GLOBAL EXISTENCE FOR 3D NAVIER-STOKES EQUATIONS IN A LONG PERIODIC DOMAIN

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Abstract. We consider the global existence of strong solutions of the 3D incompressible Navier-Stokes equations in a long periodic domain. We show by a simple argument that a strong solution exists globally in time when the initial velocity in $H^1$ and the forcing function in $L^p([0, T); L^2)$, $T > 0$, $2 \leq p \leq +\infty$ satisfy a certain condition. This condition commonly appears for the global existence in thin non-periodic domains. Larger and larger initial data and forcing functions satisfy this condition as the thickness of the domain $\epsilon$ tends to zero.

1. Introduction

We consider the incompressible Navier-Stokes equations,

$\begin{align*}
\frac{du}{dt} - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= f, \\
\nabla \cdot u &= 0,
\end{align*}$

in a periodic domain $\Omega = T^3 = [0, l_1] \times [0, l_2] \times [0, l_3]$. Here $u$ denotes the velocity of a homogeneous, viscous incompressible fluid, $f$ is the density of force per unit volume, $p$ denotes the pressure, and $\nu$ is the kinematic viscosity. We require that the forcing function $f$ and the initial data $u_0$ satisfy

$\nabla \cdot f = \nabla \cdot u_0 = 0.$

We assume in addition that

$\int_\Omega f dx = \int_\Omega u dx = 0,$

which could be achieved by the Galilean transformation with suitable vectors $c(t)$ and $e$,

$u(x, t) \rightarrow u(x + c(t) + et, t) - \frac{dc}{dt} - e.$

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Indeed, we can take
\[ c(t) = \int_0^t \int_0^r \int \mathbf{f}(x, s) \, dx \, ds \, dr, \quad e = \int u_0 \, dx. \]

By the classical results of Leray and Hopf ([11], [4]), there exists a global weak solution of the Navier-Stokes equations in a three dimensional torus. It is also known that the solution becomes necessarily strong (regular) for all regular data in a two dimensional domain. But in a three dimensional domain, global strong solutions have only been guaranteed for small initial data (See, for example, [2], [3], [14], [15] and the references therein).

In [13], Raugel and Sell treated the problem on thin periodic domain, \( \Omega = (0, l_1] \times (0, l_2] \times (0, \epsilon] \) and they obtained a significant existence result on global regular solutions. The main idea is that if the thickness of the domain is small enough, the solution of the Navier-Stokes equations is close to the 2D Navier-Stokes equations. They proved that there are large sets \( R(\epsilon) \subset H^1(\Omega) \) and \( S(\epsilon) \subset L^\infty((0, \infty), L^2(\Omega)) \) such that if \( u(0) = u_0 \in R(\epsilon) \) and \( f \in S(\epsilon) \), then there exists a strong solution \( u(t) \) that remains in \( H^1(\Omega) \) for all \( t \geq 0 \). The sets \( R(\epsilon) \) and \( S(\epsilon) \) get larger and larger as \( \epsilon \to 0 \).

Since then, there have been many improvements on the estimates of the size of these sets \( R(\epsilon) \) and \( S(\epsilon) \) under various boundary conditions (See [1], [5], [12], [6], [7], [8], [9], [16] and the references therein). Roughly, under various boundary conditions except the periodic boundary condition, it has been shown that if

\[ \|u_0\|_{H^1} \leq C\epsilon^{-1/2} \quad \text{and} \quad \|f\|_{L^\infty((0, \infty), L^2)} \leq C\epsilon^{-1/2} \]

for some constant \( C = C(\nu) \), then the corresponding global strong solution exists (See [1], [16]). We note that the above condition can cover very large initial data and forcing functions if \( \epsilon > 0 \) is small enough.

