\). Hence

\[
\sum_{n \in \mathbb{N}} |f_n|^2 \leq B \|f\|_\varphi^2(\hat{x})
\]

for a.e. \(\hat{x} \in G/\varphi(L)\), where \(B = \|T^*_\varphi\|^2\). So \((f_n)_{n \in \mathbb{N}}\) is \(\varphi\)-Bessel. \(\square\)
Let \((f_n)_{n \in \mathbb{N}}\) be a \(\varphi\)-frame. Assume that each \(f_n, n \in \mathbb{N}\) is \(\varphi\)-bounded in \(L^2(G)\). The \(\varphi\)-frame operator defined by \(S_\varphi := T_\varphi^* T_\varphi\) is bounded. Indeed,
\[
[S_\varphi f, f]_\varphi = \sum_{n \in \mathbb{N}} [f, f_n]_\varphi [f_n, f]_\varphi
= \sum_{n \in \mathbb{N}} ||[f, f_n]_\varphi||^2.
\]
So we have
\[
A[f, f]_\varphi \leq [S_\varphi f, f]_\varphi \leq B[f, f]_\varphi,
\]
which implies
\[
A \int_{G/\varphi(L)} [f, f]_\varphi(\hat{x}) d\hat{x} \leq \int_{G/\varphi(L)} [S_\varphi f, f]_\varphi(\hat{x}) d\hat{x} \leq B \int_{G/\varphi(L)} [f, f]_\varphi(\hat{x}) d\hat{x}.
\]
Therefore, \(AI \leq S_\varphi \leq BI\). By a standard argument as in the frame theory \(S_\varphi\) is invertible and \(B^{-1} I \leq S_\varphi^{-1} \leq A^{-1} I\).

We can now characterize \(\varphi\)-frames with the aid of the \(\varphi\)-preframe operator.

**Proposition 2.4.** Let \((f_n)_{n \in \mathbb{N}}\) be a \(\varphi\)-bounded sequence in \(L^2(G)\). Then \((f_n)_{n \in \mathbb{N}}\) is a \(\varphi\)-frame if and only if \(T_\varphi^* = T_\varphi^* T_\varphi\) is well defined, bounded and onto.

**Proof.** Let \(f_n\) be a \(\varphi\)-frame. Then by the above remarks \(S_\varphi\) is onto and so is \(T_\varphi^*\). The rest follows from Proposition 2.3.

Conversely, we have \(f = S_\varphi S_\varphi^{-1} f = \sum_{n \in \mathbb{N}} [S_\varphi^{-1} f, f_n]_\varphi f_n\), so
\[
\|f\|_{S_\varphi^{-1}}^2(\hat{x}) = [f, f]_\varphi(\hat{x})
= \sum_{n \in \mathbb{N}} [S_\varphi^{-1} f, f_n]_\varphi [f_n, f]_\varphi(\hat{x})
\leq \left(\sum_{n \in \mathbb{N}} ||S_\varphi^{-1} f, f_n]_\varphi(\hat{x})||^2\right)^{1/2} \left(\sum_{n \in \mathbb{N}} ||[f_n, f]_\varphi(\hat{x})||^2\right)^{1/2}
\leq \|T_\varphi S_\varphi^{-1} f\|_{\ell_2(G/\varphi(L))}(\hat{x}) \left(\sum_{n \in \mathbb{N}} ||[f_n, f]_\varphi(\hat{x})||^2\right)^{1/2}
\leq \|T_\varphi\| \|S_\varphi^{-1}\| \|f\|_{\varphi}(\hat{x}) \left(\sum_{n \in \mathbb{N}} ||[f_n, f]_\varphi(\hat{x})||^2\right)^{1/2}
\]
for a.e. \(\hat{x} \in G/\varphi(L)\). That is,
\[
A\|f\|_{S_\varphi^{-1}}^2(\hat{x}) \leq \sum_{n \in \mathbb{N}} ||[f_n, f]_\varphi(\hat{x})||^2,
\]
where $A = \left\| T_\varphi \right\|^2 \left\| S_\varphi^{-1} \right\|^2$. Now Proposition 2.3 completes the proof. $\square$

Next we consider the case when two $\varphi$-Bessel sequences may also be $\varphi$-frames.

