SUBTOURNAMENTS ISOMORPHIC TO $W_5$ OF AN INDECOMPOSABLE TOURNAMENT

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Abstract. We consider a tournament $T = (V, A)$. For each subset $X$ of $V$ is associated the subtournament $T(X) = (X, A \cap (X \times X))$ of $T$ induced by $X$. We say that a tournament $T'$ embeds into a tournament $T$ when $T'$ is isomorphic to a subtournament of $T$. Otherwise, we say that $T$ omits $T'$. A subset $X$ of $V$ is a clan of $T$ provided that for $a, b \in X$ and $x \in V \setminus X$, $(a, x) \in A$ if and only if $(b, x) \in A$. For example, $\emptyset, \{x\} (x \in V)$ and $V$ are clans of $T$, called trivial clans. A tournament is indecomposable if all its clans are trivial. In 2003, B. J. Latka characterized the class $T$ of indecomposable tournaments omitting a certain tournament $W_5$ on 5 vertices. In the case of an indecomposable tournament $T$, we will study the set $W_5(T)$ of vertices $x \in V$ for which there exists a subset $X$ of $V$ such that $x \in X$ and $T(X)$ is isomorphic to $W_5$. We prove the following: for any indecomposable tournament $T$, if $T \not\in T$, then $|W_5(T)| \geq |V| - 2$ and $|W_5(T)| \geq |V| - 1$ if $|V|$ is even. By giving examples, we also verify that this statement is optimal.

1. Introduction

A tournament $T = (V(T), A(T))$ (or $(V, A)$) consists of a finite set $V$ of vertices together with a set $A$ of ordered pairs of distinct vertices, called arcs, such that for all $x \neq y \in V$, $(x, y) \in A$ if and only if $(y, x) \notin A$. The order of $T$, denoted by $|T|$, is the cardinality of $V(T)$. Given a tournament $T = (V, A)$, with each subset $X$ of $V$ is associated the subtournament $T(X) = (X, A \cap (X \times X))$ of $T$ induced by $X$. For $X \subseteq V$ (resp. $x \in V$), the subtournament $T(V \setminus X)$ (resp. $T(V \setminus \{x\})$) is denoted by $T - X$ (resp. $T - x$). Let $T = (V, A)$ and $T' = (V', A')$ be two tournaments. A bijection $f$ from $V$ onto $V'$ is an isomorphism from $T$ onto $T'$ provided that for all $x, y \in V$, $(x, y) \in A$ if and only if $(f(x), f(y)) \in A'$. The tournaments $T$ and $T'$ are then said to be isomorphic, which is denoted by $T \simeq T'$. An isomorphism from a tournament $T$ onto itself is called an automorphism of $T$. We say that $T'$ embeds into $T$ when $T'$ is isomorphic to a subtournament of $T$. Otherwise, we say that $T$ omits $T'$. With each tournament $T = (V, A)$ is associated its dual tournament $T^* = (V, A^*)$.
The tournament $T = (V, A^*)$, where $A^* = \{(x, y) : (y, x) \in A\}$. The tournament $T$ is then said to be self-dual when $T \cong T^*$. For every $x \neq y \in V$, the notation $x \rightarrow y$ signifies that $(x, y) \in A$. Moreover, for every $x \in V$ and $Y \subseteq V \setminus \{x\}$, $x \rightarrow Y$ (resp. $Y \rightarrow x$) means that $x \rightarrow y$ (resp. $y \rightarrow x$) for every $y \in Y$. For every $x \in V$, we set $N_T^+(x) = \{y \in V : x \rightarrow y\}$ and $N_T^-(x) = \{y \in V : y \rightarrow x\}$. Furthermore, the score of a vertex $x$ of $T$, denoted by $s_T(x)$, is the cardinality of $N_T^+(x)$.

Given a tournament $T = (V, A)$, a subset $I$ of $V$ is a clan [6] (or an interval [11, 16]) of $T$ provided that for every $x \in V \setminus I$, $x \rightarrow I$ or $I \rightarrow x$. For example, $\emptyset, \{x\}$, where $x \in V$, and $V$ are clans of $T$, called trivial clans. A tournament is then said to be indecomposable [11, 16] (or primitive [6]) if all its clans are trivial and it is decomposable otherwise. Notice that a tournament $T$ and its dual $T^*$ have the same clans, in particular, $T^*$ is indecomposable precisely if the same holds for $T$.

The main result of this paper, presented in [2] without proof, concerns the subtournaments of an indecomposable tournament $T$ which are isomorphic to a tournament $W_5$ defined on $\{0, \ldots, 4\}$ by $A(W_5 - 4) = \{(i, j) : 0 \leq i < j \leq 3\}$ and $N_{W_5}^+(4) = \{0, 2\}$. Note that the tournament $W_5$ is the tournament $W_{2n+1}$ introduced in Section 3, by taking $n = 2$ (see Figure 3). In 2003, B. J. Latka characterized the indecomposable tournaments omitting $W_5$ (see Theorem 3.4). Many classes of tournaments defined by means of embedding, involving inevitable configurations or morphological descriptions, have been studied by several authors [1, 10, 12, 13]. The aim of this paper is to examine the set $W_5(T)$ of the vertices $x$ of an indecomposable tournament $T$ for which there exists a subset $X \in \binom{\binom{V}{2}}{5}$ such that $x \in X$ and $T(X) \cong W_5$. So, notice that almost all tournaments $T = (V, A)$ verify $W_5(T) = V$. It is an elementary exercise to show that. Note also that if $T$ satisfies a certain extension axiom, then it satisfies $W_5(T) = V$. The extension axioms are introduced in [8, 9], as a first order logic sentences, for the study of 0-1 laws. These axioms form an important tool in the study of the random aspects of finite structures, each of these axioms is satisfied by almost all these structures [5, 8, 9]. We recall these axioms in the case of tournaments. A tournament $T = (V, A)$ is r-existentially closed (or r-e.c. [3]), where $r \in \mathbb{N}$, when it satisfies the r-extension axiom: for all $X \in \binom{\binom{V}{2}}{r}$ and $Y \subseteq X$, there is a vertex $x \in V \setminus X$ such that $N_{T(X \cup \{x\})}^+(x) = Y$. For $r \in \mathbb{N}$, almost all tournaments are r-e.c. [3, 8, 9]. As a 4-e.c. tournament $T$ satisfies $W_5(T) = V(T)$, then almost all tournaments $T$ satisfy $W_5(T) = V(T)$. As we are interested in indecomposable tournaments, which is the case of almost all tournaments [7], we deduce the following fact.

