ESSENTIAL NORMS OF LINEAR COMBINATIONS OF COMPOSITION OPERATORS ON $h^\infty$

KEI JI IZUCHI AND KOU HEI IZUCHI

Abstract. It is studied the linear combinations of composition operators on the Banach space of bounded harmonic functions on the open unit disk. We determine the essential norm of them.

1. Introduction

Let $\mathbb{D}$ be the open unit disk and $\partial \mathbb{D}$ the unit circle. We denote by $h^\infty = h^\infty(\mathbb{D})$ and $H^\infty = H^\infty(\mathbb{D})$ the sets of bounded harmonic and analytic functions on $\mathbb{D}$, respectively. Then $h^\infty$ and $H^\infty$ are the Banach spaces with the supremum norm

$$\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}.$$

We denote by $\mathcal{S}(\mathbb{D})$ the set of analytic self-maps of $\mathbb{D}$. For $\varphi \in \mathcal{S}(\mathbb{D})$ and a harmonic function $f$, the composite function $f \circ \varphi$ is also harmonic on $\mathbb{D}$. So each self-map $\varphi$ induces the composition operator $C_\varphi$ defined on $h^\infty$ by

$$C_\varphi f = f \circ \varphi \quad \text{for } f \in h^\infty.$$

Composition operators have been investigated on various analytic function spaces (see [2, 13]). Recently, the norm, the essential norm and the topological structure of composition operators on $H^\infty$ have been studied (see [5, 6, 8, 9, 10, 12]). But the exact value of the essential norm of the difference of composition operators $\|C_\varphi - C_\psi\|_e$ on $H^\infty$ is not yet known. In [1], Choa, Ohno and the first author studied composition operators on $h^\infty$ and determined the exact value of $\|C_\varphi - C_\psi\|_e$ on $h^\infty$.

In [5], Gorkin and Mortini studied the norm and the essential norm of linear combinations of endomorphisms on uniform algebras. They gave a sufficient condition for $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ to satisfy $\|\sum_{j=1}^N \lambda_j C_{\varphi_j}\|_e = \sum_{j=1}^N |\lambda_j|$. In [11], Ohno and the first author studied the norm and the essential norm of linear

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Received January 17, 2012.
2010 Mathematics Subject Classification. 47B33, 46J15.
Key words and phrases. essential norm, linear combination of composition operators, Banach space of bounded harmonic functions.

The first author is partially supported by Grant-in-Aid for Scientific Research (No. 21540166), Japan Society for the Promotion of Science.

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combinations of composition operators on $H^\infty$. They gave a characterization
for $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ on $H^\infty$ to satisfy $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \| = \sum_{j=1}^N |\lambda_j|$ and also gave a
characterization for $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ on $H^\infty$ to satisfy $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \|_e = \sum_{j=1}^N |\lambda_j|$ under the
assumption that $\Re \lambda_j > 0$ for every $1 \leq j \leq N$.

In Section 2, we study the norm of linear combinations of composition operators on $h^\infty$, and we shall give a characterization for $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ on $h^\infty$ to satisfy $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \| = \sum_{j=1}^N |\lambda_j|$ on $h^\infty$. We also characterize the
compactness of linear combinations of composition operators on $h^\infty$. In
Section 3, we shall determine the essential norm of linear combinations of composition
operators on $h^\infty$, and give a characterization for $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ on $h^\infty$ to satisfy $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \|_e = \sum_{j=1}^N |\lambda_j|$. In [11], the essential norm $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \|_e$ on $H^\infty$ was studied, but the exact value of $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \|_e$ on $H^\infty$ is not known.

One reason is that it is not known the existence of enough many concrete comp-
act operators on $h^\infty$. But in the case of $h^\infty$, there are a lot of concrete compact operators on $h^\infty$, so we may give the exact value of $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \|_e$ on $h^\infty$.

Generally it holds
\[
\left\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \right\|_e \leq \left\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \right\| \leq \sum_{j=1}^N |\lambda_j|.
\]

In Section 4, we shall give some examples concerning with the above inequalities.

2. Norms of linear combinations

For $z, w \in \mathbb{D}$, the pseudo-hyperbolic distance between $z$ and $w$ is given by
\[
\rho(z, w) = |z - w|/|1 - \overline{w}w|.
\]
The spaces $L^1(\partial \mathbb{D})$ and $L^\infty(\partial \mathbb{D})$ stand for the standard Lebesgue spaces with the norms $\|f\|_1$ and $\|f\|_\infty$, respectively. For $f \in L^1(\partial \mathbb{D})$, let \[
\hat{f}(z) = \int_{\partial \mathbb{D}} f(e^{i\theta}) P_z(e^{i\theta}) \, d\sigma(e^{i\theta}), \quad z \in \mathbb{D},
\]
where
\[
P_z(e^{i\theta}) = \frac{1 - |z|^2}{|e^{i\theta} - z|^2}, \quad z \in \mathbb{D},
\]
is the Poisson kernel for $z \in \mathbb{D}$ and $\sigma$ is the normalized Lebesgue measure on $\partial \mathbb{D}$. For each $f \in h^\infty$, there exists the radial limit function $f^*$ on $\partial \mathbb{D}$ defined by $f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$ a.e. on $\partial \mathbb{D}$. It is well known that \{ $f^* : f \in h^\infty$ \} = $L^\infty(\partial \mathbb{D})$ and $f = \hat{f}$, so identifying $f$ with $f^*$ we may consider $h^\infty = L^\infty(\partial \mathbb{D})$.

Note that
\[
(C_{\varphi}f)^*(e^{i\theta}) = \begin{cases} 
  f(\varphi^*(e^{i\theta})) & \text{for } |\varphi^*(e^{i\theta})| < 1 \\
  f^*(\varphi^*(e^{i\theta})) & \text{for } |\varphi^*(e^{i\theta})| = 1 
\end{cases}
\]
Lemma 2.1. Let $i \neq j$. For a measurable subset $E$ of $\partial \mathbb{D}$, let $\chi_E$ denote the characteristic function for $E$.

Theorem 2.2. Let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ satisfying $\varphi_i \neq \varphi_j$ for $i \neq j$ and $\lambda_j \in \mathbb{C}$ with $\lambda_j \neq 0$ for every $1 \leq j \leq N$. Then

$$\left\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\| = \sup_{z \in \mathbb{D}} \left\| \sum_{j=1}^{N} \lambda_j P_{\varphi_j}(z) \right\|_1.$$

Proof. For $g \in B(h^\infty)$, we have

$$\left\| \left( \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right) g \right\|_{\infty} = \sup_{z \in \mathbb{D}} \left| \sum_{j=1}^{N} \lambda_j g(\varphi_j(z)) \right| = \sup_{z \in \mathbb{D}} \left| \int_{\partial \mathbb{D}} \left( \sum_{j=1}^{N} \lambda_j P_{\varphi_j}(z) \right) g^*(d\sigma) \right|.$$

Thus we get the assertion. \hfill $\square$

By the proof of Lemma 2.1, one easily sees the following.

Lemma 2.3. Let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ satisfying $\varphi_i \neq \varphi_j$ for $i \neq j$ and $\lambda_j \in \mathbb{C}$ with $\lambda_j \neq 0$ for every $1 \leq j \leq N$. Then there exists a sequence $(z_n)_n$ in $\mathbb{D}$ with $|z_n| \to 1$ as $n \to \infty$ satisfying

$$\left\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\| = \lim_{n \to \infty} \left\| \sum_{j=1}^{N} \lambda_j P_{\varphi_j}(z_n) \right\|_1.$$

The following is an elementary property of Poisson kernels.

Lemma 2.4. For $z, w \in \mathbb{D}$,

$$\|P_z - P_w\|_1 = 2 - \frac{4\cos^{-1}(\rho(z, w))}{\pi}.$$

Let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ satisfying $\varphi_i \neq \varphi_j$ for $i \neq j$. We use the same notations as in [11]. We denote by $\mathcal{Z}(\{\varphi_j\})$ the set of sequences $(z_n)_n$ in $\mathbb{D}$ satisfying the following four conditions:

(a) $(z_n)_n$ is a convergent sequence.
(b) $(\varphi_j(z_n))_n$ is a convergent sequence for every $1 \leq j \leq N$.
(c) $\lim_{n \to \infty} |\varphi_j(z_n)| = 1$ for some $1 \leq j \leq N$.
(d) $(\rho(\varphi_i(z_n), \varphi_j(z_n)))_n$ is a convergent sequence for every $1 \leq i, j \leq N$. a.e. on $\partial \mathbb{D}$.

We denote by $B(h^\infty)$ the closed unit ball of $h^\infty$. For a measurable subset $E$ of $\partial \mathbb{D}$, let $\chi_E$ denote the characteristic function for $E$. 