**Proposition 2.5.** Let $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ be two $\varphi$-bounded $\varphi$-Bessel sequences in $L^2(G)$. If $f = \sum_{n \in \mathbb{N}} f_n g_n f_n$, a.e. for all $f \in L^2(G)$, then both $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ are $\varphi$-frames.

**Proof.** Let us denote by $B$ the $\varphi$-Bessel bound of $(f_n)_{n \in \mathbb{N}}$. For all $f \in L^2(G)$, we have

$$\left\| f \right\|^2 (\hat{x}) = \left\| f, f \right\|^2 (\hat{x}) = \left( \sum_{n \in \mathbb{N}} \left\| f_n \right\|^2 \right) \left( \sum_{n \in \mathbb{N}} \left\| g_n \right\|^2 \right) = \left( \sum_{n \in \mathbb{N}} \left\| f_n \right\|^2 \right) \left( \sum_{n \in \mathbb{N}} \left\| g_n \right\|^2 \right) \leq \sum_{n \in \mathbb{N}} \left\| f_n \right\|^2 \left\| g_n \right\|^2 \leq B \left\| f \right\|^2 (\hat{x}) \sum_{n \in \mathbb{N}} \left\| f_n \right\|^2 \left\| g_n \right\|^2 .$$

That is,

$$B^{-1} \left\| f \right\|^2 (\hat{x}) \leq \sum_{n \in \mathbb{N}} \left\| f_n \right\|^2 \left\| g_n \right\|^2$$

for every $f \in L^2(G)$, for a.e. $\hat{x} \in G/\varphi(L)$. Hence $(g_n)_{n \in \mathbb{N}}$ is a $\varphi$-frame. A similar argument shows that $(f_n)_{n \in \mathbb{N}}$ is also a $\varphi$-frame. $\square$

It is clear that every $\varphi$-orthonormal basis is a Parseval $\varphi$-frame, but the converse is not true.

**Example 2.6.** Consider the LCA group $G = \mathbb{R}^+$. As a uniform lattice in $G$ we choose $L = \{2^n; n \in \mathbb{Z}\}$. Then $L^\perp = \mathbb{Z}$. We can choose $S_L := [1,2]$ as a fundamental domain for $L$ in $G$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be the topological automorphism defined by $\varphi(x) = x^2$. Let $(f_n)_{n \in \mathbb{N}}$ be a $\varphi$-orthonormal basis for $L^2(G)$ (e.g. consider the orthonormal basis $\{ M_k = \chi_{S_L}; k \in L \times L^\perp \}$, as in [23, Theorem 3.1.7] for $L^2(G)$, where $M_k$ is the modulation operator. By [19, Theorem 14], $\{ T_k \chi_{S_L}; k \in L \}$ is a $\varphi$-orthonormal basis for $L^2(G)$). Then $\{ f_1, f_2, f_2, f_2, f_3, f_3, f_3, f_3, \ldots \}$ is a Parseval $\varphi$-frame but not a $\varphi$-orthonormal basis.

It is easy to see that if $(f_n)_{n \in \mathbb{N}}$ is a Parseval $\varphi$-frame and $\left\| f_n \right\| = 1$ a.e. for every $n \in \mathbb{N}$, then $(f_n)_{n \in \mathbb{N}}$ is a $\varphi$-orthonormal basis.
3. $\varphi$-Riesz Bases in $L^2(G)$

Our goal in this section is to define and investigate $\varphi$-Riesz bases in $L^2(G)$, applying $\varphi$-factorable operators.

Riesz bases in $L^2(\mathbb{R})$ have several equivalent definitions (see [9, 15, 27]). The main result of this section (Theorem 3.4), sets out equivalent conditions for a sequence in $L^2(G)$ to be a $\varphi$-Riesz basis, where $G$ is a second countable LCA group and $\varphi$ is a topological automorphism on $G$. We start with a definition.