**Fact 1.1.** Almost all the indecomposable tournaments $T$ satisfy $W_5(T) = V(T)$.

Note that these facts extend in a natural way when one considers a tournament other than $W_5$. 
In this paper, we focus on the tournament $W_5$, we establish the following theorem and we verify that it is optimal.

**Theorem 1.2.** Let $T$ be an indecomposable tournament. If $W_5$ embeds into $T$, then $|W_5(T)| \geq |T| - 2$. If, moreover, $|T|$ is even, then $|W_5(T)| \geq |T| - 1$.

2. **The indecomposable tournaments**

**Definition.** Given a tournament $T = (V, A)$, with each subset $X$ of $V$, such that $|X| \geq 3$ and $T(X)$ is indecomposable, are associated the following subsets of $V \setminus X$.

- $[X] = \{x \in V \setminus X : x \rightarrow X$ or $X \rightarrow x\}$.
- For every $u \in X$, $X(u) = \{x \in V \setminus X : \{u,x\}$ is a clan of $T(X \cup \{x\})\}$.
- $\text{Ext}(X) = \{x \in V \setminus X : T(X \cup \{x\})$ is indecomposable\}.

**Lemma 2.1** ([6]). Let $T = (V, A)$ be a tournament and let $X$ be a subset of $V$ such that $|X| \geq 3$ and $T(X)$ is indecomposable. The nonempty elements of the family $\{X(u) : u \in X\} \cup \{\text{Ext}(X), [X]\}$ constitute a partition of $V \setminus X$ and the following assertions are satisfied.

- Let $u \in X$, $x \in X(u)$ and $y \in V \setminus (X \cup X(u))$. If $T(X \cup \{x,y\})$ is decomposable, then $\{u,x\}$ is a clan of $T(X \cup \{x,y\})$.
- Let $x \in [X]$ and $y \in V \setminus (X \cup [X])$. If $T(X \cup \{x,y\})$ is decomposable, then $X \cup \{y\}$ is a clan of $T(X \cup \{x,y\})$.
- Let $x \neq y \in \text{Ext}(X)$. If $T(X \cup \{x,y\})$ is decomposable, then $\{x,y\}$ is a clan of $T(X \cup \{x,y\})$.

From this lemma follows the next result.

**Corollary 2.2** ([6]). Let $T = (V, A)$ be an indecomposable tournament. If $X$ is a subset of $V$ such that $|X| \geq 3$, $|V \setminus X| \geq 2$ and $T(X)$ is indecomposable, then there are distinct $x, y \in V \setminus X$ such that $T(X \cup \{x,y\})$ is indecomposable.

We also recall the following result concerning the indecomposable tournaments.

**Lemma 2.3** ([15]). Let $T = (V, A)$ be an indecomposable tournament. If $X$ is a subset of $V$ such that $|X| \geq 3$, $|V \setminus X| \geq 4$ and $T(X)$ is indecomposable, then there are distinct $x, y \in V \setminus X$ such that $T - \{x,y\}$ is indecomposable.

3. **The critical tournaments and Latka’s theorem**

Given an indecomposable tournament $T$ with $V(T) \neq \emptyset$, $T$ is said to be critical if for all vertex $x$ of $T$, the tournament $T - x$ is decomposable. The critical tournaments are one of the tools of the proof of Theorem 1.2. Moreover, an important part of these tournaments form the class of indecomposable tournaments of order $> 7$ and omitting $W_5$ due to B. J. Latka [12]. In order to present the characterization of the critical tournaments due to J. H.
Schmerl and W. T. Trotter [16], we introduce the following notations and tournaments. A transitive tournament is a tournament omitting the tournament $C_3 = \{(0,1,2), (0,1), (1,2), (2,0)\}$. For $n \in \mathbb{N}$, we set $\mathbb{N}_n = \{0,\ldots,n\}$, $2\mathbb{N}_n = \{2i : i \in \mathbb{N}_n\}$ and for every finite set $X$ of $\mathbb{N}$, we denote by $X$ the transitive tournament defined on $X$ by $A(X) = \{(i,j) \in X \times X : i < j\}$. Now, we introduce the tournaments $T_{2n+1}$, $U_{2n+1}$ and $W_{2n+1}$ defined on $2\mathbb{N}_n$, where $n \geq 2$, as follows.

1. $A(T_{2n+1}) = \{(i,j) : j - i \in \{1,\ldots,n\} \text{ mod } 2n + 1\}$ (see Figure 1).
2. $U_{2n+1}(\mathbb{N}_n) = \mathbb{N}_n$, $U_{2n+1}(\mathbb{N}_n \setminus \mathbb{N}_n) = \mathbb{N}_{2n} \setminus \mathbb{N}_n$ and for $i \in \mathbb{N}_{n-1}$, \{i + 1,\ldots,n\} $\rightarrow i + n + 1 \rightarrow \mathbb{N}_i$ (see Figure 2).
3. $W_{2n+1}(\mathbb{N}_{2n-1}) = \mathbb{N}_{2n-1}$ and $N_{2n+1}^+(2n) = 2\mathbb{N}_{n-1}$ (see Figure 3).