However, under the periodic boundary condition, it is not known whether (4) implies the existence of global strong solutions. Until now, it is known that, when \( f = 0 \), the existence of the global strong solution is guaranteed under the condition ([10])

\[ \|u_0\|_{H^1} \leq C\epsilon^{-1/2} \log \epsilon^{1/2}, \]

or under the following condition ([6])

\[ \|(Nu_0)_3\| \leq C\nu \epsilon^{1/2}, \quad \|Nf\|_{L^\infty((0, \infty), L^2)} \leq C\nu^2 \epsilon^{1/2}, \]

\[ \|
abla u_0\| \leq C\nu^{-1/2}, \quad \|f\|_{L^\infty((0, \infty), L^2)} \leq C\nu^2 \epsilon^{-1/2}. \]

Here, \( N \) is the average operator with respect to the thin direction. We note that the first two conditions in the above are not so restrictive since \( Nu_0 \) and \( Nf \) are independent of the third variable and so they are in fact \( \epsilon \) independent conditions.

In this paper, we consider the global existence of strong solutions in a long periodic domain, \( \Omega = (0, \epsilon] \times (0, \epsilon] \times (0, l] \). We first prove in a simple way that
a global strong solution exists whenever the initial and the forcing functions satisfy for any $2 \leq p \leq \infty$ and $L > 0$,

$$
\|\nabla u_0\|_{L^2} \leq \frac{C \nu}{L} \quad \text{and} \quad \|f\|_{L^p((0,\infty),L^2)} \leq C \nu^{(2p-1)/p} \lambda_1^{(3p-4)/4p}
$$

(5)

together with a mild condition,

$$
\frac{1}{L}\|u_0\|_{L^2} \leq 1
$$

(6)

for some universal constant $C$. Here, $\lambda_1 = 4\pi^2/l$ is the first eigenvalue of the Stokes operator. This result is obtained simply by considering a differential inequality for a product of norms, which is comparable to $H^{1/2}$ norm. The most natural choice of $L$ in the condition (6) is $L = \sqrt{|\Omega|}$, which is not practically restrictive since it just means that the spatial average of the square of the velocity is bounded by a suitable constant. Then, when the domain is long rod type $\Omega = (0, \epsilon] \times (0, \epsilon] \times (0, l]$, the choice $L = \sqrt{|\Omega|}$ becomes of order $\epsilon$ and the bound on $H^1$ norm of the velocity in (5) is improved greatly compared to the case of thin domain. We also give a condition independent of the $L^2$ norm of the velocity. Concretely, we show that the global regularity is guaranteed if

$$
\|\nabla u_0\| \leq C \nu^{-1/2}, \quad \|f\|_{p,2} \leq C \nu^{(2p-1)/p} \epsilon^{1/2}
$$

for any $2 \leq p \leq \infty$. The above condition exactly recovers (4) even for more general $p$ and supports that the condition (4) might be enough for the global existence in a thin periodic domain under the periodic boundary condition.

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2. Preliminary estimates

Throughout the paper, $\Omega = (0, \epsilon] \times (0, \epsilon] \times (0, l]$. Here, $l$ is a fixed constant and $\epsilon > 0$ is a small parameter. For convenience’s sake, we denote the two dimensional torus $D = D_\epsilon = (0, \epsilon] \times (0, \epsilon]$. The function spaces we work with are

$$
H = \{u \in L^2(\Omega) | \nabla \cdot u = 0, \int_\Omega u = 0\}
$$

and $V = H \cap W^{1,2}(\Omega)$. It is well known that $\|\nabla u\|_{L^2}$ is an equivalent norm for $V$ due to the Poincaré inequality. For convenience’s sake, we also denote

$$
\| \cdot \|_L = \| \cdot \|_p, \quad \| \cdot \|_2 = \| \cdot \|_L, \quad \| \cdot \|_{L^p(0,\infty;L^q(\Omega))} = \| \cdot \|_{p,q},
$$

the Leray projection on $L^2(\Omega)$ into $H$ by $P$, and the Stokes operator by $A = P(-\Delta)$. We define the bilinear form $B(u,v) = P(u \cdot \nabla)v$ and the trilinear form $b(u,v,w)$ by