Lemma 2.2. Let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ satisfying $\varphi_i \neq \varphi_j$ for $i \neq j$ and $\lambda_j \in \mathbb{C}$ with $\lambda_j \neq 0$ for every $1 \leq j \leq N$. Then

$$\left\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\| = \sup_{z \in \mathbb{D}} \left\| \sum_{j=1}^{N} \lambda_j P_{\varphi_j}(z) \right\|_1.$$
In this paper, the set $\mathcal{Z}(\{\varphi_j\})$ acts an important role. Note that for a sequence $\{z_n\}_n$ in $\mathbb{D}$, if $|\varphi_j(z_n)| \to 1$ as $n \to \infty$ for some $1 \leq j \leq N$, then it is easy to see that there exists a subsequence $\{z_{n_k}\}_k$ of $\{z_n\}_n$ satisfying $\{z_{n_k}\}_k \in \mathcal{Z}(\{\varphi_j\})$.

Let $\{z_n\}_n \in \mathcal{Z}(\{\varphi_j\})$. We write $a_j = \lim_{n \to \infty} \varphi_j(z_n)$ for every $1 \leq j \leq N$. Let

\[(2.1)\quad I(\{z_n\}) = \{j : |a_j| = 1, 1 \leq j \leq N\}.\]

We define the equivalence relation $i \sim j$ in $I(\{z_n\})$ by

\[(2.2)\quad \lim_{n \to \infty} \rho(\varphi_i(z_n), \varphi_j(z_n)) = 0.\]

By condition (c) and (2.1), $I(\{z_n\}) \neq \emptyset$. For each $t \in I(\{z_n\})$, let

\[(2.3)\quad I(\{z_n\}, t) = \{j \in I(\{z_n\}) : j \sim t, 1 \leq j \leq N\}.\]

For $s, t \in I(\{z_n\})$, either $I(\{z_n\}, s) = I(\{z_n\}, t)$ or $I(\{z_n\}, s) \cap I(\{z_n\}, t) = \emptyset$ holds. Hence there is a subset $\{t_1, t_2, \ldots, t_\ell\} \subset I(\{z_n\})$ such that $I(\{z_n\}) = \bigcup_{\ell=1}^t I(\{z_n\}, t_p)$ and $I(\{z_n\}, t_p) \cap I(\{z_n\}, t_q) = \emptyset$ for $p \neq q$.

Generally, we have $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \| \leq \sum_{j=1}^N |\lambda_j|$. If $\lambda_1/|\lambda_1| = \lambda_j/|\lambda_j|$ for every $1 \leq j \leq N$, one easily sees that $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \| = \sum_{j=1}^N |\lambda_j|$. The other case, we have the following.

**Theorem 2.5.** Let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ satisfying $\varphi_i \neq \varphi_j$ for $i \neq j$ and $\lambda_j \in \mathbb{C}$ with $\lambda_j \neq 0$ for every $1 \leq j \leq N$. Suppose that $\lambda_i/|\lambda_i| \neq \lambda_j/|\lambda_j|$ for some $1 \leq i, j \leq N$. Then the following conditions are equivalent:

\[(i)\quad \| \sum_{j=1}^N \lambda_j C_{\varphi_j} \| = \sum_{j=1}^N |\lambda_j| \text{ on } \mathcal{H}^\infty.\]

\[(ii)\quad \| \sum_{j=1}^N \lambda_j C_{\varphi_j} \| = \sum_{j=1}^N |\lambda_j| \text{ on } H^\infty.\]

\[(iii)\quad \text{There exists a sequence } \{z_n\}_n \in \mathcal{Z}(\{\varphi_j\}) \text{ satisfying } \rho(\varphi_i(z_n), \varphi_j(z_n)) \to 1 \text{ as } n \to \infty \text{ for every } 1 \leq i, j \leq N \text{ with } \lambda_i/|\lambda_i| \neq \lambda_j/|\lambda_j|.\]

**Proof.** (ii) $\Leftrightarrow$ (iii) was proven in [11, Theorem 3.1]. (ii) $\Rightarrow$ (i) is trivial.

We shall prove (i) $\Rightarrow$ (iii). Suppose that (i) holds. By Lemma 2.2, there is a sequence $\{z_n\}_n$ in $\mathbb{D}$ with $|z_n| \to 1$ such that

$$\lim_{n \to \infty} \left\| \sum_{j=1}^N \lambda_j P_{\varphi_j(z_n)} \right\|_1 = \sum_{j=1}^N |\lambda_j|.$$ 

Suppose that $\lambda_{j_1}/|\lambda_{j_1}| \neq \lambda_{j_2}/|\lambda_{j_2}|$, then

$$\sum_{j=1}^N |\lambda_j| = \lim_{n \to \infty} \left\| \sum_{j=1}^N \lambda_j P_{\varphi_j(z_n)} \right\|_1$$
\[
\leq \liminf_{n \to \infty} \left\| \lambda_j P_{\varphi_j(z_n)} + \lambda_j P_{\varphi_j(z_n)} \right\| + \sum_{j \neq 1, j_2} \left| \lambda_j \right|.
\]

Hence
\[
\left\| \lambda_j P_{\varphi_j(z_n)} + \lambda_j P_{\varphi_j(z_n)} \right\| \to |\lambda_j| + \left| \lambda_j \right|
\]
as \(n \to \infty\). By Lemma 2.3, \( \| P_{\varphi_j}(z_n) - P_{\varphi_j}(z_n) \| \to 2 \). By Lemma 2.4, \( \rho(\varphi_j(z_n), \varphi_j(z_n)) \to 1 \). Hence \( \max\{|\varphi_j(z_n)|, |\varphi_j(z_n)|\} \to 1 \) as \( n \to \infty \). Thus we get (iii). \( \square \)

To study the compactness of \( \sum_{j=1}^N \lambda_j C_{\varphi_j} \), we use the following lemma which follows from that \( B(h^\infty) \) is a normal family.

**Lemma 2.6.** Let \( \varphi_1, \varphi_2, \ldots, \varphi_N \) be functions in \( S(\mathbb{D}) \) satisfying \( \varphi_i \neq \varphi_j \) for \( i \neq j \) and \( \lambda_j \in \mathbb{C} \) with \( \lambda_j \neq 0 \) for every \( 1 \leq j \leq N \). Then \( \sum_{j=1}^N \lambda_j C_{\varphi_j} \) is compact on \( h^\infty \) if and only if \( \| \sum_{j=1}^N \lambda_j C_{\varphi_j} \| \to 0 \) as \( n \to \infty \) for every sequence \( \{f_n\}_n \) in \( B(h^\infty) \) such that \( \{f_n\}_n \) converges to 0 uniformly on any compact subset of \( \mathbb{D} \).

From this, if \( \|\varphi\|_\infty < 1 \), then \( C_{\varphi} \) is compact on \( h^\infty \). The following is a characterization of the compactness of \( \sum_{j=1}^N \lambda_j C_{\varphi_j} \) on \( h^\infty \).

**Theorem 2.7.** Let \( \varphi_1, \varphi_2, \ldots, \varphi_N \) be functions in \( S(\mathbb{D}) \) with \( \|\varphi\|_\infty = 1 \) satisfying \( \varphi_i \neq \varphi_j \) for \( i \neq j \) and \( \lambda_j \in \mathbb{C} \) with \( \lambda_j \neq 0 \) for every \( 1 \leq j \leq N \). Then the following conditions are equivalent:

(i) \( \sum_{j=1}^N \lambda_j C_{\varphi_j} \) is compact on \( h^\infty \).

(ii) \( \sum_{j=1}^N \lambda_j C_{\varphi_j} \) is compact on \( H^\infty \).

(iii) \( \{\lambda_i : i \in I(\{z_n\}, t)\} = 0 \) for every \( \{z_n\}_n \in \mathcal{Z}(\{\varphi_j\}) \) and \( t \in \mathcal{I}(\{z_n\}) \).