![Figure 1. The tournament $T_{2n+1}$.](image1)

![Figure 2. The tournament $U_{2n+1}$.](image2)

**Proposition 3.1** ([16]). Up to isomorphisms, the critical tournaments are the tournaments $T_{2n+1}$, $U_{2n+1}$ and $W_{2n+1}$, where $n \geq 2$.

Notice that the critical tournaments are self-dual and recall the remarks below, which follow easily from the definitions of these tournaments.
Theorem 3.4 (\cite{12}). Up to isomorphisms, the indecomposable tournaments of $T_{2n+1}$ (resp. $U_{2n+1}, W_{2n+1}$) on at least 5 vertices, where $n \geq 2$, are the tournaments $T_{2n+1}$ (resp. $U_{2m+1}, W_{2m+1}$), where $2 \leq m \leq n$. In particular, for all integers $p, q, l \geq 2$, the tournaments $T_{2p+1}, U_{2q+1}$ and $W_{2l+1}$ are incomparable with respect to the embedding.

Remark 3.2. Up to isomorphisms, the indecomposable subtournaments of $T_{2n+1}$ (resp. $U_{2n+1}, W_{2n+1}$) on at least 5 vertices, where $n \geq 2$, are the tournaments $T_{2n+1}$ (resp. $U_{2m+1}, W_{2m+1}$), where $2 \leq m \leq n$. In particular, for all integers $p, q, l \geq 2$, the tournaments $T_{2p+1}, U_{2q+1}$ and $W_{2l+1}$ are incomparable with respect to the embedding.

Remark 3.3. Let $T = (\mathbb{N}_6, A)$ be an indecomposable tournament such that $T(\mathbb{N}_4) = U_5$. The tournament $T$ is isomorphic to $U_7$ if and only if, by interchanging the vertices 5 and 6, one of the six following configurations occurs.

- $N_T^+(5) = \{0, 1, 2\}$ and $N_T^+(6) = \{5\}$.
- $N_T^+(5) = \{1, 2, 6\}$ and $N_T^+(6) = \{0\}$.
- $N_T^+(5) = \{1, 2, 3\}$ and $N_T^+(6) = \{0, 3, 5\}$.
- $N_T^+(5) = \{2, 3, 6\}$ and $N_T^+(6) = \{1, 3, 0\}$.
- $N_T^+(5) = \{2, 3, 4\}$ and $N_T^+(6) = \{0, 1, 3, 4, 5\}$.
- $N_T^+(5) = \{3, 4, 6\}$ and $N_T^+(6) = \{0, 1, 2, 3, 4\}$.

In order to present the characterization of the indecomposable tournaments omitting $W_5$, due to B. J. Latka, we also introduce the Paley tournament $P_7$ defined on $\mathbb{N}_6$ by $A(P_7) = \{(i, j) : j - i \in \{1, 2, 4\} \mod 7\}$. Notice that for every $x \neq y \in \mathbb{N}_6$, $P_7 - x \simeq P_7 - y$ and set $B_0 = P_7 - 6$. Moreover, for all $x \neq y \in \mathbb{N}_6$, $B_0 - x \simeq B_0 - y \simeq U_5$. Notice also that the tournaments $B_0$ and $P_7$ are self-dual.

Theorem 3.4 (\cite{12}). Up to isomorphisms, the indecomposable tournaments on at least 5 vertices and omitting $W_5$ are the tournaments $B_0, P_7, T_{2n+1}$ and $U_{2n+1}$, where $n \geq 2$.

With this characterization, we obtain the following statement of Theorem 1.2: Let $T$ be an indecomposable tournament of order $\geq 5$ such that $T \ncong B_0, P_7, T_{2n+1}$ or $U_{2n+1}$ for $n \geq 2$. Then $|W_5(T)| \geq |T| - 2$. If, moreover, $|T|$ is even, then $|W_5(T)| \geq |T| - 1$.

Given five distinct vertices $x_i$ ($i \in \mathbb{N}_4$) of a tournament $T$. For convenience, we write $T(x_0, x_1, x_2, x_3, x_4) \simeq W_5$ to signify that the bijection $\tau : i \mapsto x_i$ is
an isomorphism from $W_5$ onto $T((x_0, x_1, x_2, x_3, x_4))$. Similarly, for another choice of five distinct vertices $y_i$ ($i \in \mathbb{N}_4$) of $T$, we write $T(x_0, x_1, x_2, x_3, x_4) \cong T(y_0, y_1, y_2, y_3, y_4)$ to signify that the bijection $\sigma : x_i \mapsto y_i$ is an isomorphism from $T((x_0, x_1, x_2, x_3, x_4))$ onto $T((y_0, y_1, y_2, y_3, y_4))$.

4. The minimal tournaments

The minimal tournaments are involved in the proof of Theorem 1.2. These tournaments have been introduced in 1998 by A. Cournier and P. Ille [4] as follows. Given an indecomposable tournament $T = (V, A)$ and two distinct vertices $x \neq y \in V$, $T$ is said to be minimal for $x$ and for $y$ (or $\{x, y\}$-minimal) whenever for every proper subset $X$ of $V$ ($X \neq V$), if $\{x, y\} \subset X$ ($|X| \geq 3$), then $T(X)$ is decomposable. We say that $T$ is minimal when there exist $x \neq y \in V(T)$ such that $T$ is $\{x, y\}$-minimal. A. Cournier and P. Ille characterized the minimal tournaments. In order to recall this characterization, we introduce the tournaments $F_n$ and $G_n$ in the following manner.