$$
b(u,v,w) = \langle B(u,v), w \rangle = \int_\Omega B(u,v) \cdot wdx.
$$
We now define an orthogonal projection $M$ on $L^2(\Omega)$ by
\begin{equation}
Mu = \frac{1}{\epsilon^2} \int_0^\epsilon \int_0^\epsilon u(x_1, x_2, x_3) dx_1 dx_2
\end{equation}
and denote $v \equiv Mu$ and $w \equiv (I - M)u$ for simplicity. Note that the above projection is different from the one in [6]. Here, $v = v(x_3)$ and $\nabla \cdot v = 0$. So, $v_3$ must be a constant in space. Since we assume (3) from the first, we then get
\begin{equation}
v_3 = \frac{1}{|\Omega|} \int v_3 = \frac{1}{|\Omega|} \int u_3 = 0.
\end{equation}
It is clear that the following Poincaré inequality holds for $w \in H^1$ since $Mw = 0$:
\begin{equation}
\|w\|_2^2 \leq C \epsilon^2 \|\nabla w\|^2.
\end{equation}
Further, $w$ satisfies the following inequalities, which are basically Gagliardo-Nirenberg inequalities.

**Lemma 2.1.** Given $u \in V \cap D(A)$, let $v = Mu$ and $w = (I - M)u$. We have
\begin{equation}
\|\nabla v\|_\infty \leq \frac{C}{\epsilon} \|\nabla v\|^{1/2} \|Av\|^{1/2},
\end{equation}
\begin{equation}
\|\nabla w\|_q \leq C(\|\nabla_3 w\| + \|w\|^{1/2} \|\nabla w\|^{3/q}), \quad 1 < q \leq 3.
\end{equation}
Here, all $C$’s are independent of $\epsilon$.

**Proof.** Since $w(\cdot, x_3)$ is average zero on $D$ for any $x_3 \in (0, l]$, $w$ satisfies the following two dimensional Gagliardo-Nirenberg inequality.
\begin{equation}
\|\nabla w\|_{L^q(D)}^q \leq C \|\nabla^2 w\|^{q-1}_{L^2(D)} \|w\|_{L^2(D)}.
\end{equation}
Here, $C$ is independent from $\epsilon$. In fact, the above inequality is scaling invariant. Integrating with respect to $x_3$, we have
\begin{equation}
\int_0^l dx_3 \int_D |\nabla w|^q \leq C \int_0^l dx_3 \left( \int_D |\nabla^2 w|^2 \right)^\frac{q-1}{2} \sup_{x_3} \|w\|_{L^2(D)}(x_3).
\end{equation}
While,
\begin{equation}
\|w\|_{L^2(D)}^2(b) \leq \int_a^b dx_3 \|\partial_3 \|w\|^2_{L^2(D)}(x_3) + \|w\|_{L^2(D)}^2(a)
\end{equation}
\begin{equation}
\leq \int_a^b \int_D |\partial_3 w||w|dx + \|w\|_{L^2(D)}^2(a)
\end{equation}
\begin{equation}
\leq \|\nabla w\|_q \|w\| + \|w\|_{L^2(D)}^2(a).
\end{equation}
Integrating the above with respect to $a$ over $(0, l]$, we have
\begin{equation}
\sup_{x_3} \|w\|_{L^2(D)}^2 \leq \|\nabla w\|_q \|w\| + \frac{1}{l} \|w\|^2.
\end{equation}
Plugging the above into (12) and using the Hölder inequality, we have
\[ \int_\Omega |\nabla w|^q \leq C \|\nabla^2 w\|^{q-1}(\|\nabla_3 w\| \|w\| + \|w\|^2)^{1/2}. \]
Since \( \|\nabla^2 w\| \leq C\|Aw\| \), we have the desired inequality (11). Similarly,
\[ (\partial_3 v_i)^2(b) = 2 \int_a^b dx_3 \partial_3^2 v_i \partial_3 v_i + (\partial_3 v_i)^2(a). \]

There exists \( a \) such that \( \partial_3 v_i(a) = 0 \) since \( \partial_3 v_i \) is average zero. Thus we have
\[ (\nabla v_i)^2(b) \leq C \int dx_3 |\nabla^2 v||\nabla v| \leq \frac{C}{\epsilon^2} \|\nabla v|||Av||. \]
Taking supremum with respect to \( b \) and adding them up for \( i = 1, 2 \), we have the desired result (10).

We now present the following estimates concerning the trilinear form \( b \). We use the above lemma with \( q = 3 \) to get the estimates.

