HOLONOMY DISPLACEMENTS IN THE HOPF BUNDLES OVER $\mathbb{C}H^n$ AND THE COMPLEX HEISENBERG GROUPS

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Abstract. For the “Hopf bundle” $S^1 \to S^{2n+1} \to \mathbb{C}H^n$, horizontal lifts of simple closed curves are studied. Let $\gamma$ be a piecewise smooth, simple closed curve on a complete totally geodesic surface $S$ in the base space. Then the holonomy displacement along $\gamma$ is given by

$$V(\gamma) = e^{A(\gamma)\lambda}$$

where $A(\gamma)$ is the area of the region on the surface $S$ surrounded by $\gamma$; $\lambda = 1/2$ or 0 depending on whether $S$ is a complex submanifold or not.

We also carry out a similar investigation for the complex Heisenberg group $\mathbb{R} \to \mathbb{H}^{2n+1} \to \mathbb{C}^n$.

1. Introduction

Consider the Hopf fibration $S^1 \to S^3 \to S^2$. Let $\gamma$ be a simple closed curve on $S^2$. Pick a point in $S^3$ over $\gamma(0)$, and take the unique horizontal lift $\tilde{\gamma}$ of $\gamma$. Since $\gamma(1) = \gamma(0)$, $\tilde{\gamma}$ lies in the same fiber as $\tilde{\gamma}(0)$ does. We are interested in understanding the difference between $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$. The following equality was already known [2]:

$$V(\gamma) = e^{A(\gamma)\lambda}$$

where $V(\gamma)$ is the holonomy displacement along $\gamma$, and $A(\gamma)$ is the area of the region surrounded by $\gamma$.

In this paper, we shall generalize this fact to (higher dimensional) pseudo-spheres and the complex Heisenberg group. First we look at the fibration of the pseudo-sphere $S^{2n+1}$

$$S^1 \to S^{2n+1} \to \mathbb{C}H^n$$

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a principal $S^1$-bundle over the complex hyperbolic space $\mathbb{C}H^n$. Let $S$ be a complete totally geodesic surface in the base space $\mathbb{C}H^n$, and $\xi$s be the pullback bundle over $S$. Let $\gamma$ be a piecewise smooth, simple closed curve on $S$ parametrized by $0 \leq t \leq 1$, and $\tilde{\gamma}$ its horizontal lift. The pullback over the curve $\gamma$ is called a Hopf torus (so $\tilde{\gamma}$ is a curve on the Hopf torus). Then

$$\tilde{\gamma}(1) = e^{\frac{1}{2}A(\gamma)}\cdot \tilde{\gamma}(0) \quad \text{or} \quad \tilde{\gamma}(0),$$

depending on whether $S$ is a complex submanifold or not, where $A(\gamma)$ is the area of the region on the surface $S$ surrounded by $\gamma$ (See Theorem 3.3).

We also carry out a similar investigation for the complex Heisenberg group. Let $1 \to \mathbb{R} \to \mathcal{H}^{2n+1} \to \mathbb{C}^n \to 1$ be the central short exact sequence of the complex Heisenberg group. Let $S$ be a complete totally geodesic plane in $\mathbb{C}^n$, and $\xi$s be the pullback bundle over $S$. Let $\gamma$ be a piecewise smooth, simple closed curve on $S$. Then

$$V(\gamma) = e(\xi_s) \cdot A(\gamma),$$

where $A(\gamma)$ is the area of the region on the surface $S$ surrounded by $\gamma$, and the number $e(\xi_s)$ is determined by the equality $[v, w] = e(\xi_s)v_{2n+1}$ for an orthonormal basis $\{v, w\}$ for the tangent space of $S$ (See Theorem 4.1).

2. Preliminaries

The proof of the statement in the introduction (in the case of the Hopf fibration $S^1 \to S^3 \to S^2$) uses the Gauss-Bonnet theorem. For $S^1 \to S^{2,1} \to \mathbb{C}H^1$, such is not available because the base space is not compact. Therefore, we cannot apply the arguments in [2] directly, and need to develop a new method of proof. It turns out that $S^{2,1}$ is the building blocks for higher dimensional cases.

Let $F \to E \xrightarrow{p} B$ be a principal $F$-bundle ($F = \mathbb{R}^1$ or $S^1$) of Riemannian manifolds, with $B$ a 2-dimensional complete manifold and $p$ a Riemannian submersion. For a simple closed curve $\gamma(t), 0 \leq t \leq 1$ on $B$, the holonomy displacement $V(\gamma)$ along $\gamma$ is defined as follows: Let $\tilde{\gamma}(t)$ be the horizontal lift of $\gamma$. Then

$$\tilde{\gamma}(1) = V(\gamma) \cdot \tilde{\gamma}(0)$$

for some $V(\gamma) \in F$. We shall establish a technical lemma which will be used later.