**Definition 3.1.** A sequence $(f_n)_{n \in \mathbb{N}}$ in $L^2(G)$ is said to be a $\varphi$-Riesz basis if there exists a $\varphi$-orthonormal basis $(g_n)_{n \in \mathbb{N}}$ and a $\varphi$-factorable operator $U: L^2(G) \to L^2(G)$, which is a topological automorphism such that $U(g_n) = f_n$ for every $n \in \mathbb{N}$.

We introduce a $\varphi$-complete ($\varphi$-total) sequence in $L^2(G)$ as follows:

**Definition 3.2.** Given a sequence $(f_n)_{n \in \mathbb{N}} \subseteq L^2(G)$, by $\text{span}_{\varphi}(f_n) = L^2(G)$ we mean that for every $f \in L^2(G)$ there exists a sequence $(h_n)_{n \in \mathbb{N}} \subseteq L^2(G/\varphi(L))$, such that $f = \sum_{n=1}^{\infty} h_n f_n$, a.e. We say a sequence $(f_n)_{n \in \mathbb{N}} \subseteq L^2(G)$ is $\varphi$-complete ($\varphi$-total) in $L^2(G)$, if $\text{span}_{\varphi}(f_n) = L^2(G)$.

The following lemma will be needed in the proof of Theorem 3.4.

**Lemma 3.3.** Suppose $U$ is a bounded $\varphi$-factorable operator on $L^2(G)$. For every $f \in L^2(G)$, we have $\|Uf\|_{\varphi} \leq \|U\| \|f\|_{\varphi}$ a.e.

**Proof.** For every $\varphi$-periodic $h \in L^\infty(G)$, we have

$$
\int_{G/\varphi(L)} |h(\hat{x})|^2 \|U(f)\|^2_{\varphi}(\hat{x}) d\hat{x} = \int_{G/\varphi(L)} \sum_{k \in L} |U(f)(x\varphi(k^{-1}))|^2 |h(x\varphi(k^{-1}))|^2 d\hat{x} = \int_{G/\varphi(L)} \sum_{k \in L} |U(hf)(x\varphi(k^{-1}))|^2 d\hat{x} = \|U(hf)\|^2_{\varphi} \leq \|U\|^2 \|hf\|^2_{\varphi} = \|U\|^2 \int_G |hf(x)|^2 dx = \|U\|^2 \int_{G/\varphi(L)} \sum_{k \in L} |hf(x\varphi(k^{-1}))|^2 d\hat{x} = \|U\|^2 \int_{G/\varphi(L)} |h(\hat{x})|^2 \|f\|^2_{\varphi}(\hat{x}) d\hat{x},
$$

which obviously completes the proof. $\square$
In the following theorem we establish equivalent conditions for a sequence in \( L^2(G) \) to be a \( \varphi \)-Riesz basis. As a matter of fact Theorem 3.4 gives a characterization of \( \varphi \)-Riesz bases with regard to standard Riesz bases in \( L^2(G) \), which implies that a \( \varphi \)-Riesz basis shares many useful properties with a Riesz basis.

**Theorem 3.4.** Let \((f_n)_{n \in \mathbb{N}}\) be a sequence in \( L^2(G) \). The following are equivalent.

1. \((f_n)_{n \in \mathbb{N}}\) is \( \varphi \)-complete, and there exist positive constants \( A \) and \( B \) such that for any sequence \( \{h_n\}_{n \in \mathbb{N}} \in l_2^f(G/\varphi(L)) \) one has

\[
A \sum_{n=1}^{\infty} |h_n|^2 \leq \| \sum_{n=1}^{\infty} h_n f_n \|_\varphi^2 \leq B \sum_{n=1}^{\infty} |h_n|^2 \quad \text{a.e.}
\]

(3.1)

2. \((f_n)_{n \in \mathbb{N}}\) is a \( \varphi \)-Riesz basis.

3. \((M_{\gamma} f_n)_{\gamma \in \varphi(L)} \), \( n \in \mathbb{N} \) is a \( L^2(G) \) basis.