**Lemma 2.2.** Let \( v \) and \( w \) be as before, we have
\[ |b(w, w, Aw)| \leq C\|w\|^{1/2}\|\nabla w\|^{1/2}\|Aw\|^2, \]
\[ |b(v, w, Aw)|, |b(w, v, Aw)|, |b(w, w, Av)| \leq C\|\nabla v\|^{1/2}\|Av\|^{1/2}\|w\|^{1/2}\|Aw\|^{3/2}. \]

Here, all \( C \)'s are independent from \( \epsilon \).

**Proof.** First, by integration by parts,
\[ b(v, w, Aw) = - \int (w \cdot \nabla) w \cdot \Delta w = \int (\nabla_j w \cdot \nabla) w \cdot \nabla_j w + w \cdot \nabla(\nabla_j w) \nabla_j w = \int (\nabla_j w \cdot \nabla) w \cdot \nabla_j w. \]
Thus, using (11) with \( q = 3 \), (9), and the smallness of \( \epsilon \),
\[ |b(w, w, Aw)| \leq C\|\nabla w\|^3 \leq C\|Aw\|^2(\|\nabla_3 w\|^{1/2}\|w\|^{1/2} + \|w\|) \leq C\|Aw\|^2\|\nabla w\|^{1/2}\|w\|^{1/2}. \]
By similar argument,
\[ b(v, w, Av) = \int (\nabla_j v \cdot \nabla) w \cdot \nabla_j w. \]
Then, since \( v \) depends only on \( x_3 \),
\[ |b(v, w, Av)| \leq \int_0^l dx_3 |\nabla v| \int_0^l |\nabla w|^2 \leq C\|\nabla v\|_{L^\infty(0,l)}\|\nabla w\|^2 \leq C\|\nabla v\|^{1/2}\|Av\|^{1/2}\|\nabla w\||Av|| \leq C\|\nabla v\|^{1/2}\|Av\|^{1/2}\|w\|^{1/2}\|Av\|^{3/2}. \]
Here, we used in the last line the interpolation inequality
\begin{equation}
\|\nabla f\|^2 = -\int f \Delta f \leq \|f\| \|Af\|.
\end{equation}

Similarly,
\begin{align*}
|b(w, v, Aw)| &\leq \int_0^t dx |\nabla v||w||_{L^2(D)}||Aw||_{L^2(D)} \\
&\leq C \|\nabla v\|_{L^\infty(0,t)} \|w\| \|Aw\| \\
&\leq C \|\nabla v\|^{1/2} \|Av\|^{1/2} \|w\| \|Aw\|^{3/2},
\end{align*}
\begin{align*}
|b(w, w, Av)| &\leq \left| \int \nabla_j w \cdot \nabla v + w \cdot \nabla_j \nabla_j v \right| \\
&\leq C \|\nabla v\|^{1/2} \|Av\|^{1/2} \|w\|^{1/2} \|Aw\|^{3/2}.
\end{align*}

\section{3. Regularity}

In this section, we give our main result. We first reformulate (1)-(2) in the standard nonlinear evolutionary equation on the Hilbert space $V$,
\begin{equation}
u_t + \nu Au + B(u, u) = Pf.
\end{equation}
We shall consider solutions of (16) with the initial data $u_0$ and $f = f(t)$ in the class
\begin{equation}
\begin{aligned}
&u_0 \in V, \\
&f(t) \in L^p([0, \infty), H), \; p \geq 2.
\end{aligned}
\end{equation}

We first present the following theorem, which is simple and shows the underlying idea of our result.

\textbf{Theorem 3.1.} Given any $p \geq 2$, the Navier-Stokes evolutionary equation (16) has a solution
\begin{equation}
\begin{aligned}
u \in C^0([0, \infty), H) \cap L^\infty((0, \infty), V)
\end{aligned}
\end{equation}

if
\begin{equation}
\|u_0\| \|\nabla u_0\| + 2\nu^{-\frac{2p-3}{p-2}} \lambda_1^{-\frac{2p-4}{p-2}} \|f\|_{p,2}^2 \leq \frac{\nu^2}{C^2}.
\end{equation}

Here, $\lambda_1$ is the first eigenvalue of the Stokes operator, $C$ is an absolute constant independent of $\epsilon$. Moreover, in this case
\begin{equation}
\|\nabla u\|^2(t) \leq \|\nabla u_0\|^2 + 4\nu^{-\frac{2p-3}{p-2}} \lambda_1^{-\frac{2p-2}{p-2}} \|f\|_{p,2}^2
\end{equation}
for all $t > 0$.