(1) For $n \geq 4$, $F_n$ is defined on $\mathbb{N}_{n-1}$ as follows: for $i, j \in \mathbb{N}_{n-1}$, $(i, j) \in A(F_n)$ if and only if $j = i + 1$ or $i \geq j + 2$ (see Figure 4).

(2) For $n \geq 6$, $G_n$ is defined on $\mathbb{N}_{n-1}$ as follows: $G_n(\mathbb{N}_{n-3}) = F_{n-2}$,

$N^+_G(n - 2) = \{n - 3\}$ and $N^+_G(n - 1) = \{n - 2\}$ (see Figure 5).

![Figure 4. The tournament $F_n$.](image-url)

Proposition 4.1 ([4]). Up to isomorphisms, the minimal tournaments of order $\geq 3$ are the tournaments $C_3, U_5, W_5, F_n, G_n$ and $G^*_n$, where $n \geq 6$.

Corollary 4.2. Given a minimal tournament $T$ of order $n \geq 6$, we have $|W_5(T)| \geq n - 1$.

Proof. As $W_5$ is self-dual, it suffices to prove the result for the tournaments $F_n$ and $G_n$ for $n \geq 6$. For $n \geq 5$, we have $|W_5(F_n)| = n$ because for all $i \in \mathbb{N}_{n-5}$, $F_n(i + 3, i + 4, i, i + 1, i + 2) \cong W_5$. For $n = 6$, $|W_5(G_6)| \geq 5$ because $G_6(1, 2, 5, 4, 3) \cong W_5$. For $n \geq 7$, $|W_5(G_n)| = n$ because $G_n(\mathbb{N}_{n-5}, n-4, n-1, n-2, n-3) \cong W_5$ and $|W_5(G_n(\mathbb{N}_{n-3}))| = |W_5(F_{n-2})| = n - 2$.

$\square$
5. **Proof of Theorem 1.2 for \(|T| \leq 8\)**

Theorem 1.2 is trivial for \(|T| = 7\). In this section, we establish this theorem for \(|T| = 8\). Up to isomorphisms, there are 6880 tournaments of 8 vertices, 3785 tournaments of them are indecomposable [14], but our verification, made by hand, is not exhaustive. We begin by the case where \(B_6\) embeds into \(T\).

So, notice the following additional remarks concerning \(B_6\) and \(P_7\). The Paley tournament \(P_7\) is regular: for every \(x \in \mathbb{N}_6\), \(s_{P_7}(x) = 3\). The tournament \(B_6\) is quasi-regular: for every \(x \in \mathbb{N}_5\), \(s_{B_6}(x) = 2\) if \(x \in \{2, 4, 5\}\), and \(s_{B_6}(x) = 3\) if \(x \in \{0, 1, 3\}\). Moreover, the automorphism group of \(B_6\) is generated by the permutation \(\pi = (013)(254)\).

**Lemma 5.1.** If \(B_6\) embeds into an indecomposable tournament \(T\) on 7 vertices and if \(T \not\cong P_7\), then \(|W_5(T)| = 7\).

**Proof.** We set \(V(T) = \mathbb{N}_6\) and \(T(\mathbb{N}_5) = B_6\). By interchanging \(T\) and \(T^*\), we can assume that \(s_T(6) \leq 3\). The automorphisms of \(B_6\) restrict the proof to the following cases. When \(s_T(6) = 1\), we can assume that \(N_T^+(6) = \{0\}\) or \(\{2\}\). If \(N_T^+(6) = \{0\}\), then \(T(1, 5, 2, 6, 0) \cong T(2, 3, 4, 6, 0) \cong W_5\). If \(N_T^+(6) = \{2\}\), then \(T(0, 1, 6, 2, 3) \cong T(5, 0, 6, 2, 4) \cong W_5\). When \(s_T(6) = 2\), we can assume that \(N_T^+(6) = \{4, 5\}, \{0, 1\}, \{0, 5\}, \{3, 5\}\) or \(\{1, 5\}\). If \(N_T^+(6) = \{1, 5\}\), then \(T\) is decomposable because \(6 \in N_6(4)\). If \(N_T^+(6) = \{4, 5\}\), then \(T(3, 0, 6, 4, 1) \cong T(0, 1, 2, 6, 5) \cong W_5\). If \(N_T^+(6) = \{0, 1\}\), then \(T(3, 5, 6, 0, 2) \cong T(3, 4, 5, 6, 1) \cong W_5\). If \(N_T^+(6) = \{0, 5\}\), then \(T(2, 3, 4, 6, 0) \cong T(4, 1, 6, 5, 2) \cong W_5\). If \(N_T^+(6) = \{3, 5\}\), then \(T(0, 1, 2, 6, 5) \cong T(3, 4, 5, 6, 1) \cong W_5\).

**Figure 5.** The tournament \(G_n\).
Lemma 5.2. If $T(6) = 3$, we can assume that $N^+_T(6) = \{0, 1, 2, 6, 5\}$. When $s_T(6) = 3$, we can assume that $N^+_T(6) = \{0, 1, 2, 6, 5\}$. When $s_T(6) = 3$, we can assume that $N^+_T(6) = \{0, 1, 2, 6, 5\}$. When $s_T(6) = 3$, we can assume that $N^+_T(6) = \{0, 1, 2, 6, 5\}$.

Lemma 5.3. We set $T(7) = 2$, then $W_5(T) = 3$. Therefore, $T = 3$. We have $T(7) = 2$, then $W_5(T) = 3$. Therefore, $T = 3$. We have $T(7) = 2$, then $W_5(T) = 3$. Therefore, $T = 3$. We have $T(7) = 2$, then $W_5(T) = 3$. Therefore, $T = 3$.