**Lemma 2.1.** Suppose $V(\gamma) = \lambda(\gamma)A(\gamma) (F = \mathbb{R}^1)$, or $e^{\lambda(\gamma)A(\gamma)} (F = S^1)$ for a constant $\lambda(\gamma)$, where $A(\gamma)$ is the area of the region on $B$ surrounded by a piecewise smooth simple closed curve $\gamma$. If $\lambda(\gamma)$ is constant for all $\gamma$'s which are the boundaries of rectangular regions, then it is constant for every piecewise smooth simple closed curve $\gamma$. 
Proof. Let us assume that \( F = \mathbb{R} \). The case of \( F = S^1 \) will be similar. Let \( \gamma_0 \) be a curve on \( B \). Since the region surrounded by \( \gamma_0 \) is compact, we may assume that this region is contained completely in one local patch. Let 
\[
\varphi : \mathbb{R}^2 \to U \subset B
\]
be a local chart, and \( p^{-1}(U) \approx U \times F \). For notational simplicity, we shall identify \( \mathbb{R}^2 \) with \( U \) (and suppress \( \varphi \)). Let \( \Omega(U, \gamma_0(0)) \) and \( \Omega(\mathbb{R}, 0) \) be the space of paths emanating from \( \gamma_0(0) \) and 0 \( \in \mathbb{R} \), respectively. For each \( \gamma \in \Omega(U, \gamma_0(0)) \), let \( \omega_\gamma \) be the unique curve in \( \mathbb{R} \) so that \( \eta(t) = \gamma(t) \cdot \omega_\gamma(t) \) is the horizontal lift of \( \gamma \). This defines a map 
\[
3 : \Omega(U) \to \Omega(\mathbb{R})
\]
by \( 3(\gamma)(t) = \omega_\gamma(t) \). We use the sup metrics \( \rho \) on both \( \Omega(U) \) and \( \Omega(\mathbb{R}) \). That is, 
\[
\rho(\gamma_1, \gamma_2) = \sup_{t \in [0,1]} d(\gamma_1(t), \gamma_2(t)),
\]
where \( d \) is the distance function on \( U \). A similar definition holds for \( \Omega(\mathbb{R}) \). We wish to show that \( 3 \) is continuous at \( \gamma_0 \). Let \( \epsilon > 0 \) be given. By the continuity of the connection, for each \( t \in [0,1] \), there is an open neighborhood \( W \) of \( \gamma(t) \) such that any piecewise smooth curve in \( W \) has a horizontal lift which lies in \( W \times (-\epsilon/2, \epsilon/2) \). Since \( \gamma(0) \) is compact, we can find \( \delta > 0 \) such that if \( \rho(\gamma_0, \gamma) < \delta \), then \( \rho(\omega_{\gamma_0}, \omega_\gamma) < \epsilon \). This proves that \( 3 \) is continuous.

Any piecewise smooth simple closed curve can be approximated by a sequence of piecewise linear curves which are sums of boundaries of rectangular regions. Since \( \lambda(\gamma) \) is constant for rectangular regions, the same is true for any piecewise smooth simple closed curve. \( \square \)

Next, we need to know all complete totally geodesic submanifolds of the base space of the principal bundle \( S^1 \to S^{2n,1} \to \mathbb{C}H^n \). Since \( S^{2n,1} \) is a symmetric space, the following gives a complete answer.

**Proposition 2.2** ([1, XI Theorem 4.3]). Let \((G, H, \sigma)\) be a symmetric space and \( g = h + m \) the canonical decomposition. Then there is a natural one-to-one correspondence between the set of linear subspaces \( m' \) of \( m \) such that \([m', m'], m' \subset m' \) and the set of complete totally geodesic submanifolds \( M' \) through the origin \( 0 \) of the affine symmetric space \( M = G/H \), the correspondence being given by \( m' = T_0(M') \).