**Proof.** (1) \(\Rightarrow\) (2) Let \((e_n)_{n \in \mathbb{N}}\) be a \( \varphi \)-orthonormal basis in \( L^2(G) \). Then by [19, Theorem 14], \( \text{span}\{\varphi(e_n)\} = L^2(G) \). Define \( U : L^2(G) \to L^2(G) \) by \( U(\sum_{n=1}^{\infty} h_n e_n) = \sum_{n=1}^{\infty} h_n f_n \), where \( \{h_n\}_{n \in \mathbb{N}} \in l_2^f(G/\varphi(L)) \). Then \( U \) is bounded. In fact, by (3.1)

\[
\|U(\sum_{n=1}^{\infty} h_n e_n)\|_\varphi^2 = \| \sum_{n=1}^{\infty} h_n f_n \|_\varphi^2
\]

\[
\leq B \sum_{n=1}^{\infty} |h_n|^2
\]

\[
= B \| \sum_{n=1}^{\infty} h_n e_n \|_\varphi^2, \quad \text{a.e.,}
\]

and so

\[
\|U(\sum_{n=1}^{\infty} h_n e_n)\|_2^2 = \int_{G/\varphi(L)} \|U(\sum_{n=1}^{\infty} h_n e_n)\|_{\varphi}^2 (\hat{x}) \, d\hat{x}
\]

\[
\leq B \int_{G/\varphi(L)} \| \sum_{n=1}^{\infty} h_n e_n \|_{\varphi}^2 (\hat{x}) \, d\hat{x}
\]

\[
= B \| \sum_{n=1}^{\infty} h_n e_n \|_2^2
\]

for any \( \{h_n\}_{n \in \mathbb{N}} \in l_2^f(G/\varphi(L)) \). That is, \( \|U\| \leq \sqrt{B} \). Now define \( S : L^2(G) \to L^2(G) \) by \( S(\sum_{n=1}^{\infty} h_n f_n) = \sum_{n=1}^{\infty} h_n e_n \), where \( \{h_n\}_{n \in \mathbb{N}} \in l_2^f(G/\varphi(L)) \). Hence by (3.1) we get

\[
\|S(\sum_{n=1}^{\infty} h_n f_n)\|_\varphi^2 = \| \sum_{n=1}^{\infty} h_n e_n \|_\varphi^2
\]
\[
\sum_{n=1}^{\infty} |h_n|^2 = \sum_{n=1}^{\infty} |h_n|^2 \\
\leq 1/A \sum_{n=1}^{\infty} h_n f_n \| \varphi, \text{ a.e.}
\]

This implies that \( S \) is bounded on \( L^2(G) \) and \( \| S \| \leq \sqrt{1/A} \). Also obviously, \( SU = I \) and \( US = I \) on \( L^2(G) \). Hence \( U \) is a topological isomorphism, which is clearly \( \varphi \)-factorable and \( U(e_n) = f_n \) for every \( n \in \mathbb{N} \).

(2)\( \Rightarrow \) (3) Choose a \( \varphi \)-orthonormal basis \((e_n)_{n \in \mathbb{N}}\) for \( L^2(G) \) and the corresponding topological automorphism \( U \) which is a \( \varphi \)-factorable operator and \( U(e_n) = f_n \) for every \( n \in \mathbb{N} \), as in Definition 3.1. By [19, Theorem 14], \((M_s f_n)_{\gamma \in \varphi(L)^+}, n \in \mathbb{N}\) is an orthonormal basis for \( L^2(G) \), and since \( U \) is \( \varphi \)-factorable

\[
U(M_s e_n) = M_s U(e_n) = M_s f_n
\]
for every \( n \in \mathbb{N}, \gamma \in \varphi(L)^+ \). So \((M_s f_n)_{\gamma \in \varphi(L)^+}, n \in \mathbb{N}\) is a Riesz basis.