**Proof.** (ii) \( \Leftrightarrow \) (iii) was proven in [11, Theorem 2.2]. (i) \( \Rightarrow \) (ii) is trivial.

The proof (iii) \( \Rightarrow \) (i) is the same as the one in [11, Theorem 2.2] essentially. Suppose that \( \sum_{j=1}^N \lambda_j C_{\varphi_j} \) is not compact on \( h^\infty \). By Lemma 2.6, there is a sequence \( \{f_n\}_n \) in \( B(h^\infty) \) such that \( f_n \to 0 \) uniformly on any compact subset of \( \mathbb{D} \) and \( \| \sum_{j=1}^N \lambda_j f_n \circ \varphi_j \| \to 0 \) as \( n \to \infty \). Considering a subsequence of \( \{f_n\}_n \), we may assume that there exists \( \delta > 0 \) such that

\[
\left\| \sum_{j=1}^N \lambda_j f_n \circ \varphi_j \right\| > \delta \quad \text{for every } n \geq 1.
\]

Take a sequence \( \{z_n\}_n \) in \( \mathbb{D} \) satisfying \( |z_n| \to 1 \) and

\[
\left| \sum_{j=1}^N \lambda_j f_n(\varphi_j(z_n)) \right| > \delta \quad \text{for every } n \geq 1.
\]

We may assume that \( \varphi_j(z_n) \to \alpha_j \in \overline{\mathbb{D}} \) for every \( 1 \leq j \leq N \). Since \( f_n \to 0 \) uniformly on any compact subset of \( \mathbb{D} \), \( |\alpha_j| = 1 \) for some \( j \). Considering
a subsequence of \( \{f_n\}_n \), we may assume that \( \{z_n\}_n \in \mathcal{Z}(\{\varphi_j\}) \). By (2.1), \( I(\{z_n\}) = \{j : |\alpha_j| = 1, 1 \leq j \leq N\} \). Then we have

\[
\liminf_{k \to \infty} \left| \sum_{j \in I(\{z_n\})} \lambda_j f_k(\varphi_j(z_k)) \right| \geq \delta.
\]

Let \( \{t_1, t_2, \ldots, t_\ell\} \subset I(\{z_n\}) \) such that \( I(\{z_n\}) = \bigcup_{p=1}^\ell I(\{z_n\}, t_p) \) and \( I(\{z_n\}, t_p) \cap I(\{z_n\}, t_q) = \emptyset \) for \( p \neq q \). Let \( j \in I(\{z_n\}, t_p) \). By (2.2) and (2.3), \( \rho(\varphi_j(z_k), \varphi_{t_p}(z_k)) \to 0 \) as \( k \to \infty \). By Lemma 2.4, \( \|P_{\varphi_j(z_k)} - P_{\varphi_{t_p}(z_k)}\|_1 \to 0 \).

Since \( \{f_k(\varphi_j(z_k))\}_k \) is a bounded sequence, considering a subsequence of \( \{f_k\}_k \), we may assume that \( f_k(\varphi_j(z_k)) \to \gamma_j \in \mathbb{T} \) as \( k \to \infty \) for every \( 1 \leq j \leq N \). We have

\[
|f_k(\varphi_j(z_k)) - f_k(\varphi_{t_p}(z_k))| = \left| \int_{\partial D} f_k^*(P_{\varphi_j(z_k)} - P_{\varphi_{t_p}(z_k)}) \, d\sigma \right|
\leq \|P_{\varphi_j(z_k)} - P_{\varphi_{t_p}(z_k)}\|_1
\to 0 \quad \text{as} \quad k \to \infty.
\]

Thus we get \( \gamma_j = \gamma_{t_p} \) for every \( j \in I(\{z_n\}, t_p) \). Therefore

\[
\lim_{k \to \infty} \sum_{j \in I(\{z_n\})} \lambda_j f_k(\varphi_j(z_k)) = \lim_{k \to \infty} \sum_{p=1}^\ell \sum_{j \in I(\{z_n\}, t_p)} \lambda_j f_k(\varphi_j(z_k))
= \sum_{p=1}^\ell \gamma_{t_p} \sum_{j \in I(\{z_n\}, t_p)} \lambda_j
= 0 \quad \text{by condition (iii)}.
\]

This contradicts with (2.4). Thus we get (iii) \( \Rightarrow \) (i).

3. Essential norms of linear combinations

Let \( \varphi_1, \varphi_2, \ldots, \varphi_N \) be functions in \( \mathcal{S}(D) \) with \( \|\varphi_j\|_\infty = 1 \) satisfying \( \varphi_i \neq \varphi_j \) for \( i \neq j \) and \( \lambda_j \in \mathbb{C} \) with \( \lambda_j \neq 0 \) for every \( 1 \leq j \leq N \). Let \( \mathcal{K} \) be the set of compact operators on \( h^\infty \). The essential norm is defined by

\[
\left\| \sum_{j=1}^n \lambda_j C_{\varphi_j} \right\|_c = \inf_{K \in \mathcal{K}} \left\| K + \sum_{j=1}^n \lambda_j C_{\varphi_j} \right\|.
\]

For each \( \{z_n\}_n \in \mathcal{Z}(\{\varphi_j\}) \), we define

\[
\Gamma(\{z_n\}) = \liminf_{k \to \infty} \left\| \sum_{j \in I(\{z_n\})} \lambda_j P_{\varphi_j(z_k)} \right\|_1.
\]

This term is used to determine the values of essential norms of linear combinations of composition operators. The following is an elementary property of Poisson kernels.
Lemma 3.1. Let \( \{z_{j,k}\}_k \) be sequences in \( \mathbb{D} \) such that \( z_{j,k} \to \alpha_j \) as \( k \to \infty \) and \( |\alpha_j| = 1 \) for every \( 1 \leq j \leq N \). Let \( U \) be an open subset of \( \partial \mathbb{D} \) satisfying \( \{\alpha_j\}_{j=1}^N \subset U \). Then for \( \lambda_j \in \mathbb{C}, 1 \leq j \leq N \), we have
\[
\lim_{k \to \infty} \int_{\partial \mathbb{D}} \chi_U \left| \sum_{j=1}^N \lambda_j P_{z_{j,k}} \right| d\sigma = \lim_{k \to \infty} \left\| \sum_{j=1}^N \lambda_j P_{z_{j,k}} \right\|_1.
\]

First, we give a lower estimate of \( \| \sum_{j=1}^N \lambda_j C_{\varphi_j} \|_e \).

Theorem 3.2. Let \( \varphi_1, \varphi_2, \ldots, \varphi_N \) be functions in \( S(\mathbb{D}) \) with \( \| \varphi_j \|_\infty = 1 \) satisfying \( \varphi_i \neq \varphi_j \) for \( i \neq j \) and \( \lambda_j \in \mathbb{C} \) with \( \lambda_j \neq 0 \) for every \( 1 \leq j \leq N \). Then
\[
\sup_{\{z_n\}_n} \Gamma(\{z_n\}) \leq \left\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \right\|_e.
\]

Proof. Let \( \{z_n\}_n \in \mathcal{Z}(\{\varphi_j\}) \). We may assume that \( \Gamma(\{z_n\}) > 0 \). Considering a subsequence of \( \{z_n\}_n \), we may assume that
\[
\Gamma(\{z_n\}) = \lim_{k \to \infty} \left\| \sum_{j \in I(\{z_n\})} \lambda_j P_{\varphi_j(z_n)} \right\|_1
\]
and
\[
\left\| \sum_{j \in I(\{z_n\})} \lambda_j P_{\varphi_j(z_n)} \right\|_1 \neq 0 \quad \text{for every } k \geq 1.
\]

By condition (b) in Section 2, \( \varphi_j(z_k) \to \alpha_j \in \overline{\mathbb{D}} \) as \( k \to \infty \) for every \( j \). Recall that \( |\alpha_j| = 1 \) for \( j \in I(\{z_n\}) \) and \( |\alpha_j| < 1 \) for \( j \notin I(\{z_n\}) \).

By induction, we shall take a subsequence \( \{z_{n_k}\}_k \) of \( \{z_n\}_n \) and a sequence of open subsets \( \{U_k\}_k \) of \( \partial \mathbb{D} \) satisfying the following two conditions;
\[
\{\alpha_j : j \in I(\{z_n\})\} \subset U_{k+1} \subset U_k, \quad \{\alpha_j : j \in I(\{z_n\})\} = \bigcap_{k=1}^\infty U_k
\]
and
\[
\int_{\partial \mathbb{D}} \chi_{(U_k \setminus U_{k+1})} \left| \sum_{j \in I(\{z_n\})} \lambda_j P_{\varphi_j(z_{n_k})} \right| d\sigma > \Gamma(\{z_n\}) \left(1 - \frac{1}{k}\right).
\]

Put \( n_1 = 1 \). Then there is an open subset \( U_1 \) of \( \partial \mathbb{D} \) with \( \{\alpha_j : j \in I(\{z_n\})\} \subset U_1 \) such that
\[
\int_{\partial \mathbb{D}} \chi_{U_1} \left| \sum_{j \in I(\{z_n\})} \lambda_j P_{\varphi_j(z_{n_k})} \right| d\sigma > 0.
\]

We may take an open subset \( U_2 \) of \( \partial \mathbb{D} \) with \( \{\alpha_j : j \in I(\{z_n\})\} \subset U_2 \subset U_1 \) such that
\[
\int_{\partial \mathbb{D}} \chi_{(U_1 \setminus U_2)} \left| \sum_{j \in I(\{z_n\})} \lambda_j P_{\varphi_j(z_{n_k})} \right| d\sigma > 0.
\]
Let \( m \) be a positive integer. We assume that \( \{z_{n_1}, z_{n_2}, \ldots, z_{n_m}\} \) and \( \{U_1, U_2, \ldots, U_{m+1}\} \) are taken satisfying conditions (3.3) and (3.4). We have

\[
\lim_{k \to \infty} \int_{\partial \mathbb{D}} \chi_{U_{m+1}} \left| \sum_{j \in I(z_{n_k})} \lambda_j P_{\varphi_j(z_k)} \right| d\sigma = \Gamma(\{z_n\}) \quad \text{by Lemma 3.1}
\]

Hence there exists a positive integer \( n_{m+1} \) such that

\[
\int_{\partial \mathbb{D}} \chi_{U_{m+1}} \left| \sum_{j \in I(z_{n_k})} \lambda_j P_{\varphi_j(z_{n_{m+1}})} \right| d\sigma > \Gamma(\{z_n\}) \left(1 - \frac{1}{m+1}\right).
\]

It is not difficult to take an open subset \( U_{m+2} \) of \( \partial \mathbb{D} \) with \( \{\alpha_j : j \in I(\{z_n\})\} \subset U_{m+2} \subset U_{m+1} \) such that

\[
\int_{\partial \mathbb{D}} \chi_{U_{m+1} \setminus U_{m+2}} \left| \sum_{j \in I(z_{n_k})} \lambda_j P_{\varphi_j(z_{n_{m+1}})} \right| d\sigma > \Gamma(\{z_n\}) \left(1 - \frac{1}{m+1}\right).
\]

Of course we may take \( \{U_k\}_k \) satisfying the second condition in (3.3). This completes the induction.