Lemma 5.4. If $B_0$ embeds into an indecomposable tournament $T$ on 8 vertices, then $|W_5(T)| \geq 7$.

Proof. We set $V(T) = N_7$ and $T(N_6) = P_7$. By interchanging $T$ and $T^*$, we can assume that $s_T(7) \leq 3$. First, assume that $s_T(7) = 1$. We take $N^+_T(7) = \{x\}$ and $y \in N_0 \setminus \{x\}$. We have $T - (7, y) \simeq B_0$, $T - y \not\simeq P_7$ and $T - y$ is indecomposable by Lemma 2.1. It follows from Lemma 5.1 that $|W_5(T - y)| = 7$. By changing $y$ by $z \in N_0 \setminus \{x, y\}$, we obtain $|W_5(T - z)| = 7$. Therefore, $|W_5(T)| = 8$. Second, assume that $s_T(7) = 2$ and set $N^+_T(7) = \{x, y\}$. For $\alpha \in \{x, y, z\}$, we have $T - \{7, \alpha\} \simeq B_0$, $T - \alpha \not\simeq P_7$ and $T - \alpha$ is indecomposable by Lemma 2.1. It follows from Lemma 5.1 that $|W_5(T - \alpha)| = 7$ and hence $|W_5(T)| = 8$. Finally, assume that $s_T(7) = 3$, set $N^+_T(7) = \{x, y, z\}$ and let $\alpha \in \{x, y, z\}$. We have $T(X) \simeq B_0$, where $X = N_6 \setminus \{\alpha\}$ and $T - \alpha \not\simeq P_7$. Moreover, $T - \alpha$ is indecomposable. Otherwise, as $7 \notin [X]$, then by Lemma 2.1, there is $u \in X$ such that $7 \in X(u)$. Since $s_T(-\alpha)(7) = 2$, then $s_T(X)(u) = 2$. As moreover, $\{u, T\} \rightarrow \alpha$, then $\{u, T\}$ is a nontrivial clan of $T$, which contradicts the indecomposability of $T$. It follows from Lemma 5.1 that $|W_5(T - \alpha)| = 7$ and thus $|W_5(T)| = 8$. □

Lemma 5.5. If $B_0$ embeds into an indecomposable tournament $T$ on 8 vertices, then $|W_5(T)| \geq 7$.

Proof. We set $V(T) = N_7$ and $T(N_6) = B_0$. By Lemma 5.1 and Lemma 5.2 we obtain the following remark. If $Ext(N_6) \neq \emptyset$, then $|W_5(T)| \geq 7$. So, assume that $Ext(N_6) = \emptyset$. By Lemma 2.1, assume first that $6 \in N_5(u)$, where $u \in N_5$, and that $7 \in [N_5]$. By interchanging $T$ and $T^*$, we can assume that $7 \rightarrow N_5$ and hence $6 \rightarrow 7$ by Lemma 2.1. We have $T(X) \simeq B_0$ where $X = N_6 \setminus \{u\}$. Since $s_T(-u)(7) = 5 \notin [2, 3, 4, 0, 6]$, then $7 \in Ext(X)$ by Lemma 2.1. It follows, from the remark above, that $|W_5(T)| \geq 7$. Now, suppose that $6 \notin N_5(u)$ and $7 \in N_5(v)$, where $u \neq v \in N_5$. By interchanging $T$ and $T^*$ by considering the automorphisms of $B_0$, we may assume that $u \in \{0, 1, 3, 4\}$ and $v = 3$. We have $T(Y) \simeq U_5$ and $7 \in Y(5)$, where $Y = N_5 \setminus \{u\}$. Moreover, $6 \notin Y(5)$. Indeed, if $u = 1$ or 4, then $5 \rightarrow 0 \rightarrow 6$ and if $u = 0$ or 3, then $6 \rightarrow 4 \rightarrow 5$. As furthermore, $\{5, 7\}$ is not a clan of $T - u$, it ensues, from Lemma 2.1, that $T - u$ is indecomposable. Therefore, $|W_5(T)| \geq 7$ by the remark above. □
Lemma 5.4. Let $T = (N_6, A)$ be an indecomposable tournament such that $T(N_4) = U_5$. If $T \not\cong U_7$ and $\text{Ext}(N_4) = \emptyset$, then $W_5(T) \cap \{3, 4\} \neq \emptyset$.

Proof. Notice first that $\text{Ext}(N_4) = \emptyset$ if and only if $\{N^+_{T-\alpha}(5), N^+_{T-\alpha}(6)\} \subset C = \{\emptyset, N_4, \{0\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{0, 3\}, \{0, 1, 2\}, \{1, 2, 3\}, \{2, 3, 4\}, \{0, 1, 3\}, \{0, 1, 3, 4\}\}$. As $T - 6$ is decomposable, then $T \not\cong T_7$. Similarly, $T \not\cong T_7$ by Remark 3.2. It follows from Theorem 3.4 that $W_5$ embeds into $T$. If $T - \{3, 4\} \not\cong W_5$, then there exists $X \in \binom{\{0, \ldots, 6\}}{3}$ such that $T(X) \cong W_5$ and $X \cap \{3, 4\} \neq \emptyset$. So, assume that $T - \{3, 4\} \cong W_5$. Since $T(N_2) = N_2$, then, by considering the subtournaments of $W_5$ which are isomorphic to $N_2$ and by taking $5 \rightarrow 6$, we obtain that $W_5 \cong T(5, 0, 6, 1, 2) \cup \{0\}, T(5, 0, 2, 6, 1), T(6, 0, 1, 2, 5), T(1, 5, 2, 6, 0), T(0, 1, 2, 5, 6), T(0, 1, 6, 2, 5), T(0, 5, 1, 2, 6)$. If $W_5 \cong T(6, 0, 1, 2, 5)$ or $T(1, 5, 2, 6, 0)$ (resp. $W_5 \cong T(5, 0, 6, 1, 2), T(5, 0, 2, 6, 1)$ or $T(0, 1, 2, 5, 6)$, then $N^+_{T-\alpha}(5) \not\in C$ (resp. $N^+_{T}(6) \not\in C$). If $W_5 \cong T(0, 1, 6, 2, 5)$, then $N^+_{T}(6) \not\equiv \{2, 3\}$ or $\{2, 3, 4\}$ and $N^+_{T-\alpha}(5) \equiv \{0\}$ or $\{0, 3\}$.