(3)\( \Rightarrow \) (2) Let \( S_{\varphi(L)} \) be a fundamental domain for \( \varphi(L) \). By [23, Theorem 3.1.7], the system \((M_s T_{\varphi(k)} \chi_{S_{\varphi(L)}}, k \in L, \gamma \in \varphi(L)^+)\) is an orthonormal basis for \( L^2(G) \), where \( T_{\varphi(k)} \chi_{S_{\varphi(L)}} \) is the translation of \( \chi_{S_{\varphi(L)}} \) by \( \varphi(k) \). Define \( U : L^2(G) \to L^2(G) \) by \( U(M_{\gamma n} T_{\varphi(k_n)} \chi_{S_{\varphi(L)}}) = M_{\gamma n} f_n, m, n \in \mathbb{N} \). Obviously, \( U \) is a \( \varphi \)-factorable operator. Moreover, by [19, Theorem 14], \((T_{\varphi(k)} \chi_{S_{\varphi(L)}}, k \in L)\) is a \( \varphi \)-orthonormal basis for \( L^2(G) \), and obviously \( U(T_{\varphi(k_n)} \chi_{S_{\varphi(L)}}) = f_n \) for every \( n \in \mathbb{N} \). Finally since \((M_s f_n)_{\gamma \in \varphi(L)^+}, n \in \mathbb{N}\) is a Riesz basis, \( U \) is a topological automorphism.

(2)\( \Rightarrow \) (1) Suppose \((e_n)_{n \in \mathbb{N}}\) is a \( \varphi \)-orthonormal basis and \( U \) is the corresponding topological automorphism which is a \( \varphi \)-factorable operator and \( U(e_n) = f_n \) for every \( n \in \mathbb{N} \), as in the Definition 3.1. Let \((h_n)_{n \in \mathbb{N}} \in l^2(G/\varphi(L))\). Then using Lemma 3.3

\[
\sum_{n=1}^{\infty} \| h_n f_n \|_{\varphi}^2 = \sum_{n=1}^{\infty} \| h_n U(e_n) \|_{\varphi}^2 \\
= \| U(\sum_{n=1}^{\infty} h_n e_n) \|_{\varphi}^2 \\
\leq \| U \|^2 \sum_{n=1}^{\infty} \| h_n e_n \|_{\varphi}^2 \\
= \| U \|^2 \sum_{n=1}^{\infty} |h_n|^2, \text{ a.e.}
\]

On the other hand

\[
\sum_{n=1}^{\infty} |h_n|^2 = \sum_{n=1}^{\infty} \| h_n e_n \|_{\varphi}^2
\]
\[\|U^{-1}U(\sum_{n=1}^{\infty} h_n e_n)\|_{\varphi}^2 \leq \|U^{-1}\|_2^2 \|\sum_{n=1}^{\infty} h_n e_n\|_{\varphi}^2 = \|U^{-1}\|_2^2 \|\sum_{n=1}^{\infty} h_n f_n\|_{\varphi}^2, \text{ a.e.}\]

So (3.1) holds. Moreover \((f_n)_{n \in \mathbb{N}}\) is \(\varphi\)-complete. Indeed, given any \(f \in L^2(G)\), there exists a unique \(g \in L^2(G)\) with \(U(g) = f\) (since \(U\) is one-to-one and onto). Write \(g = \sum_{n=1}^{\infty} [g, e_n] \varphi e_n\) as in [19, Theorem 18]. Then \(h_n = [g, e_n] \varphi \in L^\infty(G/\varphi(L))\) for every \(n \in \mathbb{N}\) and by Bessel’s Inequality ([19, Theorem 11])
\[
\sum_{n=1}^{\infty} |h_n(\hat{x})|^2 \leq \|f\|_{\varphi}(\hat{x}) < \infty
\]
for a.e. \(\hat{x} \in G/\varphi(L)\). Also
\[f = U(g) = U(\sum_{n=1}^{\infty} h_n e_n) = \sum_{n=1}^{\infty} h_n U(e_n) = \sum_{n=1}^{\infty} h_n f_n,\]
showing that \(\text{span} \{f_n\} = L^2(G)\). This completes the proof. \(\square\)

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References

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