For each positive integer \( k \), there exists a function \( g_k \in L^\infty(\partial \mathbb{D}) \) such that

\[
g_k \left( \sum_{j \in I(z_{n_k})} \lambda_j P_{\varphi_j(z_{n_k})} \right) = \left| \sum_{j \in I(z_{n_k})} \lambda_j P_{\varphi_j(z_{n_k})} \right| \quad \text{a.e. on } \partial \mathbb{D}.
\]

Note that \( |g_k| = 1 \) a.e. on \( \partial \mathbb{D} \). Let \( h_k = \chi_{U_k \setminus U_{k+1}} g_k \). Then \( \|h_k\|_\infty = 1 \), and by (3.3) \( h_k \to 0 \) weakly in \( L^\infty(\partial \mathbb{D}) \), so \( \hat{h}_k \to 0 \) weakly in \( \mathcal{H}^\infty \). Let \( K \) be a compact operator on \( \mathcal{H}^\infty \). Then we have \( \|K h_k\|_\infty \to 0 \) and

\[
\left| \sum_{j \notin I(z_{n_k})} \lambda_j \hat{h}_k(\varphi_j(z_{n_k})) \right| \to 0 \quad \text{as } k \to \infty.
\]

Therefore

\[
\left\| \sum_{j=1}^N \lambda_j C_{\varphi_j} + K \right\| \\
\geq \limsup_{k \to \infty} \left\| \left( \sum_{j=1}^N \lambda_j C_{\varphi_j} \right) \hat{h}_k + K \hat{h}_k \right\|_\infty \\
= \limsup_{k \to \infty} \left\| \sum_{j=1}^N \lambda_j \hat{h}_k \circ \varphi_j \right\|_\infty
\]
Thus we get the assertion.

Next we shall study an upper estimate of \( \| \sum_{j=1}^{n} \lambda_j C_{\varphi_j} \|_e \) and this is the main subject of this paper. For \( g \in L^\infty(\partial \mathbb{D}) \), we define the bounded linear operator \( M_g \) on \( h^\infty \) by

\[
(M_g f)(z) = \int_{\partial \mathbb{D}} f^* g P_z \, d\sigma, \quad f \in h^\infty, \quad z \in \mathbb{D}.
\]

Let \( U, V \) be measurable subsets of \( \partial \mathbb{D} \). Then \( M_{X_U} M_{X_V} = M_{X_U \cap X_V} \) and \( I = M_{X_U} + M_{X_V} \), where \( I \) is the identity operator on \( h^\infty \).

**Lemma 3.3.** Let \( \varphi \in \mathcal{S}(\mathbb{D}) \) with \( \| \varphi \|_\infty = 1 \). For \( 0 < \delta < 1 \), let \( U \) be a measurable subset of \( \partial \mathbb{D} \) with \( U \subset \{ e^{i\theta} \in \partial \mathbb{D} : |\varphi^*(e^{i\theta})| \leq \delta \} \). Then \( M_{X_U} C_\varphi \) is compact on \( h^\infty \).

**Proof.** Let \( \{ f_n \}_n \) be a sequence in \( B(h^\infty) \) such that \( \{ f_n \}_n \) converges uniformly on any compact subset of \( \mathbb{D} \). Then \( \sup_{e^{i\theta} \in U} |f_n(\varphi^*(e^{i\theta}))| \to 0 \) as \( n \to \infty \). Hence

\[
\| M_{X_U} C_\varphi f_n \| = \| \chi_U(f_n \circ \varphi) \| = \sup_{z \in \mathbb{D}} \left| \int_{U} f_n(\varphi^*) P_z \, d\sigma \right| \to 0
\]

as \( n \to \infty \). By Lemma 2.6, \( M_{X_U} C_\varphi \) is compact on \( h^\infty \). \( \square \)

One easily checks the following.

**Lemma 3.4.** Let \( U_1, U_2, \ldots, U_m \) be measurable subsets of \( \partial \mathbb{D} \) with \( U_i \cap U_j = \emptyset \) for \( i \neq j \). For every bounded linear operators \( T_1, T_2, \ldots, T_m \) on \( h^\infty \), we have

\[
\left\| \sum_{j=1}^{m} M_{X_{U_j}} T_j \right\| = \max_{1 \leq j \leq m} \| M_{X_{U_j}} T_j \|.
\]

**Lemma 3.5.** Let \( \varphi_1, \varphi_2, \ldots, \varphi_N \) be functions in \( \mathcal{S}(\mathbb{D}) \) satisfying \( \varphi_i \neq \varphi_j \) for \( i \neq j \) and \( \lambda_j \in \mathbb{C} \) with \( \lambda_j \neq 0 \) for every \( 1 \leq j \leq N \). Then

\[
\left\| M_{X_U} \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\|_e = \inf_{K \in \mathbb{K}} \left\| M_{X_U} \left( K + \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right) \right\|.
\]
Proof. It is trivial that
\[ \| M_{X_U} \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \| \leq \inf_{K \in \mathcal{K}} \left\| M_{X_U} \left( K + \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right) \right\|. \]

Let \( K \in \mathcal{K} \). Then
\[ \| K + M_{X_U} \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \| = \left\| (M_{X_U} + M_{X_U^c})K + M_{X_U} \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\| = \left\| M_{X_U} \left( K + \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right) \right\| + M_{X_U} K \]
\[ \geq \left\| M_{X_U} \left( K + \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right) \right\| \text{ by Lemma 3.4.} \]
Therefore we get the assertion. \( \square \)

Let \( \varphi_1, \varphi_2, \ldots, \varphi_N \) be functions in \( \mathcal{S}(D) \) with \( \| \varphi_j \|_\infty = 1 \) satisfying \( \varphi_i \neq \varphi_j \) for \( i \neq j \). For \( 0 < \delta < 1 \), write
\[ W_{\delta,j} = \{ e^{i\theta} \in \partial D : |\varphi_j^*(e^{i\theta})| > \delta \}. \]
We define the family \( \Lambda \) by
\[ \Lambda = \{ p = (p_1, p_2, \ldots, p_N) : p_j = 0 \text{ or } 1, 1 \leq j \leq N \}. \]
We use the following notations;
(3.5) \[ W^0_{\delta,j} = W_{\delta,j} \quad \text{and} \quad W^1_{\delta,j} = W^c_{\delta,j} = \partial D \setminus W_{\delta,j}. \]
For each \( p = (p_1, p_2, \ldots, p_N) \in \Lambda \), write
(3.6) \[ W_{\delta,p} = \bigcap_{j=1}^{N} W^p_{\delta,j} \quad \text{and} \quad \tilde{p} = \{ j : p_j = 0, 1 \leq j \leq N \}. \]
Note that \( W_{\delta,p} \) may be an empty set for some \( p \in \Lambda \), and \( W_{\delta,p} \cap W_{\delta,q} = \emptyset \) holds for \( p, q \in \Lambda \) with \( p \neq q \). We have
\[ 1 = \prod_{j=1}^{N} (\chi_{W_{\delta,j}} + \chi_{W_{\delta,j}^c}) = \sum_{p \in \Lambda} \chi_{W_{\delta,p}} \text{ on } \partial D. \]
Hence \( I = \sum_{p \in \Lambda} M_{X_{W_{\delta,p}}} \) on \( h^\infty \).
Lemma 3.6. Let \( \varphi_1, \varphi_2, \ldots, \varphi_N \) be functions in \( S(D) \) with \( \| \varphi_j \|_\infty = 1 \) satisfying \( \varphi_i \neq \varphi_j \) for \( i \neq j \) and \( \lambda_j \in \mathbb{C} \) with \( \lambda_j \neq 0 \) for every \( 1 \leq j \leq N \). For a measurable subset \( U \) of \( \partial D \), \( 0 < \delta < 1 \) and a nonempty subset \( L \subseteq \{1, 2, \ldots, N\} \), we have

\[
\left\| M_{X_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \right\|_e = \max_{p \in \Lambda} \left\| M_{X_{(U \cap W_{k,p})}} \sum_{j \in L \cap \hat{p}} \lambda_j C_{\varphi_j} \right\|_e.
\]

