AXIOMS FOR THE THEORY OF RANDOM VARIABLE STRUCTURES: AN ELEMENTARY APPROACH

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Abstract. The theory of random variable structures was first studied by Ben Yaacov in [2]. Ben Yaacov's axiomatization of the theory of random variable structures used an early result on the completeness theorem for Łukasiewicz's [0,1]-valued propositional logic. In this paper, we give an elementary approach to axiomatizing the theory of random variable structures. Only well-known results from probability theory are required here.

1. Introduction

The study of the theory of random variable structures was initiated by Ben Yaacov in [2]. He proved that the class of random variable structures is elementary and gave axioms for the theory of random variable structures, but his axiomatization of the theory used an early result on the completeness theorem for Łukasiewicz's [0,1]-valued propositional logic. In this paper, we use only well-known results from probability theory to give an elementary approach to axiomatizing the theory of random variable structures. Our approach is built on the axiomatization of the theory of probability algebras (e.g., see [5]).

In the rest of this section, we introduce the definitions and notations in this paper. In Section 2, we axiomatize the theory of random variable structures. Only basic measure theoretic probability theory is required. The main result is Theorem 2.10.

Definitions and notations

We follow the notations in [3, Chapter 16] (see [5] for more details). Let the signature $\mathcal{L}_{Pr}$ denote the set $\{0, 1, \overline{0}, \cap, \cup, \mu\}$, where $0$ and $1$ are constant symbols, $\overline{0}$ is a unary function symbol, $\cap$ and $\cup$ are binary function symbols,
and μ is a unary predicate symbol. Among those symbols, ∈ and μ are 1-Lipschitz, and ∩ and ∪ are 2-Lipschitz. Let the theory of probability algebras Pr consist of the following axioms:

(i) boolean algebra axioms
(ii) measure axioms:

\[ \mu(\mathbf{1}) = 1 \text{ and } \sup_x \sup_y \left| \frac{\mu(x) + \mu(y)}{2} - \frac{\mu(x \cup y) + \mu(x \cap y)}{2} \right| = 0 \]

(iii) \( \sup_x \sup_y |d(x, y) - \mu(x \triangle y)| = 0 \), where \( x \triangle y \) denotes the symmetric difference: \( x \triangle y = (x \cap y^c) \cup (x^c \cap y) \).

The theory of atomless probability algebras APr consists of axioms in Pr and the following one:

(iv) \( \sup_x \inf_y |\mu(x \cap y) - \frac{\mu(x)}{2}| = 0 \).

Let \( (\Omega, \mathcal{F}, \mu) \) be a probability space. For \( A_1, A_2 \in \mathcal{F} \), we write \( A_1 \sim_\mu A_2 \) if the symmetric difference \( A_1 \triangle A_2 \) has measure zero. Clearly we see that \( \sim_\mu \) is an equivalence relation. Let \( \hat{\mathcal{F}} \) denote the collection of equivalence classes of \( \mathcal{F} \) modulo \( \sim_\mu \). We call elements in \( \hat{\mathcal{F}} \) events. Naturally, \( \hat{\mathcal{F}} \) is a σ-algebra and \( \mu \) induces a well-defined countably additive probability measure on \( \hat{\mathcal{F}} \). We call \( \hat{\mathcal{F}} \) the measure algebra associated to \( (\Omega, \mathcal{F}, \mu) \). The \( L_1 \)-structure \( \mathcal{M} = (\hat{\mathcal{F}}, 0, 1, \triangle, \cap, \cup, \mu) \) is called a probability algebra. It is called an atomless probability algebra if the probability space \( (\Omega, \mathcal{F}, \mu) \) is atomless; that is, for every \( F \in \mathcal{F} \) with \( \mu(F) > 0 \) there is \( G \in \mathcal{F} \) with \( G \subseteq F \) such that \( 0 < \mu(G) < \mu(F) \).