Proposition 5.5. Given an indecomposable tournament $T$ of order 8, we have $|W_5(T)| \geq 7$.

Proof. Suppose, by contradiction, that there are $x \neq y \in V(T)$ such that $\{x, y\} \cap W_5(T) = \emptyset$. Let $X$ be a minimal subset of $V(T)$ such that $\{x, y\} \subset X (|X| \geq 3)$ and $T(X)$ is indecomposable. $T(X)$ is $\{x, y\}$-minimal. By Proposition 4.1 and Corollary 4.2, $T(X) \cong C_3$ or $U_5$. If $T(X) \cong C_3$, then, by Lemma 2.3 and Theorem 3.4, $B_6$ embeds into $T$. By Lemma 5.3, $|W_5(T)| \geq 7$, a contradiction. Therefore, $T(X) \cong U_5$. We take $V(T) = N_7$ and $T(X) = U_5$. By observing the subtournaments of $U_5$ which are isomorphic to $C_3$, we obtain that $\{x, y\} = \{3, 4\}$. We have $\text{Ext}(N_4) = \emptyset$. Otherwise, by Theorem 3.4, there is $\alpha \in \{5, 6, 7\}$ such that $T(N_4 \cup \{\alpha\}) \cong B_6$, contradiction by Lemma 5.3. By Corollary 2.2, we may assume that $T - 7$ is indecomposable. If $T - 7 \not\cong U_7$, then, by Lemma 5.4, we have $W_5(T - 7) \cap \{3, 4\} \neq \emptyset$, a contradiction. To finish, it remains to examine the case where $T - 7 \cong U_7$. By interchanging $T$ and $T^*$ and by using Remark 3.3, it suffices to consider the following three cases: $(N^+_{T-\gamma}(5), N^+_{T-\gamma}(6)) = (\{0, 1, 2\}, \{5\}), (\{1, 2, 6\}, \{0\})$ or $(\{1, 2, 3\}, \{0, 3, 5\})$. If $(N^+_{T-\gamma}(5), N^+_{T-\gamma}(6)) = (\{0, 1, 2\}, \{5\})$ (resp. $(\{1, 2, 6\}, \{0\}), (\{1, 2, 3\}, \{0, 3, 5\})$), then $5 \in N_4(0)$ and $6 \in N_4(3)$ (resp. $5 \in N_4(0)$ and $6 \in N_4(3)$, $5 \in N_4(1)$ and $6 \in N_4(3)$). It follows that $7 \in N_4(0)$ or $N_4$ (resp. $7 \in N_4(u)$ for $u \in \{0, 3\}, 7 \in N_4(u)$ for $u \in \{1, 3\}$). Otherwise, since $\{v, 7\}$, where $v \in \{1, 2, 3, 4\}$ (resp. $v \in \{1, 2, 4\}, v \in \{0, 2, 4\}$), and $[N_6]$ are not clans of $T$, then, by Lemma 2.1, there is $\alpha \in \{5, 6\}$ such that $T - \alpha$ is
indecomposable. By Remark 3.3, \( T - \alpha \not\cong U_7 \), which contradicts Lemma 5.4. Thus, we distinguish the following cases.

- \( N_{T-\gamma}^+(5) = \{0, 1, 2, 5\} \) and \( 7 \in N_4(0) \) or \([N_4]\). First, suppose that \( 7 \in N_4(0) \). If \( 6 \rightarrow 7 \), then \( 0 \rightarrow 7 \) because \( \{5, 7\} \) is not a clan of \( T \). Thus, \( T(3, 0, 6, 7, 1) \cong W_5 \), a contradiction. If \( 7 \rightarrow 6 \), as \( \{0, 7\} \) is not a clan of \( T \), then \( 7 \rightarrow 5 \). Thus, \( T(3, 7, 6, 5, 2) \cong W_5 \), a contradiction. Now, assume that \( 7 \in [N_4] \). If \( 7 \rightarrow N_4 \), then \( 7 \rightarrow 5 \), otherwise \( T(5, 7, 0, 1, 3) \cong W_5 \). Since \( N_4 \) is not a clan of \( T \), then \( 6 \rightarrow 7 \) and hence \( T(7, 4, 5, 1, 6) \cong W_5 \), a contradiction. If \( N_4 \rightarrow 7 \), as \( \{6, 7\} \) and \( N_4 \) are not clans of \( T \), then \( 5 \rightarrow 7 \rightarrow 6 \) and thus \( T(1, 3, 7, 6, 5) \cong W_5 \), a contradiction.