Proof. Let \( p \in \Lambda \). If \( j \notin \hat{p} \), then by (3.5) and (3.6) \( |\chi_{W_{k,p}} \varphi_j^*| \leq \delta \) on \( \partial D \). Hence by Lemma 3.3, \( M_{X_{W_{k,p}}} \varphi_j \in \mathcal{K} \). Let \( K_p \in \mathcal{K} \) for every \( p \in \Lambda \). We have

\[
\left\| M_{X_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \right\|_e = \left\| M_{X_U} \left( \sum_{p \in \Lambda} M_{X_{W_{k,p}}} \right) \sum_{j \in L} \lambda_j C_{\varphi_j} \right\|_e
\]

\[
= \left\| M_{X_U} \sum_{p \in \Lambda} \left( M_{X_{W_{k,p}}} \sum_{j \in L \cap \hat{p}} \lambda_j C_{\varphi_j} \right) \right\|_e
\]

\[
\leq \left\| \sum_{p \in \Lambda} M_{X_{(U \cap W_{k,p})}} \left( K_p + \sum_{j \in L \cap \hat{p}} \lambda_j C_{\varphi_j} \right) \right\|
\]

\[
= \max_{p \in \Lambda} \left\| M_{X_{(U \cap W_{k,p})}} \left( K_p + \sum_{j \in L \cap \hat{p}} \lambda_j C_{\varphi_j} \right) \right\| \quad \text{by Lemma 3.4.}
\]

Hence

\[
\left\| M_{X_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \right\|_e \leq \max_{p \in \Lambda} \inf_{K_p \in \mathcal{K}} \left\| M_{X_{(U \cap W_{k,p})}} \left( K_p + \sum_{j \in L \cap \hat{p}} \lambda_j C_{\varphi_j} \right) \right\|
\]

\[
= \max_{p \in \Lambda} \left\| M_{X_{W_{k,p}}} \sum_{j \in L \cap \hat{p}} \lambda_j C_{\varphi_j} \right\|_e \quad \text{by Lemma 3.5.}
\]

There also exists a sequence \( \{K_n\}_n \) in \( \mathcal{K} \) satisfying

\[
\left\| M_{X_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \right\|_e = \lim_{n \to \infty} \left\| K_n + M_{X_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \right\|.
\]

We have

\[
\left\| K_n + M_{X_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \right\| = \left\| \left( \sum_{p \in \Lambda} M_{X_{W_{k,p}}} \right) \left( K_n + M_{X_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \right) \right\|
\]

\[
= \max_{p \in \Lambda} \left\| M_{X_{W_{k,p}}} \left( K_n + M_{X_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \right) \right\| \quad \text{by Lemma 3.4}
\]
\[
\begin{align*}
= \max_{p \in \Lambda} \left\| M_{\chi_{W_p}} \left( K_n + M_{\chi_U} \sum_{j \in L \cap \partial p} \lambda_j C_{\varphi_j} + M_{\chi_U} \sum_{j \in L \setminus \partial p} \lambda_j C_{\varphi_j} \right) \right\| \\
\geq \max_{p \in \Lambda} \left\| M_{\chi_{(U \cap W_p)}} \sum_{j \in L \cap \partial p} \lambda_j C_{\varphi_j} \right\|_e.
\end{align*}
\]

Therefore we get
\[
\left\| M_{\chi_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \right\|_e \geq \max_{p \in \Lambda} \left\| M_{\chi_{(U \cap W_p)}} \sum_{j \in L \cap \partial p} \lambda_j C_{\varphi_j} \right\|_e. \tag{3.7}
\]

**Lemma 3.7.** Let \( \varphi_1, \varphi_2, \ldots, \varphi_N \) be functions in \( \mathcal{S}(\mathbb{D}) \) with \( \| \varphi_j \|_\infty = 1 \) satisfying \( \varphi_i \neq \varphi_j \) for \( i \neq j \) and \( \lambda_j \in \mathbb{C} \) with \( \lambda_j \neq 0 \) for every \( 1 \leq j \leq N \). Let \( U \) be a measurable subset of \( \partial \mathbb{D} \) with \( \sigma(U) > 0 \) and a nonempty subset \( L \subset \{1, 2, \ldots, N\} \). Let \( 0 < \delta_1 < 1 \) for \( i = 1, 2 \). Suppose that \( |\varphi_j^*| > \delta_1 \) a.e. on \( U \) for every \( j \in L \) and \( |\varphi_j^*| \leq \delta_2 \) a.e. on \( U \) for every \( j \notin L \). Then there is a sequence \( \{z_n\}_n \in \mathbb{D} \) satisfying the following conditions:

(i) \( \left\| M_{\chi_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \right\| \leq \liminf_{n \to \infty} \left\| \sum_{j \in L} \lambda_j P_{\varphi_j(z_n)} \right\|_1, \)

(ii) \( \left( \prod_{j \in L} \varphi_j \right)(z_n) > \delta_1^N \) for every \( n \geq 1. \)

(iii) \( |\varphi_j(z_n)| < (1 + \delta_2)/2 \) for every \( n \geq 1 \) and \( j \notin L. \)

**Proof.** Note that
\[
\prod_{j \in L} \varphi_j^* > \delta_1^N \quad \text{a.e. on } U
\]
and
\[
A := \left\| M_{\chi_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \right\| > 0.
\]

For each positive integer \( n \), there exists \( f_n \in B(h^\infty) \) satisfying
\[
A - \frac{1}{n} < \left\| \left( M_{\chi_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \right) f_n \right\|_\infty \leq A,
\]
that is,
\[
A - \frac{1}{n} < \left\| \chi_U \sum_{j \in L} \lambda_j (f_n \circ \varphi_j)^* \right\|_\infty \leq A.
\]

By (3.7), there is \( z_n \in \mathbb{D} \) such that
\[
\left| \left( \prod_{j \in L} \varphi_j \right)(z_n) \right| > \delta_1^N
\]
and
\[
A - \frac{1}{n} < \left| \int_{\partial \mathbb{D}} \chi_U \left( \sum_{j \in L} \lambda_j (f_n \circ \varphi_j)^* \right) P_{z_n} \, d\sigma \right| < A. \tag{3.8}
\]
We have
\[ A - \frac{1}{n} < \left| \int_U \sum_{j \in L} \lambda_j (f_n \circ \varphi_j)^* P_{z_n} \, d\sigma \right| \]
\[ = \left| \int_{\partial D} \sum_{j \in L} \lambda_j (f_n \circ \varphi_j)^* P_{z_n} \, d\sigma - \int_{U^c} \sum_{j \in L} \lambda_j (f_n \circ \varphi_j)^* P_{z_n} \, d\sigma \right| \]
\[ \leq \left| \sum_{j \in L} \lambda_j f_n (\varphi_j(z_n)) \right| + \left| \int_{U^c} \sum_{j \in L} \lambda_j (f_n \circ \varphi_j)^* P_{z_n} \, d\sigma \right| \]
\[ := I_1(n) + I_2(n) \] say.
We have
\[ I_1(n) \leq \left\| \sum_{j \in L} \lambda_j P_{\varphi_j(z_n)} \right\|_1. \]
Also by (3.8),
\[ A - \frac{1}{n} < \left| \int_U \sum_{j \in L} \lambda_j (f_n \circ \varphi_j)^* P_{z_n} \, d\sigma \right| \leq A \int_U P_{z_n} \, d\sigma, \]
so we get \( A \int_{U^c} P_{z_n} \, d\sigma < 1/n \). Since \( A \neq 0 \), \( \int_{U^c} P_{z_n} \, d\sigma \to 0 \). Hence we have \( I_2(n) \to 0 \) as \( n \to \infty \). Therefore
\[ A \leq \lim \inf_{n \to \infty} \left\| \sum_{j \in L} \lambda_j P_{\varphi_j(z_n)} \right\|_1. \]
Thus we get (i).
Let \( j \notin L \). By the assumption, \( |\varphi_j^*| \leq \delta_2 \) a.e. on \( U \). Then
\[ |\varphi_j(z_n)| \leq \left| \int_U \varphi_j^* P_{z_n} \, d\sigma \right| + \int_{U^c} \varphi_j^* P_{z_n} \, d\sigma \leq \delta_2 + \left| \int_{U^c} P_{z_n} \, d\sigma \right|. \]
Thus we get
\[ \limsup_{n \to \infty} |\varphi_j(z_n)| \leq \delta_2. \]
Considering a subsequence of \( \{z_n\}_n \), we have (iii).