Let \( (\Omega, \mathcal{F}, \mu) \) be a probability space. Consider the set of all \( \mathcal{F} \)-measurable functions \( f: \Omega \to [0, 1] \). Define the \( L^1 \)-metric \( d_1(f, g) := \int_\Omega |f - g|d\mu \) for all \( \mathcal{F} \)-measurable \( f, g: \Omega \to [0, 1] \). The set of such functions together with \( d_1 \) forms a pseudometric space, which is denoted by \( L^1(\Omega, \mathcal{F}, \mu, [0, 1]) \), or simply by \( L^1(\mu, [0, 1]) \). For all \( f, g \in L^1(\mu, [0, 1]) \), we say that \( f \) is equal to \( g \) almost surely, and write \( f =_{a.s.} g \) (or \( f = g \ a.s. \)), if \( f \) is equal to \( g \) up to a null set. We denote the equivalence class of \( f \) under \( =_{a.s.} \) by \([f]_{a.s.}\). For each \( F \in \mathcal{F} \), let \( \chi_F \) denote the characteristic function of \( F \), and let \( 1_F \) denote \([\chi_F]_{a.s.}\). Let \( N \) be \( \{ f \mid \int_{\Omega} |f|d\mu = 0 \} = \{ f \mid f = 0 \ a.s. \} \). Then the quotient space

\[ L^1(\Omega, \mathcal{F}, \mu, [0, 1]) = L^1(\hat{\Omega}, \mathcal{F}, \mu, [0, 1])/N \]

is a metric space with the \( L^1 \)-metric \( d_1 \), called an \( L^1 \)-space. It is well known that the space \( L^1(\Omega, \mathcal{F}, \mu, [0, 1]) \) with the \( L^1 \)-metric \( d_1 \) is a complete metric space. When the underlying probability space is clear, \( L^1(\Omega, \mathcal{F}, \mu, [0, 1]) \) is often abbreviated as \( L^1(\mu, [0, 1]) \), or just \( L^1(\mathcal{F}, [0, 1]) \) when the underlying set \( \Omega \) and the probability measure \( \mu \) are clear. We write \( L^1(\Omega, \mathcal{F}, \mu, [0, 1]) \) for the set of equivalence classes of characteristic functions in the space \( L^1(\hat{\Omega}, \mathcal{F}, \mu, [0, 1]) \). Let \( \mathbb{D} \) denote the dyadic numbers in \([0, 1] \). We write \( L^1(\Omega, \mathcal{F}, \mu, \mathbb{D}) \) for the set of equivalence classes of \( \mathbb{D} \)-valued simple functions in \( L^1(\hat{\Omega}, \mathcal{F}, \mu, [0, 1]) \).
Clearly,
\[
L^1((\Omega, \mathcal{F}, \mu), \{0, 1\}) \subseteq L^1(\Omega, \mathcal{F}, \mu, \mathbb{D}) \subseteq L^1((\Omega, \mathcal{F}, \mu), [0, 1]).
\]
Moreover, we have that \(L^1((\Omega, \mathcal{F}, \mu), \{0, 1\})\) is closed in \(L^1((\Omega, \mathcal{F}, \mu), [0, 1])\), and \(L^1((\Omega, \mathcal{F}, \mu), \mathbb{D})\) is dense in \(L^1((\Omega, \mathcal{F}, \mu), [0, 1])\). Let \(A\) be a subset of \(L^1((\Omega, \mathcal{F}, \mu), [0, 1])\). Let \(\sigma(A) \subseteq \mathcal{F}\) denote the \(\sigma\)-subalgebra of \(\mathcal{F}\)-measurable sets generated by the random variables in the equivalence classes in \(A\). We call \(\sigma(A)\) the \(\sigma\)-algebra generated by \(A\).

The elements in \(L^1((\Omega, \mathcal{F}, \mu), [0, 1])\) are not \(\mathcal{F}\)-measurable functions, but equivalence classes of them. In probability theory, most useful functions, relations, and maps (such as continuous functions, integrals, inequality relations, conditional expectations) on measurable functions are well-defined on the equivalence classes of those functions. Therefore, it causes no harm (and is more readable) to denote an equivalence class in \(L^1((\Omega, \mathcal{F}, \mu), [0, 1])\) by a member of the class.

A ([0,1]-valued) random variable structure is based on a set of the form \(L^1((\Omega, \mathcal{F}, \mu), [0, 1])\), where \((\Omega, \mathcal{F}, \mu)\) is a probability space. It is called an atomless random variable structure, if its underlying probability space is atomless. We use the setting of continuous logic [3] ([4] is also a good reference) to discuss the model theory of random variable structures. Here we consider the signature \(L_{\text{RV}} = \{0, \neg, \frac{1}{2}, I\}\), where \(0\) is a constant symbol, \(\neg\) is a binary function symbol, \(\frac{1}{2}\) and 2 are unary function symbols, and \(I\) is a unary predicate symbol. Recall that on \(M^n\), we take the maximum metric. Among those symbols, \(\neg\) is 1-Lipschitz, \(\frac{1}{2}\) is \(\frac{1}{2}\)-Lipschitz and \(I\) is 1-Lipschitz.