3. The bundle \( S^1 \to S^{2n,1} \to \mathbb{C}H^n \)

We shall study the bundle 
\[
U(1) \to U(1,n)/U(n) \to U(1,n)/(U(1) \times U(n)).
\]
Note that \( U(1,n)/U(n) \cong S^{2n-1} \), and \( U(1,n)/(U(1) \times U(n)) \cong \mathbb{C}H^n \), where \( S^{2n-1} = H^{1,2n} = \{(z_0, \ldots, z_n) \in \mathbb{C}^n : -|z_0|^2 + \sum_{i=1}^n |z_i|^2 = -1\} \). For more
information on $S^{2n,1}$, see [3]. We first consider the case when $n = 1$. Rather than using $U(1) \to U(1,1)/U(1) \to U(1,1)/(U(1) \times U(1))$, we shall use

\[ U(1) \to SU(1,1) \to SU(1,1)/U(1). \]

Here $SU(1,1) = \{ A \in GL(2, \mathbb{C}) : AJA^* = J \text{ and } \det(A) = 1 \}$ where $J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

From now on, we shall use the convention of $\mathfrak{gl}(n, \mathbb{C}) \subset \mathfrak{gl}(2n, \mathbb{R})$ by

\[
\begin{bmatrix}
z_{11} & z_{12} \\
z_{21} & z_{22}
\end{bmatrix} \mapsto
\begin{bmatrix}
x_{11} & -y_{11} & x_{12} & -y_{12} \\
y_{11} & x_{11} & y_{12} & x_{12} \\
x_{21} & -y_{21} & x_{22} & -y_{22} \\
y_{21} & x_{21} & y_{22} & x_{22}
\end{bmatrix}.
\]

The group $SU(1,1)$ has the following natural representation into $GL(4, \mathbb{R})$:

\[
w = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \\
-w_2 & w_1 & -w_3 & w_4 \\
w_3 & -w_4 & w_1 & -w_2 \\
w_4 & w_3 & w_2 & w_1
\end{bmatrix}
\]

with the condition $w_1^2 + w_2^2 - w_3^2 - w_4^2 = 1$. In fact, the map

\[ w_1 + w_2 i + w_3 j + w_4 k \mapsto w \]

is a monomorphism from the unit quaternions into $GL(4, \mathbb{R})$. Therefore,

$SU(1,1) \cong S^{2,1}$.

The circle group

$S^1 = \left\{ \begin{bmatrix} e^{iz} & 0 \\ 0 & e^{-iz} \end{bmatrix} : 0 \leq z \leq 2\pi \right\}$

is a subgroup of $SU(1,1)$, and acts on $SU(1,1)$ as right translations, freely with quotient $CH^1$, the complex hyperbolic line, giving rise to the fibration

$S^1 \to SU(1,1) \to CH^1$.

In order to understand the projection map better, let $\tilde{w}$ be the “$i$-conjugate” of $w$ (replace $w_2$ by $-w_2$). That is,

\[
\tilde{\tilde{w}} = \begin{bmatrix} w_1 - w_2 & w_3 & w_4 \\
w_2 & w_1 - w_4 & w_3 \\
w_3 & -w_4 & w_1 - w_2 \\
w_4 & w_3 & -w_2 & w_1
\end{bmatrix}.
\]

Then,

\[
w\tilde{\tilde{w}} = \begin{bmatrix}
w_1^2 + w_2^2 + w_3^2 + w_4^2 & 0 & 2(w_1 w_3 - w_2 w_4) & 2(w_2 w_3 + w_1 w_4) \\
0 & w_1^2 + w_2^2 + w_3^2 + w_4^2 & -2(w_2 w_3 + w_1 w_4) & 2(w_1 w_3 - w_2 w_4) \\
2(w_1 w_3 - w_2 w_4) & -2(w_2 w_3 + w_1 w_4) & w_1^2 + w_2^2 + w_3^2 + w_4^2 & 0 \\
2(w_2 w_3 + w_1 w_4) & 2(w_1 w_3 - w_2 w_4) & 0 & w_1^2 + w_2^2 + w_3^2 + w_4^2
\end{bmatrix}
\]

and

\[(w_1^2 + w_2^2 + w_3^2 + w_4^2)^2 - (2w_1 w_3 - 2w_2 w_4)^2 - (2w_2 w_3 + 2w_1 w_4)^2 = 1.\]
Clearly, $\mathbb{C}H^1$ can be identified with the following

$$\mathbb{C}H^1 = \left\{ \begin{pmatrix} x & 0 & y & z \\ 0 & x & -z & y \\ y & -z & x & 0 \\ z & y & 0 & x \end{pmatrix} : x^2 - y^2 - z^2 = 1, \; x > 0 \right\}.$$ 

Therefore, the map

$$p: SU(1,1) \longrightarrow \mathbb{C}H^1$$

defined by $p(w) = w\tilde{w}$ has the following properties:

$$p(wv) = wp(v)\tilde{w}$$ \quad for all $w, v \in SU(1,1)$,

$$p(wv) = p(w)$$ \quad if and only if \quad $v \in S^1$.