**Theorem 3.8.** Let \( \varphi_1, \varphi_2, \ldots, \varphi_N \) be functions in \( S(D) \) with \( \|\varphi_j\|_\infty = 1 \) satisfying \( \varphi_i \neq \varphi_j \) for \( i \neq j \) and \( \lambda_j \in \mathbb{C} \) with \( \lambda_j \neq 0 \) for every \( 1 \leq j \leq N \). Then there exists \( \{z_n\}_n \in \mathcal{Z}(\{\varphi_j\}) \) satisfying
\[ \left\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \right\|_e \leq \Gamma(\{z_n\}). \]

**Proof.** We may assume that
\[ (3.9) \quad \left\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \right\|_e > 0. \]
Take a sequence \(\{\delta_n\}_n\) satisfying \(0 < \delta_n < \delta_{n+1} < 1\) and \(\delta_n \to 1\). By Lemma 3.6,
\[
\left\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\|_e = \max_{p \in \Lambda} \left\| M_{X_{W_{\delta_1, p}}} \sum_{j \in p} \lambda_j C_{\varphi_j} \right\|_e.
\]
Hence there exists \(p^1 \in \Lambda\) satisfying
\[
\left\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\|_e = \left\| M_{X_{W_{\delta_1, p^1}}} \sum_{j \in p^1} \lambda_j C_{\varphi_j} \right\|_e.
\]
By Lemma 3.6 again,
\[
\left\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\|_e = \max_{p \in \Lambda} \left\| M_{X_{W_{\delta_1, p} \cap W_{\delta_2, p}}} \sum_{j \in p \cap p^2} \lambda_j C_{\varphi_j} \right\|_e.
\]
Hence there exists \(p^2 \in \Lambda\) satisfying
\[
\left\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\|_e = \left\| M_{X_{W_{\delta_1, p^1} \cap W_{\delta_2, p}}} \sum_{j \in p^1 \cap p^2} \lambda_j C_{\varphi_j} \right\|_e.
\]
Repeating the same argument, there exists a sequence \(\{p^\ell\}_\ell\) in \(\Lambda\) satisfying
\[
\left\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\|_e = \left\| M_{X_{W_{\delta_1, p^1} \cap \cdots \cap W_{\delta_k, p^k}}} \sum_{\ell=1}^{k} \lambda_j C_{\varphi_j} \right\|_e.
\]
Since \(\tilde{p} \subset \{1, 2, \ldots, N\}\), there exists a positive integer \(k_0\) satisfying
\[
(3.10) \quad L_0 := \bigcap_{\ell=1}^{k_0} p^\ell = \bigcap_{\ell=1}^{k} \tilde{p}^\ell \quad \text{for every } k \geq k_0.
\]
By (3.9), we have \(L_0 \neq \emptyset\),
\[
\sigma \left( \bigcap_{\ell=1}^{k} W_{\delta_{\ell}, p^\ell} \right) > 0
\]
and
\[
(3.11) \quad \left\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\|_e = \left\| M_{X_{W_{\delta_1, p^1} \cap \cdots \cap W_{\delta_k, p^k}}} \sum_{\ell=1}^{k} \lambda_j C_{\varphi_j} \right\|_e
\]
for every \(k \geq k_0\).
Let \(k \geq k_0\) and \(j \in L_0\). By (3.10), \(j \in \tilde{p}^k\), so by (3.5), \(p^k_j = 0\) and by (3.6),
\[
\bigcap_{\ell=1}^{k} W_{\delta_{\ell}, p^\ell} \subset W_{\delta_{k}, p^k} \subset W_{\delta_{\ell}, p^\ell} = W_{\delta_{\ell}, p^\ell}.
\]
Since \( W_{δ_k,j} = \{ e^{iθ} ∈ \partial \mathbb{D} : |φ_j^*(e^{iθ})| > δ_k \} \), we get
\[
|φ_j^*| > δ_k \text{ a.e. on } \bigcap_{ℓ=1}^k W_{δ_ℓ,p_ℓ} \text{ for } j \in L_0.
\]

Let \( j \notin L_0 \). By (3.10), there is an integer \( i \) with \( 1 ≤ i ≤ k_0 \) such that \( j \notin \tilde{p}_i \), so \( p_δ^j = 1 \). Hence by (3.5) and (3.6),
\[
\bigcap_{ℓ=1}^k W_{δ_ℓ,p_ℓ} \subset W_{δ_i,p_i} \subset W_{δ^i_j} = \delta_j^i.
\]

Let \( 0 = j / i \). By (3.10), there is an integer \( j / i \) such that
\[
|φ_j^*| ≤ δ_i \text{ a.e. on } W_{δ^i_j} \text{ and } \delta_i ≤ δ_k_0, \text{ we have}
\]
\[
|φ_j^*| ≤ δ_k_0 \text{ a.e. on } \bigcap_{ℓ=1}^k W_{δ_ℓ,p_ℓ} \text{ for } j \notin L_0.
\]

Applying Lemma 3.7, for each \( k ≥ k_0 \) there is a sequence \( \{z_n \}_{n} \) in \( \mathbb{D} \) satisfying
\[
\begin{align}
(3.12) \quad \left\| M_{\cap_{ℓ=1}^k W_{δ_ℓ,p_ℓ}} \sum_{j \in L_0} λ_j Cφ_j \right\| &= \liminf_{n \to ∞} \left\| \sum_{j \in L_0} λ_j Pφ_j(z_{n,j}) \right\|_1, \\
(3.13) \quad \left| \left( \prod_{j \in L_0} φ_j \right)(z_{k,n}) \right| ≥ δ_k^N \quad \text{for every } n ≥ 1
\end{align}
\]
and
\[
(3.14) \quad |φ_j(z_{k,n})| < (1 + δ_k)/2 \quad \text{for every } n ≥ 1 \text{ and } j \notin L_0.
\]

We have
\[
\left| \sum_{j=1}^N λ_j Cφ_j \right|_c = \left| M_{\cap_{ℓ=1}^k W_{δ_ℓ,p_ℓ}} \sum_{j \in L_0} λ_j Cφ_j \right|_c \quad \text{by (3.11)}
\]
\[
\leq \left| M_{\cap_{ℓ=1}^k W_{δ_ℓ,p_ℓ}} \sum_{j \in L_0} λ_j Cφ_j \right|
\]
\[
= \liminf_{n \to ∞} \left| \sum_{j \in L_0} λ_j Pφ_j(z_{n,j}) \right|_1 \quad \text{by (3.12)}.
\]

For each \( k ≥ k_0 \), we may take a positive integer \( n_k \) satisfying
\[
\liminf_{n \to ∞} \left| \sum_{j \in L_0} λ_j Pφ_j(z_{n,j}) \right|_1 - \frac{1}{k} \leq \left| \sum_{j \in L_0} λ_j Pφ_j(z_{n,k,j}) \right|_1.
\]
Then
\[
(3.15) \quad \left| \sum_{j=1}^N λ_j Cφ_j \right|_c ≤ \liminf_{k \to ∞} \left| \sum_{j \in L_0} λ_j Pφ_j(z_{n,k,j}) \right|_1.
\]
By (3.13),
\[
\left| \prod_{j \in L_0} \varphi_j \left( z_{k,n} \right) \right| > \delta_k^N \quad \text{for every } k \geq k_0.
\]
Since \( \delta_k \to 1 \),
\[
\lim_{k \to \infty} \left| \prod_{j \in L_0} \varphi_j \left( z_{k,n} \right) \right| = 1.
\]
By (3.14),
\[
\limsup_{k \to \infty} |\varphi_j(z_{k,n})| \leq \left( 1 + \delta_{k_0} \right) / 2 \quad \text{for every } j \notin L_0.
\]
Considering a subsequence of \( \{ z_{k,n} \} \), we may assume that \( \{ z_{k,n} \} \in Z(\{ \varphi_j \}) \) and \( I(\{ z_{k,n} \}) = L_0 \). Hence
\[
\left\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \right\|_e \leq \liminf_{m \to \infty} \left\| \sum_{j \in I(\{ z_{k,n} \})} \lambda_j P_{\varphi_j(z_{m,n})} \right\|_1 \quad \text{by (3.15)}
\]
\[
= \Gamma(\{ z_n \}) \quad \text{by (3.1)}.
\]
This completes the proof. \( \square \)

Combining Theorems 3.2 with 3.8, we have the main theorem.

**Theorem 3.9.** Let \( \varphi_1, \varphi_2, \ldots, \varphi_N \) be functions in \( S(\mathbb{D}) \) with \( \| \varphi_j \|_{\infty} = 1 \) satisfying \( \varphi_i \neq \varphi_j \) for \( i \neq j \) and \( \lambda_j \in \mathbb{C} \) with \( \lambda_j \neq 0 \) for every \( 1 \leq j \leq N \). Then
\[
\left\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \right\|_e = \max_{\{ z_n \} \in \mathcal{Z}(\{ \varphi_j \})} \Gamma(\{ z_n \}).
\]

**Corollary 3.10.** Let \( \varphi_1, \varphi_2, \ldots, \varphi_N \) be functions in \( S(\mathbb{D}) \) with \( \| \varphi_j \|_{\infty} = 1 \) satisfying \( \varphi_i \neq \varphi_j \) for \( i \neq j \) and \( \lambda_j \in \mathbb{C} \) with \( \lambda_j \neq 0 \) for every \( 1 \leq j \leq N \). Then \( \sum_{j=1}^N \lambda_j C_{\varphi_j} \) is a compact operator on \( h^\infty \) if and only if \( \Gamma(\{ z_n \}) = 0 \) for every \( \{ z_n \} \in \mathcal{Z}(\{ \varphi_j \}) \).