\textbf{Proof.} By taking the scalar product of (16) with $u$ and using the fact that
\begin{equation}
\int B(u, u) dx = 0,
\end{equation}
we find that
\begin{equation}
\frac{d}{dt} ||u||^2 + 2\nu \|\nabla u\|^2 \leq 2 \|f\| \|u\|.
\end{equation}
Therefore, by the Grönwall inequality,

\[ b \nu \| \nabla u \|^2 \leq 2 \int f \| u \| \| A u \| \]  

(21)

Now, we multiply (20) by \( \| \nabla u \|^2 \) and (21) by \( \| u \|^2 \) and adding them to have

\[ \frac{d}{dt} (\| u \|^2 \| \nabla u \|^2) + 2 \nu \| \nabla u \|^4 + 2 \nu \| u \|^2 \| A u \|^2 \]

(22)

\[ \leq 2 \| f \| \| u \| (\| \nabla u \|^2 + \| u \| \| A u \|) + C(\| u \| \| \nabla u \|^4/2 \| u \|^2 \| A u \|^2). \]

By the Young inequality and (15), we have

\[ 2 \| f \| \| u \| (\| \nabla u \|^2 + \| u \| \| A u \|) \leq 4 \| f \| \| u \|^2 \| A u \|
\]

\[ \leq 4 \| f \| \| \nabla u \|^4 \| A u \|/\lambda_1^{1/2} \| u \|^{1/2} \| u \| \| A u \|
\]

\[ \leq \nu \| u \|^2 \| A u \|^2 + \frac{4}{\nu \lambda_1^{1/2}} \| f \|^2 \| u \| \| \nabla u \|. \]

Denoting \( G^2 = \| u \|^2 \| \nabla u \|^2 \), we thus arrive at

\[ \frac{d}{dt} G^2 + \nu \lambda_1 G^2 \leq \left[ CG^{1/2} - \nu \right] \| u \|^2 \| A u \|^2 + \frac{4}{\nu \lambda_1^{1/2}} \| f \|^2 G. \]

If \( G(t) \leq \frac{\nu}{\nu - \nu} \) for all \( t > 0 \),

\[ \frac{d}{dt} G + \nu \lambda_1 G \leq \frac{2}{\nu \lambda_1^{1/2}} \| f \|^2. \]

Therefore, by the Grönwall inequality,

\[ G \leq G(0) e^{-\nu \lambda_1 t} + \frac{2}{\nu \lambda_1^{1/2}} \int_0^t \| f \|^2(s) e^{\nu \lambda_1 (s-t)} ds \]

\[ \leq G(0) + \frac{2}{\nu \lambda_1^{1/2}} \| f \|_{L^2}^2 (p - 2)/p \]

\[ \leq G(0) + 2 \left( \frac{p - 2}{p} \right) \nu^{2 - \frac{2}{p}} \lambda_1^{-\frac{2(p - 2)}{p}} \| f \|_{L^2}^2. \]
for any $p \geq 2$. Note that the above estimate holds true even for $p = 2$ and $\infty$. Since $((p - 2)/p)^{(p-2)/p} \leq 1$, the typical continuation argument and (18) justifies the above argument and we indeed have $G(t) \leq \frac{\nu^2}{4C^2}$ for all $t > 0$.

Furthermore, if $G(t) \leq \frac{\nu^2}{4C^2}$, we apply the Hölder inequality to (21) to have

$$
\frac{d}{dt} \|\nabla u\|^2 + \frac{\nu}{2} \frac{\lambda_1}{\nu} \|\nabla u\|^2 \leq \frac{2}{\nu} \|f\|^2.
$$

Again, by the Grönwall inequality,

$$
\|\nabla u\|^2(t) \leq \|\nabla u_0\|^2 + \int_0^t \frac{\nu}{2} \|f\|^2(s) e^{\nu \lambda_1 (s-t)/2} ds,
$$

which gives (19) and finishes the proof. □

The condition (18) is in a sense a condition of smallness of the initial data and external force. However, this condition allows for initial data with large $H^1$ norm provided that the $L^2$ norm of the initial data and $f$ are small enough. In particular, when $f = 0$, the above theorem tells that there exists a globally regular solution if $\|u_0\|$ is small enough compared with $\nu^2/\|\nabla u_0\|^{-1}$. As a corollary of the above theorem, we have the following.