This shows that the map $p$ is, indeed, the orbit map of the principal bundle $S^1 \to SU(1,1) \to \mathbb{C}H^1$. The Lie group $SU(1,1)$ will have a left-invariant Riemannian metric given by the following orthonormal basis on the Lie algebra $\mathfrak{su}(1,1)$

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$ 

Notice that $e_1$ and $e_2$ correspond to $[0 \; 1]$ and $[0 \; i]$ in $\mathfrak{gl}(2,\mathbb{C})$ and $[e_1, e_2] = -2e_3$. Consider the subset of $SU(1,1)$:

$$T = \left\{ \begin{pmatrix} \cosh x & (\sinh x)e^{-iy} \\ (\sinh x)e^{iy} & \cosh x \end{pmatrix} : x \geq 0, \; 0 \leq y \leq 2\pi \right\}$$

$$= \left\{ \begin{pmatrix} \cosh x & 0 & (\sinh x)(\cos y) & (\sinh x)(\sin y) \\ 0 & \cosh x & - (\sinh x)(\sin y) & (\sinh x)(\cos y) \\ (\sinh x)(\cos y) & -(\sinh x)(\sin y) & \cosh x & 0 \\ (\sinh x)(\sin y) & (\sinh x)(\cos y) & 0 & \cosh x \end{pmatrix} \right\}$$ 

which is the exponential image of

$$m = \left\{ \begin{pmatrix} 0 & \xi^* \\ \xi & 0 \end{pmatrix} : \xi \in \mathbb{C} \right\}.$$ 

Of course, $SU(1,1)$ is topologically a product $S^1 \times \mathbb{C}H^1$. The map $p$ restricted to $T$ is just the squaring map; that is,

$$p(w) = w^2, \quad w \in T.$$ 

**Theorem 3.1.** Let $S^1 \to SU(1,1) \to \mathbb{C}H^1$ be the natural fibration. Let $\gamma$ be a piecewise smooth, simple closed curve on $\mathbb{C}H^1$. Then the holonomy displacement along $\gamma$ is given by

$$V(\gamma) = e^{\frac{1}{2}A(\gamma)i} \in S^1,$$

where $A(\gamma)$ is the area of the region on $\mathbb{C}H^1$ enclosed by $\gamma$. 
Proof. Let $\gamma(t)$ be a closed loop on $\mathbb{C}H^1$ with $\gamma(0) = p(I_4)$. Therefore,

$$\gamma(t) = \begin{bmatrix} \cosh 2x(t) & 0 & \sinh 2x(t) \cos y(t) & \sinh 2x(t) \sin y(t) \\ 0 & \cosh 2x(t) & -\sinh 2x(t) \sin y(t) & \sinh 2x(t) \cos y(t) \\ \sinh 2x(t) \cos y(t) & -\sinh 2x(t) \sin y(t) & \cosh 2x(t) & 0 \\ \sinh 2x(t) \sin y(t) & \sinh 2x(t) \cos y(t) & 0 & \cosh 2x(t) \end{bmatrix}.$$ 

Let

$$\tilde{\gamma}(t) = \begin{bmatrix} \cosh x(t) & 0 & \sinh x(t) \cos y(t) & \sinh x(t) \sin y(t) \\ 0 & \cosh x(t) & -\sinh x(t) \sin y(t) & \sinh x(t) \cos y(t) \\ \sinh x(t) \cos y(t) & -\sinh x(t) \sin y(t) & \cosh x(t) & 0 \\ \sinh x(t) \sin y(t) & \sinh x(t) \cos y(t) & 0 & \cosh x(t) \end{bmatrix}$$

with $x(t) \geq 0$ so that $p(\tilde{\gamma}(t)) = \gamma(t)$ ($\tilde{\gamma}$ is a lift of $\gamma$), and let

$$\omega(t) = \begin{bmatrix} \cos z(t) & -\sin z(t) & 0 & 0 \\ \sin z(t) & \cos z(t) & 0 & 0 \\ 0 & 0 & \cos z(t) & \sin z(t) \\ 0 & 0 & -\sin z(t) & \cos z(t) \end{bmatrix}.$$ 

Put $\eta(t) = \tilde{\gamma}(t) \cdot \omega(t)$. We still have $p(\eta(t)) = \gamma(t)$, and $\eta$ is another lift of $\gamma$. We wish $\eta$ to be the horizontal lift of $\gamma$. That is, we want $\eta'(t)$ to be orthogonal to the fiber at $\eta(t)$. The condition is that $\langle \eta'(t), (\ell_{\eta(t)})(e_3) \rangle = 0$, or equivalently, $\langle (\ell_{\eta(t)}^{-1})^* \eta'(t), e_3 \rangle = 0$. That is,

$$\eta(t)^{-1} \cdot \eta'(t) = \alpha_1 e_1 + \alpha_2 e_2$$

for some $\alpha_1, \alpha_2 \in \mathbb{R}$. From this, we get the following equation:

(3–1) \quad \quad \quad z'(t) = \sinh^2 x(t) y'(t).