In the last part of this section, we give a characterization for \( \sum_{j=1}^N \lambda_j C_{\varphi_j} \) to satisfy
\[
\left\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \right\|_e = \sum_{j=1}^N |\lambda_j|.
\]
Theorem 3.11. Let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be functions in $S(D)$ with $\|\varphi_j\|_\infty = 1$ satisfying $\varphi_i \neq \varphi_j$ for $i \neq j$ and $\lambda_j \in \mathbb{C}$ with $\lambda_j \neq 0$ for every $1 \leq j \leq N$. Then
\[
\left\| \sum_{j=1}^{N} \lambda_j C\varphi_j \right\|_e = \sum_{j=1}^{N} |\lambda_j|
\]
if and only if there is $\{z_n\}_n \in \mathcal{Z}((\varphi_j))$ satisfying the following conditions:

(i) $\left| \left( \prod_{j=1}^{N} \varphi_j \right)(z_n) \right| \to 1$ as $n \to \infty$.

(ii) $\rho(\varphi_i(z_n), \varphi_j(z_n)) \to 1$ as $n \to \infty$ for every $1 \leq i, j \leq N$ with $\lambda_i/|\lambda_i| \neq \lambda_j/|\lambda_j|$.

To prove the above theorem, we need a lemma.

Lemma 3.12. Let $\{F_{j,n}\}_n, 1 \leq j \leq m$, be sequences of positive functions in $L^1(\partial D)$. Suppose that $\|F_{i,n}\|_1 \to c_i \neq \infty$ for every $1 \leq j \leq N$ and $\|F_{i,n} - F_{j,n}\|_1 \to c_i + c_j$ for every $1 \leq i, j \leq N$ with $i \neq j$ as $n \to \infty$. Then for $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{C}$, we have
\[
\left\| \sum_{j=1}^{m} \lambda_j F_{j,n} \right\|_1 \to \sum_{j=1}^{m} |\lambda_j|c_j \quad \text{as } n \to \infty.
\]

Proof. Let $E_{i,j,n} = \{e^{i\theta} \in \partial D : (F_{i,n} - F_{j,n})(e^{i\theta}) > 0\}$, $i \neq j$. Then $E_{i,j,n}^c = E_{j,i,n}$. Since
\[
\|F_{i,n} - F_{j,n}\|_1 = \int_{E_{i,j,n}} F_{i,n} - F_{j,n} d\sigma + \int_{E_{i,j,n}^c} F_{j,n} - F_{i,n} d\sigma
\]
and $\|F_{i,n} - F_{j,n}\|_1 \to c_i + c_j$, we have
\[
\int_{E_{i,j,n}} F_{i,n} d\sigma \to c_i \quad \text{and} \quad \int_{E_{i,j,n}} F_{j,n} d\sigma \to c_j.
\]
Hence
\[
\int_{E_{i,j,n}} F_{j,n} d\sigma \to c_j \quad \text{for } i \neq j.
\]

For each $1 \leq j \leq N$, we write
\[
E_{j,n} = \bigcap_{i: i \neq j} E_{j,i,n}.
\]
Then
\[
\int_{E_{j,n}} F_{j,n} d\sigma \to c_j \quad \text{and} \quad \int_{E_{j,n}} F_{j,n} d\sigma \to 0.
\]
For \( j_1 \neq j_2 \), we have
\[
\tilde{E}_{j_1,n} \cap \tilde{E}_{j_2,n} = \left( \bigcap_{t,t \neq j_1} E_{j_1,t,n} \right) \cap \left( \bigcap_{s,s \neq j_2} E_{j_2,s,n} \right) \\
\subseteq E_{j_1,j_2,n} \cap E_{j_2,j_1,n} \\
= E_{j_1,j_2,n} \cap E_{j_1,j_2,n}^c = \emptyset.
\]
Hence
\[
\sum_{j=1}^{m} \lambda_j F_{j,n} \geq \sum_{j=1}^{m} \int_{E_{j,n}} \lambda_j d\sigma \\
\geq \sum_{j=1}^{m} \left( \int_{E_{j,n}} |\lambda_j| d\sigma - \sum_{i \neq j} |\lambda_i| \int_{E_{j,n}} F_{i,n} d\sigma \right) \\
\to \sum_{j=1}^{m} |\lambda_j| \epsilon_j \quad \text{as } n \to \infty \text{ by (3.16)}.
\]
This completes the proof. \( \square \)

**Proof of Theorem 3.11.** Suppose that
\[
\left\| \sum_{j=1}^{N} \lambda_j C_{j,n} \right\|_{C_{1}} = \sum_{j=1}^{N} |\lambda_j|.
\]
By Theorem 3.9, there exists a sequence \( \{z_n\}_n \in \mathcal{Z}(\{\varphi_j\}) \) satisfying
\[
\Gamma(\{z_n\}) = \sum_{j=1}^{N} |\lambda_j|.
\]
By (3.1),
\[
\liminf_{k \to \infty} \left\| \sum_{j \in I(\{z_n\})} \lambda_j P_{j,n}(z_k) \right\|_{1} = \sum_{j=1}^{N} |\lambda_j|.
\]
This shows that \( I(\{z_n\}) = \{1, 2, \ldots, N\} \), so (i) holds, and we have
\[
\lim_{k \to \infty} \left\| \sum_{j=1}^{N} \lambda_j P_{j,n}(z_k) \right\|_{1} = \sum_{j=1}^{N} |\lambda_j|.
\]
By the proof of Theorem 2.5, we get condition (ii).
Suppose that there is \( \{z_n\}_n \in \mathcal{Z}(\{\varphi_j\}) \) satisfying (i) and (ii). Then \( I(\{z_n\}) = \{1, 2, \ldots, N\} \). For each \( 1 \leq j \leq N \), let
\[
J_j = \{i : \lambda_i/|\lambda_i| = \lambda_j/|\lambda_j|\}.
\]
Then there exist \( j_1, j_2, \ldots, j_\ell \) such that \( J_{j_t} \cap J_{j_s} = \emptyset \) for \( t \neq s \) and \( \bigcup_{t=1}^{\ell} J_{j_t} = \{1, 2, \ldots, N\} \). We have

\[
\left\| \sum_{j=1}^{N} \lambda_j P_{\varphi_j(z_n)} \right\|_1 = \left\| \sum_{t=1}^{\ell} \sum_{i \in J_{j_t}} \lambda_i P_{\varphi_i(z_n)} \right\|_1
\]

\[
= \left\| \sum_{t=1}^{\ell} \frac{\lambda_j}{|\lambda_j|} \sum_{i \in J_{j_t}} |\lambda_i| P_{\varphi_i(z_n)} \right\|_1.
\]

Let \( 1 \leq t, s \leq \ell \) with \( t \neq s \). By condition (ii), for every \( i_1 \in J_{j_t} \) and \( i_2 \in J_{j_s} \), we have \( \rho(\varphi_{i_1}(z_n), \varphi_{i_2}(z_n)) \to 1 \). By Lemma 2.4,

\[
\| P_{\varphi_{i_1}(z_n)} - P_{\varphi_{i_2}(z_n)} \|_1 \to 2 \quad \text{as } n \to \infty.
\]

We write

\[ F_{t,n} = \sum_{i \in J_{j_t}} |\lambda_i| P_{\varphi_i(z_n)}. \]

Then \( \| F_{t,n} \|_1 = \sum_{i \in J_{j_t}} |\lambda_i| \), and

\[
\| F_{t,n} - F_{s,n} \|_1 \to \left( \sum_{i \in J_{j_t}} |\lambda_i| \right) + \left( \sum_{i \in J_{j_s}} |\lambda_i| \right) \quad \text{as } n \to \infty.
\]

Therefore by Lemma 3.12,

\[
\left\| \sum_{j=1}^{N} \lambda_j P_{\varphi_j(z_n)} \right\|_1 = \left\| \sum_{t=1}^{\ell} \frac{\lambda_j}{|\lambda_j|} \sum_{i \in J_{j_t}} |\lambda_i| P_{\varphi_i(z_n)} \right\|_1
\]

\[
\to \sum_{t=1}^{\ell} \sum_{i \in J_{j_t}} |\lambda_i| \quad \text{as } n \to \infty
\]

\[
= \sum_{j=1}^{N} |\lambda_j|.
\]