- \( N_{T-\gamma}^+(5) = \{1, 2, 6, 7\} \) and \( 7 \in N_4(u) \) for \( u \in \{0, 3\} \). If \( 7 \in N_4(0) \) with \( 7 \rightarrow 6 \) (resp. \( 7 \in N_4(3) \) with \( 5 \rightarrow 7 \)), as \( \{5, 7\} \) (resp. \( \{6, 7\} \)) is not a clan of \( T \), then \( 7 \rightarrow 0 \) (resp. \( 7 \rightarrow 3 \)). It follows from Lemma 2.1 and Remark 3.3 that \( T - 5 \) (resp. \( T - 6 \)) is indecomposable and not isomorphic to \( U_7 \), which contradicts Lemma 5.4. Now, if \( 7 \in N_4(0) \) with \( 6 \rightarrow 7 \) (resp. \( 7 \in N_4(3) \) with \( 7 \rightarrow 5 \)), since \( \{0, 7\} \) (resp. \( \{3, 7\} \)) is not a clan of \( T \), then \( 5 \rightarrow 7 \) (resp. \( 6 \rightarrow 7 \)). So, \( T(3, 5, 6, 7, 1) \cong W_5 \) (resp. \( T(2, 4, 6, 7, 5) \cong W_5 \)), a contradiction.

- \( N_{T-\gamma}^+(5) = \{1, 2, 3, 5\} \) and \( 7 \in N_4(u) \) for \( u \in \{1, 3\} \). If \( 7 \in N_4(1) \) and \( 6 \rightarrow 7 \) (resp. \( 7 \in N_4(3) \) and \( 7 \rightarrow 5 \)), as \( \{5\} \) (resp. \( \{6, 7\} \)) is not a clan of \( T \), then \( 1 \rightarrow 7 \) (resp. \( 3 \rightarrow 7 \)). It ensues from Lemma 2.1 and Remark 3.3 that \( T - 5 \) (resp. \( T - 6 \)) is indecomposable and not isomorphic to \( U_7 \). This contradicts Lemma 5.4. Now, if \( 7 \in N_4(1) \) and \( 7 \rightarrow 6 \) (resp. \( 7 \in N_4(3) \) and \( 5 \rightarrow 7 \)), then \( 7 \rightarrow 5 \) (resp. \( 7 \rightarrow 6 \)) because \( \{1, 7\} \) (resp. \( \{3, 7\} \)) is not a clan of \( T \). Therefore, \( T(4, 7, 6, 5, 2) \cong W_5 \) (resp. \( T(2, 4, 7, 6, 5) \cong W_5 \)), a contradiction. \( \square \)

6. Theorem 1.2: Proof and optimality

**Theorem 1.2.** Let \( T \) be an indecomposable tournament. If \( W_5 \) embeds into \( T \), then \( |W_5(T)| \geq |T| - 2 \). If, moreover, \( |T| \) is even, then \( |W_5(T)| \geq |T| - 1 \).

**Proof.** The result is trivial for \( |T| \leq 7 \). By Proposition 5.5, we can assume that \( |T| = n \geq 9 \). First, assume that \( n \) is even. Suppose, by contradiction, that \( |W_5(T)| \leq n - 2 \) and consider \( x \neq y \in V(T) \) such that \( \{x, y\} \cap W_5(T) = \emptyset \). Let \( X \) be a minimal subset of \( V(T) \) such that \( \{x, y\} \subset X \) \((|X| \geq 3)\) and \( T(X) \) is indecomposable, so that \( T(X) \) is \( \{x, y\}\)-minimal. By Proposition 4.1 and Corollary 4.2, \( T(X) \simeq C_3 \) or \( U_5 \). By applying several times Lemma 2.3, there exists a subset \( Y \in (V(T)^{\complement}) \) such that \( X \subset Y \) and \( T(Y) \) is indecomposable. This contradicts Proposition 5.5. Now, assume that \( n \) is odd. If \( T \) is critical, then, by Remark 3.2, \( T \simeq W_n \) and hence \( |W_5(T)| = n \). If \( T \) is not critical, then there is \( x \in V(T) \) such that \( T - x \) is indecomposable. We have \( |T - x| \) is even and \( W_5 \) embeds into \( T - x \) by Theorem 3.4. By the first case, \( |W_5(T - x)| \geq n - 2 \), so that \( |W_5(T)| \geq n - 2 \). \( \square \)
By constructing examples, we verify that Theorem 1.2 is optimal. By Fact 1.1, we only construct for each integer \( m \geq 6 \), an indecomposable tournament \( T \) of order \( m \) with \( |W_5(T)| = m - 1 \) and, when \( m \) is odd, another indecomposable tournament \( T' \) of order \( m \) with \( |W_5(T')| = m - 2 \). We then introduce the tournaments \( Q_{2n+3}, R_{2n+3} \) defined on \( N_{2n+2} \), where \( n \geq 2 \), in the following manner.

- \( Q_{2n+3}(N_{2n}) = T_{2n+1}, N_{Q_{2n+3}}(2n + 1) = \{1, \ldots, n\} \cup \{2n + 2\} \) and \( N_{Q_{2n+3}}^+(2n + 2) = N_{2n} \).
- \( R_{2n+3} - \{2n + 1\} = Q_{2n+3} - \{2n + 1\} \) and \( N_{R_{2n+3}}^+(2n + 1) = \{0, 2n + 2\} \). The tournaments \( Q_{2n+3}, R_{2n+3} \) and \( R_{2n+3} - \{2n + 2\} \) form the required constructions:

**Proposition 6.1.** For \( n \geq 2 \), the tournaments \( Q_{2n+3}, R_{2n+3} \) and \( R_{2n+3} - \{2n + 2\} \) are indecomposable and satisfy: \( W_5(Q_{2n+3}) = N_{2n+2} \setminus \{0, n + 1\} \), \( W_5(R_{2n+3}) = N_{2n+2} \setminus \{n\} \) and \( W_5(R_{2n+3} - \{2n + 2\}) = N_{2n+1} \setminus \{n\} \).