By virtue of Lemma 2.1, it will be enough to prove the statement for a particular type of curves as follows: Suppose we are given a rectangular region in the $xy$-plane

$$p \leq x \leq p + a, \quad q \leq y \leq q + b.$$ 

Consider the image $R$ of this rectangle in $\mathbb{C}H^1$ by the map

$$(x, y) \mapsto r(x, y) = (\cosh 2x, (\sinh 2x) \cos y, (\sinh 2x) \sin y),$$

$\mathbb{C}H^1$ with the $(+- -)$ metric. The area of $R$ can be calculated as follows:

$$r_x \times r_y = ((2 \cosh 2x)(\sinh 2x), -(2 \sinh^2 2x)(\cos y), -(2 \sinh^2 2x)(\sin y)).$$

Now

$$||r_x \times r_y|| = 2|\sinh 2x|, \quad (+ - -)-norm$$

$$= 2 \sinh 2x \quad \text{(because } x \geq 0).$$

Thus, the area is

$$\int_q^{q+b} \int_p^{p+a} 2 \sinh 2x \, dx \, dy = \left[ (2 \sinh^2 x)_p^{p+a} \right]_q^{q+b} = 2b(\sinh^2 (p + a) - \sinh^2 (p)).$$

On the other hand, the change of $z(t)$ along the boundary of this region can be calculated using condition (3–1). Label the four vertices by $A(p,q)$,
\(B(p + a, q), C(p + a, q + b),\) and \(D(p, q + b).\) AB can be parametrized by \(x(t) = p + at, y(t) = q, t \in [0, 1]\) so that \(y'(t) = 0.\) For BC, \(x(t) = p + a, y(t) = q + bt, t \in [0, 1].\) Then
\[
z(1) - z(0) = \int_0^1 z'(t) dt = \int \sinh^2(p + a) b dt = b \cdot \sinh^2(p + a).
\]
Similarly, \(z(t)\) does not change along CD, but on DA, \(x(t) = p, y(t) = q + b - bt, t \in [0, 1].\) So
\[
z(1) - z(0) = \int_0^1 z'(t) dt = \int \sinh^2(p) (-b) dt = -b \cdot \sinh^2(p).
\]
Thus the total vertical change of \(z\)-values, \((z(1) - z(0),\) along the perimeter of this rectangle is \(b \cdot (\sinh^2(p + a) - \sinh^2(p))\) which is \(1/2\) times the area. \(\square\)

Now we turn to the general case
\[
S^1 \longrightarrow S^{2n, 1} \xrightarrow{p} \mathbb{C}H^n.
\]
We are viewing \(S^{2n, 1} \cong U(1, n)/U(n),\) and \(\mathbb{C}H^n \cong U(1, n)/(U(1) \times U(n)).\) The Lie algebra of \(U(1, n)\) is \(\mathfrak{u}(1, n),\) and has the following canonical decomposition: \(\mathfrak{g} = \mathfrak{h} + \mathfrak{m},\) where
\[
\mathfrak{h} = \mathfrak{u}(1) + \mathfrak{u}(n) = \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix} : \lambda + \lambda = 0, B \in \mathfrak{u}(n) \right\}
\]
and
\[
\mathfrak{m} = \left\{ \begin{bmatrix} 0 & \xi^t \\ \xi & 0 \end{bmatrix} : \xi \in \mathbb{C}^n \right\}.
\]

**Lemma 3.2.** A 2-dimensional subspace \(\mathfrak{m}'\) of \(\mathfrak{m} \subset \mathfrak{u}(1, n)\) gives rise to a complete totally geodesic submanifold of \(\mathbb{C}H^n\) if and only if either

1. \(\mathfrak{m}'\) is \(J\)-invariant (i.e., has a complex structure), or
2. \(\mathfrak{m}'\) has tangent vectors \(\mathbf{v}\) and \(\mathbf{w}\) such that \(\nabla \mathbf{w} = \mathbf{v} \times \mathbf{w} = 0.\)

Furthermore, for each of these cases, the pullback of the bundle \(S^1 \rightarrow S^{2n, 1} \rightarrow \mathbb{C}H^n\) by the inclusion is isomorphic to the standard bundle \(S^1 \rightarrow SU(1, 1) \rightarrow \mathbb{C}H^1\) for (1), or the product bundle \(S^1 \times \mathbb{C}H^1\) for (2), respectively.