Thus we get \( \Gamma(\{z_n\}) = \sum_{j=1}^{N} |\lambda_j| \). By Theorem 3.9,

\[
\Gamma(\{z_n\}) \leq \left\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\|_e \leq \sum_{j=1}^{N} |\lambda_j|,
\]

so we get

\[
\left\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\|_e = \sum_{j=1}^{N} |\lambda_j|,
\]

\[ \square \]
4. Examples

Let \( \varphi_1, \varphi_2, \ldots, \varphi_N \) be functions in \( S(D) \) with \( \| \varphi_j \|_\infty = 1 \) satisfying \( \varphi_i \neq \varphi_j \) for \( i \neq j \) and \( \lambda_j \in \mathbb{C} \) with \( \lambda_j \neq 0 \) for every \( 1 \leq j \leq N \). Then we have

\[
\left\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\|_e \leq \left\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\|_c \leq \sum_{j=1}^{N} |\lambda_j|.
\]

In this section, we shall give examples \( \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \) satisfying the following conditions, respectively:

\[
\left\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\|_e < \left\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\|_c < \sum_{j=1}^{N} |\lambda_j|.
\]

\[
\left\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\|_e < \sum_{j=1}^{N} |\lambda_j|.
\]

\[
\left\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\|_e = \sum_{j=1}^{N} |\lambda_j|.
\]

\[
\left\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \right\|_e = \sum_{j=1}^{N} |\lambda_j|.
\]

Example 4.1. Let \( \varphi_1(z) = sz + 1 - s \) for \( 0 < s < 1 \) and \( \varphi_2(z) = \varphi_1(z) + t(z-1)^6 \). For \( b > 2 \) and \( t \) is real and \( |t| \) is so small, we have \( \varphi_1 \in S(D) \). By [12, Example 1], \( C_{\varphi_1} - C_{\varphi_2} \) is compact on \( H^\infty \). By Theorem 2.7, \( C_{\varphi_1} - C_{\varphi_2} \) is compact on \( h^\infty \). Hence \( \| C_{\varphi_1} - C_{\varphi_2} \|_e = 0 \). By [12, Theorem 3],

\[
\lim_{z \to 1} \rho(\varphi_1(z), \varphi_2(z)) = 0.
\]

By Theorem 2.5, \( \| C_{\varphi_1} - C_{\varphi_2} \| < 2 \). It is easy to see that \( 0 < \| C_{\varphi_1} - C_{\varphi_2} \| \). Thus \( C_{\varphi_1} - C_{\varphi_2} \) is an example satisfying (4.2).

Example 4.2. Let \( \varphi_1(z) = (z + 2)/3 \) and \( \varphi_2(z) = (z - 2)/3 \). Then \( \| \varphi_1 \|_\infty = \| \varphi_2 \|_\infty = 1 > \| \varphi_1 \varphi_2 \|_\infty \). By Theorem 2.5, it is easy to see that \( \| C_{\varphi_1} - C_{\varphi_2} \| = 2 \). Let \( \{z_n\}_n \in \mathcal{Z}(\{\varphi_1, \varphi_2\}) \). If \( 1 \in I(\{z_n\}) \), then \( \varphi_1(z_n) \to 1 \), so \( z_n \to 1 \). Hence \( \varphi_2(z_n) \to 0 \) as \( n \to \infty \). Thus \( I(\{z_n\}) = \{1\} \). Similarly if \( 2 \in I(\{z_n\}) \), then \( I(\{z_n\}) = \{2\} \). Hence by (3.1), \( \Gamma(\{z_n\}) = 1 \). Therefore

\[
\max_{\{z_n\}_n \in \mathcal{Z}(\{\varphi_1, \varphi_2\})} \Gamma(\{z_n\}) = 1.
\]

By Theorem 3.9, \( \| C_{\varphi_1} - C_{\varphi_2} \| = 1 \). So \( C_{\varphi_1} - C_{\varphi_2} \) satisfies (4.3).
Example 4.3. This example is similar to the one given in [9]. Let \( \varphi_1(z) = z \) and \( \varphi_2(z) = z^2 \). Let \( \{z_n\} \in \mathbb{D} \) with \( z_n \to -1 \). Then \( \{z_n\} \in Z(\{\varphi_1, \varphi_2\}) \) and \( I(\{z_n\}) = \{1, 2\} \). Since \( \varphi_1(z_n) \to -1 \) and \( \varphi_2(z_n) \to 1 \) as \( n \to \infty \), we have
\[
\Gamma(\{z_n\}) = \liminf_{n \to \infty} \|P_{\varphi_1(z_n)} - P_{\varphi_2(z_n)}\|_1 = 2.
\]
By Theorem 3.9, \( \|C_{\varphi_1} - C_{\varphi_2}\|_e = 2 \). By (4.1), \( C_{\varphi_1} - C_{\varphi_2} \) satisfies (4.5).

Example 4.4. Take \( \varphi_1 \in S(\mathbb{D}) \) satisfying \( \|\varphi_1\|_\infty = 1 \) and
\[
\int_{\partial \mathbb{D}} \log(1 - |\varphi_1^*|) \, d\sigma > -\infty.
\]
Then there exists an outer function \( \omega(z) \in H^\infty \) satisfying (4.6)
\[
|\omega^*| = 1 - |\varphi_1^*| \quad \text{a.e. on } \partial \mathbb{D}
\]
(see [3, 7]). We have (4.7)
\[
|\omega| + |\varphi_1| \leq 1 \quad \text{on } \mathbb{D}.
\]
By (4.6), there is a sequence \( \{z_n\} \in \mathbb{D} \) satisfying
\[
\frac{1 - |\varphi_1(z_n)|}{|\omega(z_n)|} \to 1 \quad \text{as } n \to \infty
\]
and (4.8)
\[
|\varphi_1(z_n)| \to 1 \quad \text{as } n \to \infty.
\]
Here we may assume that (4.9)
\[
\frac{1 - |\varphi_1(z_n)|}{\omega(z_n)} \to 1 \quad \text{as } n \to \infty.
\]
For \( 0 < t < 1 \), let
\[
\varphi_2(z) = \varphi_1(z) + t\omega(z)\varphi_1(z).
\]
By (4.6), \( \varphi_2 \in S(\mathbb{D}) \). Since \( \omega(z_n) \to 0 \), \( |\varphi_2(z_n)| \to 1 \). We have
\[
\rho(\varphi_1(z), \varphi_2(z)) = \left| \frac{t\omega(z)\varphi_1(z)}{1 - |\varphi_1(z)|^2 - t\omega(z)|\varphi_1(z)|^2} \right|
\leq \frac{1 - |\varphi_2(z_n)|^2}{\omega(z)} - t|\varphi_1(z_n)|^2
\leq \frac{t|\varphi_1(z_n)|}{1 + |\varphi_1(z_n)| - t|\varphi_1(z_n)|^2} \quad \text{by (4.7)}
\leq \frac{t}{2 - t}.
\]
Hence
\[
\rho(\varphi_1(z), \varphi_2(z)) \leq \frac{t}{2 - t}, \quad z \in \mathbb{D}.
\]
On the other hand,
\[
\limsup_{n \to \infty} \rho(\varphi_1(z_n), \varphi_2(z_n)) = \limsup_{n \to \infty} \left| \frac{t\varphi_1(z_n)}{\omega(z_n)} - t|\varphi_1(z_n)|^2 \right|
\]
\[
= \frac{t}{2-t} \quad \text{by (4.8) and (4.9)}.
\]
Therefore
\[
\sup_{z \in \mathbb{D}} \rho(\varphi_1(z), \varphi_2(z)) = \limsup_{n \to \infty} \rho(\varphi_1(z_n), \varphi_2(z_n)) = \frac{t}{2-t}.
\]
By Lemmas 2.1 and 2.4,
\[
\|C_{\varphi_1} - C_{\varphi_2}\| = \sup_{z \in \mathbb{D}} \|P_{\varphi_1}(z) - P_{\varphi_2}(z)\|_1
\]
\[
= \limsup_{n \to \infty} \|P_{\varphi_1(z_n)} - P_{\varphi_2(z_n)}\|_1
\]
\[
= 2 - \frac{4 \cos^{-1} \frac{t}{\pi}}{\pi} < 2.
\]
By (3.1) and Theorem 3.2, we have
\[
\|C_{\varphi_1} - C_{\varphi_2}\|_e \geq \limsup_{n \to \infty} \|P_{\varphi_1(z_n)} - P_{\varphi_2(z_n)}\|_1 = 2 - \frac{4 \cos^{-1} \frac{t}{\pi}}{\pi}.
\]
Hence \( \|C_{\varphi_1} - C_{\varphi_2}\| = \|C_{\varphi_1} - C_{\varphi_2}\|_e \). Therefore \( C_{\varphi_1} - C_{\varphi_2} \) satisfies (4.4).

References


Kei Ji Izuchi
Department of Mathematics
Niigata University
Niigata 950-2181, Japan
E-mail address: izuchi@math.sc.niigata-u.ac.jp

Kou Hei Izuchi
Department of Mathematics
Faculty of Education
Yamaguchi University
Yamaguchi 753-8511, Japan
E-mail address: izuchi@yamaguchi-u.ac.jp