**Corollary 3.2.** There exists a globally regular solution if initial data satisfies (5) and (6) with $\epsilon = \epsilon$ when $\Omega = [0,l] \times [0,\epsilon] \times [0,\epsilon]$.

Applying the projections $M$ and $(I - M)$ to the equation (16) and using $MB(v,v) = B(v,v) = 0$ and $MB(v,w) = MB(w,v) = 0$, we get the equation for $v$,

$$
\frac{dv}{dt} + \nu Av = Mf - MB(w,w),
$$

and the equation for $w$,

$$
\frac{dw}{dt} + \nu Aw = (I - M)f - B(v,v) - B(v,w) - (I - M)B(w,w).
$$

**Theorem 3.3.** There exists a globally regular solution $u$ of (16) in $\Omega$ if, for some constant $C$,

$$
\|\nabla u_0\|^2 + 2\nu \frac{2^{\frac{n-2}{p}}}{p} \lambda_1 \frac{2^{\frac{n-2}{p}}}{p} \|f\|^2 < \frac{\nu^2}{4C^2} \epsilon^{-1}.
$$

**Proof.** We start from (21). By (9), (21) becomes

$$
\frac{d}{dt} \|\nabla u\|^2 + 2\nu \|A u\|^2 \leq 2\|f\| \|A u\| + C(\epsilon \|\nabla w\| \|\nabla u\|)^{1/2} \|A u\|^2.
$$

Then,

$$
\frac{d}{dt} \|\nabla u\|^2 + (\nu - C \epsilon^{1/2}) \|\nabla u\|^2 \|A u\|^2 \leq \frac{1}{\nu} \|f\|^2
$$

since $\|\nabla u\| \geq \|\nabla w\|$. Now, we apply the Grönwall lemma to the above inequality with typical smallness argument. That is, since $\|\nabla u_0\| < \frac{\nu}{2C} \epsilon^{-1/2}$
from (25), if \( \| \nabla u \| (t) > \frac{\nu}{2C} \epsilon^{-1/2} \) for some \( t > 0 \), there would be the first time \( t = T \) such that \( \| \nabla u \| (T) = \frac{\nu}{2C} \epsilon^{-1/2} \). However, for \( 0 < t < T \),
\[
\frac{d}{dt} \| \nabla u \| ^2 + \frac{\nu}{2} \lambda_1 \| \nabla u \|^2 < \frac{1}{\nu} \| f \|^2.
\]
Applying the Grönwall lemma to the above inequality, we would have
\[
\| \nabla u \| ^2 (T) < \| \nabla u_0 \| ^2 + \frac{1}{\nu} \int_0^T \| f \|^2 e^{\frac{\nu}{2} \lambda_1 (s-T)/2} ds
\]
\[
\leq \| \nabla u_0 \| ^2 + \frac{1}{\nu} \left( \frac{2(p-2)}{p \lambda_1 \nu} \right)^{(p-2)/p} \| f \| _{p,2}^2
\]
\[
\leq \| \nabla u_0 \| ^2 + 2 \nu^{-\frac{2p-2}{p}} \lambda_1^{-\frac{2}{p}} \| f \| _{p,2}^2.
\]
If (25) holds true, this leads a contraction. Therefore, \( \| \nabla u \| ^2 < \frac{\nu^2}{2C^2} \epsilon^{-1} \) for all \( t > 0 \) and we finish the proof. \( \square \)

Clearly, the condition (25) in particular implies that there exists a globally regular solution if, for suitable \( C \),
\[
\| \nabla u_0 \| \leq C \nu \epsilon^{-1/2}, \quad \| f \| _{p,2} \leq C \nu^{(2p-1)/p} \epsilon^{-1/2}
\]
since \( \lambda_1 = \frac{\lambda_2}{\nu} \) is fixed.

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