**Proof.** With the notation \(m\) as above, let \(\mathbf{v}\) and \(\mathbf{w}\) be elements of \(\mathfrak{m}\) whose \(\xi\)'s are given by
\[
\begin{bmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ \vdots \\ x_n + iy_n \end{bmatrix} \quad \text{for } \mathbf{v} \quad \text{and} \quad \begin{bmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ \vdots \\ a_n + ib_n \end{bmatrix} \quad \text{for } \mathbf{w}.
\]
Then by Proposition 2.2, \(\mathfrak{m}'\) is a totally geodesic sub-manifold if and only if \([\mathfrak{m}', \mathfrak{m}'] \subset \mathfrak{m}'\). Some calculations show the following equality
\[
[[\mathbf{v}, \mathbf{w}], \mathbf{v}] = \sum_{k=1}^n (x_k a_k + y_k b_k) \mathbf{v} - \sum_{k=1}^n (x_k^2 + y_k^2) \mathbf{w} - 3 \sum_{k=1}^n (x_k b_k - y_k a_k)(i \mathbf{v})
\]
holds. Therefore, $[[\mathbf{v}, \mathbf{w}], \mathbf{v}] = p\mathbf{v} + q\mathbf{w}$ for some real $p$ and $q$ if and only if $i\mathbf{v} = p\mathbf{v} + q\mathbf{w}$ has solution for some real $p$ and $q$.

Suppose $\mathbf{m}'$ has a complex structure. Then we can take $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{m}'$ so that $i\mathbf{v} = \mathbf{w}$ (so $a_k = -y_k$ and $b_k = x_k$ for all $k = 1, 2, \ldots, n$). Thus, $[[\mathbf{v}, \mathbf{w}], \mathbf{v}] = p\mathbf{v} + q\mathbf{w}$ has a solution for $p$ and $q$. Suppose $\nabla \mathbf{w} - \nabla \mathbf{v} = 2\text{Im}(\nabla \mathbf{w}) = 2\sum_{k=1}^{n}(x_k b_k - y_k a_k) = 0$. Then clearly $[[\mathbf{v}, \mathbf{w}], \mathbf{v}] = p\mathbf{v} + q\mathbf{w}$ has a solution for $p$ and $q$.

Conversely, suppose $[[\mathbf{v}, \mathbf{w}], \mathbf{v}] = p\mathbf{v} + q\mathbf{w}$ has a solution for $p$ and $q$. Then $i\mathbf{v} = p\mathbf{v} + q\mathbf{w}$ must have a real solution for $p$ and $q$. Suppose $\sum_{k=1}^{n}(x_k b_k - y_k a_k) \neq 0$. Then, at least one of the summands is non-zero, say $x_1 b_1 - y_1 a_1 \neq 0$. This means that we can find a new basis for $\text{span}\{\mathbf{v}, \mathbf{w}\}$ with

$$x_1 = 1, \quad y_1 = 0; \quad a_1 = 0, \quad b_1 = 1.$$ 

Then the equation $p\mathbf{v} + q\mathbf{w} = i\mathbf{v}$ is quickly reduced to $p = 0$ and $q = 1$ (from $k = 1$), and hence we obtain $x_k = b_k$, $y_k = -a_k$ for all $k = 2, \ldots, n$. This shows $\mathbf{w} = i\mathbf{v}$, and the space spanned by $\mathbf{v}$ and $\mathbf{w}$ has a complex structure.

For the second part of the statement, it is enough to observe that

$$[\mathbf{v}, \mathbf{w}] = \left[ \begin{array}{cc} \lambda & 0 \\ 0 & 0 \end{array} \right],$$

where $\lambda = \nabla \mathbf{w} - \nabla \mathbf{v}$. If $\lambda = 0$, then the distribution $\mathbf{m}'$ is integrable, and the bundle is trivial. $\square$

By combining Theorem 3.1 and Lemma 3.2, we have now:

**Theorem 3.3.** Let $S^1 \to S^{2n,1} \to \mathbb{C}H^n$ be the natural fibration. Let $S$ be a complete totally geodesic 2-dimensional surface in $\mathbb{C}H^n$, and $\xi_S$ be the pullback bundle over $S$. Let $\gamma$ be a piecewise smooth, simple closed curve on $S$. Then the holonomy displacement along $\gamma$ is given by

$$V(\gamma) = e^{\frac{i}{2}A(\gamma)i} \text{ or } e^{\alpha_i}\in S^1,$$

where $A(\gamma)$ is the area of the region on the surface $S$ surrounded by $\gamma$, depending on whether $S$ is a complex submanifold or not.