**Proof.** We begin by verifying the indecomposability of these tournaments by using Lemma 2.1. The tournament \( Q_{2n+3} \) is indecomposable because \( Q_{2n+3}(N_{2n}) \) is indecomposable, \( 2n + 2 \rightarrow N_{2n} \) and, in this tournament, we have \( 2n + 1 \in N_{2n}(0) \) with \( 2n + 1 \rightarrow 2n + 2 \). The tournaments \( R_{2n+3} \) and \( R_{2n+3} - \{2n + 2\} \) are indecomposable by remarking that \( R_{2n+3}(N_{2n}) \) is indecomposable, and the vertex \( 2n + 1 \notin \{N_{2n}[i], 2n + 1 \notin \{N_{2n}(n)\} \), for a certain \( n \in N_{2n} \) and \( 2n + 2 \rightarrow N_{2n} \) with \( 2n + 1 \rightarrow 2n + 2 \). At present, we verify that \( W_5(Q_{2n+3}) = N_{2n+2} \setminus \{0, n + 1\} \). Since \( Q_{2n+3}(2n + 1, 2n + 2, 1, 2, n + 2) \cong W_5 \), then, by Theorem 1.2, it suffices to prove that \( \{0, n + 1\} \cap W_5(Q_{2n+3}) = \emptyset \). So, let \( x \in \{0, n + 1\} \) and suppose, by contradiction, that \( x \in W_5(Q_{2n+3}) \). By Remark 2.3, there exist \( i \neq j \in N_{2n} \setminus \{x\} \) with \( i \rightarrow j \) and \( Q_{2n+3}(\{x, 2n + 1, 2n + 2, i, j\}) \cong W_5 \). As \( Q_{2n+3}(\{x, 2n + 1, 2n + 2\} \cong C_3 \) and \( 2n + 2 \rightarrow N_{2n} \), then, by observing the subtournaments of \( W_5 \) which are isomorphic to \( C_3 \), we obtain that \( W_5 \cong Q_{2n+3}(2n + 1, 2n + 2, i, j, x) \cong Q_{2n+3}(2n + 2, x, i, j, 2n + 1) \) or \( Q_{2n+3}(2n + 2, i, j, x, 2n + 1) \). It follows that \( x \neq 0 \). Otherwise, \( \{x, 2n + 1\} = \{0, 2n + 1\} \) is not a clan of \( Q_{2n+3}(\{0, 2n + 1, j\}) \), a contradiction because \( 2n + 1 \in N_{2n}(0) \). If \( W_5 \cong Q_{2n+3}(2n + 1, 2n + 2, i, j, x, i, j, 2n + 1) \) or \( Q_{2n+3}(2n + 2, x, i, j, 2n + 1) \) (resp. \( W_5 \cong Q_{2n+3}(2n + 2, i, j, x, 2n + 1) \)) then \( i \in N_{Q_{2n+3}}^+(2n + 1) \cap N_{Q_{2n+3}}^+(n + 1) \) (resp. \( i \notin N_{Q_{2n+3}}^+(2n + 1) \cap N_{Q_{2n+3}}^+(n + 1) \)). A contradiction because \( N_{Q_{2n+3}}^+(2n + 1) \cap N_{Q_{2n+3}}^+(n + 1) = N_{Q_{2n+3}}^+(2n + 1) \cap N_{Q_{2n+3}}^+(n + 1) = \emptyset \). Now, we verify that \( W_5(R_{2n+3}) = N_{2n+2} \setminus \{n\} \). For all \( \alpha \in \{n + 2, 2n, 2n + 2\} \) and for all \( \beta \in \{1, \ldots, n - 1\} \), we have \( R_{2n+3}(2n + 2, \alpha, 0, 1, 2n + 1) \cong R_{2n+3}(n + 1, 2n + 1, 0, \beta) \cong W_5 \). Thus, \( N_{2n+2} \setminus \{n\} \subseteq W_5(R_{2n+3}) \). So, suppose, by contradiction, that \( n \in W_5(R_{2n+3}) \). Since \( R_{2n+3} - \{2n + 1\} \) omits \( W_5 \), then there exist \( k \neq l \in N_{2n+2} \setminus \{0, n, 2n + 1\} \) with \( k \rightarrow l \) and \( R_{2n+3}(\{0, 2n + 1, k, l\}) \cong W_5 \). Since \( R_{2n+3}(\{0, n, 2n + 1\}) \cong C_3 \) and \( s_{R_{2n+3}}(2n + 1) = 2 \), then, by observing again the subtournaments of \( W_5 \) which are isomorphic to \( C_3 \), we obtain that \( W_5 \cong R_{2n+3}(0, n, k, l, 2n + 1), R_{2n+3}(0, k, l, n, 2n + 1), R_{2n+3}(n, k, l, 2n + 1, 0) \),
$R_{2n+3}(k, l, 0, n, 2n + 1)$, $R_{2n+3}(k, l, 2n + 1, 0, n)$, or $R_{2n+3}(k, l, n, 2n + 1, 0)$. Thus, $R_{2n+3}\{0, n, l\}$ is a transitive tournament. This contradicts the fact that for all $l \in N_{2n} \setminus \{0, n\}$, $R_{2n+3}\{0, n, l\} \simeq C_3$. Finally we can deduce that $W_5(R_{2n+3} = \{2n + 2\}) = N_{2n+1} \setminus \{n\}$ by Theorem 1.2 and by the fact that $W_5(R_{2n+3} = N_{2n+2} \setminus \{n\})$.

We end by posing the following problems, motivated by Theorem 1.2, Fact 1.1 and Proposition 6.1.

**Problem 6.2.** Characterize the indecomposable tournaments $T$ such that $|W_5(T)| = |T| - 2$.

**Problem 6.3.** Characterize the indecomposable tournaments $T$ such that $|W_5(T)| = |T| - 1$.

References


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