Since the length of $\eta$ is half of the length of $\gamma$, we have:

**Corollary 3.4.** Suppose $\gamma$ is a piecewise smooth, simple closed curve parametrized by arc length. Then the Hopf torus in $S^{2n,1}$ over $\gamma$ is isometric to the torus generated by the lattice $\{(2\pi, 0), (A(\gamma)/2, L(\gamma)/2)\}$ in $\mathbb{R}^2$, where $L(\gamma)$ is the length of $\gamma$.

### 4. The complex Heisenberg group $\mathcal{H}^{2n+1}$

We consider $\mathcal{H}^{2n+1}$. This is $\mathbb{R} \times \mathbb{C}^n$ with group operation given by

$$(s, \mathbf{z})(t, \mathbf{z}') = (s + t + 2\text{Im}\{\mathbf{z}\}, \mathbf{z} + \mathbf{z}'),$$

where $\text{Im}\{\mathbf{z}\}$ is the imaginary part of the complex number $\mathbf{z} = (z_1, z_2, \ldots, z_n)$, $\mathbf{z}' = (z'_1, z'_2, \ldots, z'_n) \in \mathbb{C}^n$. This is a 2-step
nilpotent Lie group with center \( Z(\mathcal{H}^{2n+1}) = \mathbb{R} \). In the case of \( n = 1 \), \( \mathcal{H}^1 \) is isomorphic to the ordinary 3-dimensional Heisenberg group by

\[
(4z - 2xy, x + iy) \mapsto \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.
\]

For the sake of computations, we use the following affine representation of \( \mathcal{H}^{2n+1} \) into \( \text{Aff}(2n+1) \subset \text{GL}(2n+2) \):

\[
\begin{pmatrix}
x_1 + iy_1 \\
x_2 + iy_2 \\
\vdots \\
x_n + iy_n
\end{pmatrix}
\mapsto \begin{bmatrix}
1 & -2y_1 & 2x_1 & \cdots & -2y_n & 2x_n & s \\
0 & 1 & 0 & \cdots & 0 & 0 & x_1 \\
0 & 0 & 1 & \cdots & 0 & 0 & y_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & x_n \\
0 & 0 & 0 & \cdots & 0 & 1 & y_n \\
0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{bmatrix}.
\]

The Lie algebra has a following orthonormal basis

\[
e_1 = \begin{bmatrix}
0 & 0 & 2 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix},
\]

\[
e_2 = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix},
\]

\[
e_{2n} = \begin{bmatrix}
0 & 0 & 0 & \cdots & -2 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix},
\]

\[
e_{2n+1} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix},
\]

which defines a left-invariant Riemannian metric on \( \mathcal{H}^{2n+1} \). The short exact sequence of groups

\[
1 \longrightarrow \mathbb{R} \longrightarrow \mathcal{H}^{2n+1} \xrightarrow{p} \mathbb{C}^n \longrightarrow 1
\]

is a fiber bundle, which is topologically trivial. The left invariant metric naturally induces a connection on this principal \( \mathbb{R} \)-bundle. There is a unique metric on \( \mathbb{C}^n \) (standard Euclidean metric) which makes the projection map \( p \) a Riemannian submersion.
Theorem 4.1. Let $1 \to \mathbb{R} \to \mathcal{H}^{2n+1} \to \mathbb{C}^n \to 1$ be the central short exact sequence of the complex Heisenberg group. Let $S$ be a complete totally geodesic plane in $\mathbb{C}^n$, and $\xi_S$ be the pullback bundle over $S$. Let $\gamma$ be a piecewise smooth, simple closed curve on $S$. Then

$$V(\gamma) = e(\xi_S) \cdot A(\gamma),$$

where $A(\gamma)$ is the area of the region on the surface $S$ surrounded by $\gamma$, and the number $e(\xi_S)$ is determined by the equality $[v, w] = e(\xi_S)e_{2n+1}$ for an orthonormal basis $\{v, w\}$ for the tangent space of $S$.

Proof. Every complete totally geodesic submanifold of $\mathbb{C}^n$ is an $\mathbb{R}$-linear subspace of $\mathbb{C}^n$. Therefore $S = \text{span}\{v, w\}$ for some orthonormal basis $v, w$ where $v = \sum_{j=1}^n (a_j + ib_j)$ and $w = \sum_{j=1}^n (c_j + id_j) \in \mathbb{C}^n$. Then $\gamma$ is of the form

$$\gamma(t) = x(t)v + y(t)w \in S \subset \mathbb{C}^n,$$

where $x(t)$ and $y(t)$ are scalars. We want to find a curve $z(t)$ in $\mathbb{R}$ so that $\eta(t) = (z(t), \gamma(t))$ is orthogonal to the fiber for every $t$. In other words,

$$\langle \eta'(t), (\ell_{\eta(t)})^* e_{2n+1} \rangle = 0,$$

where $\ell$ is the left translation. This is equivalent to $\langle (\ell_{\eta(t)}^{-1})^* \eta'(t), e_{2n+1} \rangle = 0$. Note that $\eta(t)^{-1} = (-z(t) + 2 \text{Im}(\gamma(t)), -\gamma(t))$. Using the affine representation, $(\ell_{\eta(t)}^{-1})^* \eta'(t)$ is

$$\begin{bmatrix}
0 & -2(x'(t)b_1 + y'(t)d_1) & \cdots & z'(t) - 2(x'(t)y'(t) - x'(t)g'(t)) \text{Im} \{\nabla w\} \\
0 & 0 & \cdots & x'(t)\alpha_1 + y'(t)c_1 \\
0 & 0 & \cdots & x'(t)b_1 + y'(t)d_1 \\
0 & 0 & \cdots & \cdot \\
0 & 0 & \cdots & x'(t)\alpha_n + y'(t)c_n \\
0 & 0 & \cdots & x'(t)b_n + y'(t)d_n \\
0 & 0 & \cdots & 0
\end{bmatrix},$$

where $\text{Im} \{\nabla w\} = \sum_{j=1}^n (a_jd_j - c_jb_j)$. Note that $\text{Im} \{\nabla w\} = \text{Im} \{\nabla w'\}$ for any orthonormal basis $\{\nu', \omega'\}$. The equation $\langle (\ell_{\eta(t)}^{-1})^* \eta'(t), e_{2n+1} \rangle = 0$ gives rise to

$$(4.1) \quad z'(t) - 2(x'(t)y'(t) - x'(t)g'(t)) \text{Im} \{\nabla w\} = 0.$$

Suppose we are given a rectangular region on $xy$-plane

$$p \leq x \leq p + a, \quad q \leq y \leq q + b.$$

Consider the image $R$ of this rectangle in $S \subset \mathbb{C}^n$ by the map

$$(x, y) \mapsto x\nu + y\omega.$$  

Then $R$ is a rectangle with vertices $pv + qw$, $(p+a)v + qw$, $(p+a)v + (q+b)w$, $p\nu + (q+b)\omega$. Let $\gamma(t)$ be the piecewise linear boundary curve. It can be represented by $((p+4at)\nu, q\omega)$ for $0 \leq t \leq 1/4$, $((p+a)\nu, (q+b(4t-1))\omega)$ for
$1/4 \leq t \leq 1/2$, $((p+a(3-4t))v, (q+b)w)$ for $1/2 \leq t \leq 3/4$, $(pv, (q+b(4-4t))w)$ for $3/4 \leq t \leq 1$.

Then, from the equation (4–1),

$$z(1) - z(0) = 2 \int_0^1 (x(t)y'(t) - x'(t)y(t)) \text{Im}(\nabla w) dt = 4ab \text{Im}(\nabla v).$$

On the other hand,

$$[v, w] = 4 \begin{bmatrix}
    0 & 0 & 0 & \cdots & 0 & 0 & \text{Im}(\nabla w) \\
    0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
    0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
    \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
    0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
    0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}.$$ 

This means that $[v, w] = 4 \text{Im}(\nabla v)e_{2n+1} = e(\xi_S) e_{2n+1}$ so that $V(\gamma) = e(\xi_S) \cdot A(\gamma)$ with $e(\xi_S) = 4 \text{Im}(\nabla w)$.

Having shown the statement for rectangular regions, now we apply Lemma 2.1 to conclude that the same formula holds for any piecewise smooth, simple closed curve.

**Corollary 4.2.** Suppose $\gamma$ is a piecewise smooth, simple closed curve parametrized by arc-length. Then the Hopf cylinder in $H^{2n+1}$ over $\gamma$ is isometric to the cylinder generated by the translation $(x, y) \mapsto (x + e(\xi_S) A(\gamma), y + L(\gamma))$ on $\mathbb{R}^2$, where $L(\gamma)$ is the length of $\gamma$.

**References**


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