We interpret the symbols of \(L_{\text{RV}}\) in \(M\) as follows:

\[
\begin{align*}
0^\mathcal{M}(\omega) &= 0 \text{ for all } \omega \in \Omega \\
\neg^\mathcal{M}(f) &= 1 - f \text{ for all } f \in M \\
(\frac{1}{2})^\mathcal{M}(f, g) &= f + g = \max(f, g) \text{ for all } f, g \in M \\
I^\mathcal{M}(f) &= \int f d\mu \text{ for all } f \in M \\
d^\mathcal{M}(f, g) &= \int |f - g| d\mu \text{ for all } f, g \in M
\end{align*}
\]

Then \(\mathcal{M} = \big(L^1((\Omega, \mathcal{F}, \mu), [0, 1]), 0, \neg, \frac{1}{2}, I, d\big)\) is an \(L_{\text{RV}}\)-structure. Note that the \(L_{\text{RV}}\)-prestructure associated to \(\mathcal{M}\) is \(\big(L^1((\Omega, \mathcal{F}, \mu), [0, 1]), 0, \neg, \frac{1}{2}, I, d\big)\).

Let \(\mathcal{R}_{\text{V}}\) denote the class of all random variable structures as \(L_{\text{RV}}\)-structures and let \(\mathcal{AR}_{\text{V}}\) denote the class of all atomless random variable structures as \(L_{\text{RV}}\)-structures. In Section 2, we show that the classes \(\mathcal{R}_{\text{V}}\) and \(\mathcal{AR}_{\text{V}}\) are elementary.

In the signature \(L_{\text{RV}}\), we also use the following symbols as shorthand for expressions built from symbols in \(L_{\text{RV}}\):

\[
1 = \neg 0 \\
x + y = \neg(\neg x \land y)
\]
The main result is Theorem 2.10. Only basic measure theoretic probability theory is assumed. The RVs when \( r = 0 \) or 1, we write 0x for 0 and 1x for x.

2. Axioms for RV

In this section, we give axioms for the theory of (atomless) random variable structures. Only basic measure theoretic probability theory is assumed. The main result is Theorem 2.10.

The theory RV consists of the following axioms:

(E1): \( \sup_{x} \inf_{y} \max \left( I(y \wedge -y), |I(x \wedge -x) - d(x,y)| \right) = 0 \)

(E2): \( \sup_{x} \left| I(x \wedge -x) - \inf_{y} (I(y \wedge -y) + d(x,y)) \right| = 0 \)

(APP): \( \sup_{x} \inf_{y_{1}, \ldots, y_{2^{n}}} \left( d(x, \frac{1}{2^{n}} y_{1} + \frac{1}{2^{n}} y_{2} + \cdots + \frac{1}{2^{n}} y_{2^{n}}) + \max_{1 \leq i \leq 2^{n}} I(y_{i} \wedge -y_{i}) \right) = \frac{1}{2^{n}} = 0 \)

(ADD): \( \sup_{x} \sup_{y} \frac{1}{2} \left| I(x) - (I(x \wedge y) + I(y \wedge x)) \right| = 0 \)

(C): \( \sup_{x} I(0 \vee x) = 0; \quad \sup_{x} d(x \wedge 0, x) = 0; \quad |I(1) - 1| = 0 \)

(H1): \( \sup_{x} \sup_{y} d(\frac{1}{2} x + \frac{1}{2} y, \frac{1}{2} x + \frac{1}{2} y) = 0 \)

(H2): \( \sup_{x} \sup_{y} d(\frac{1}{2} x + \frac{1}{2} y, \frac{1}{2} x + \frac{1}{2} y) = 0; \quad \sup_{x} d(\frac{1}{2} x + \frac{1}{2} y, x) = 0 \)

(H3): \( \sup_{x} \sup_{y} \left( I\left( \frac{1}{2} x \wedge y \right) \wedge (x \wedge y) \right) = 0 \)

(H4): \( \sup_{x} \sup_{y} \left( I\left( \frac{1}{2} x \wedge y \right) \wedge (x \wedge y) \right) = 0 \)

(MET): \( \sup_{x} \sup_{y} \frac{1}{2} \left| d(x,y) - (I(x \wedge y) + I(y \wedge x)) \right| = 0 \)

(N): \( d(\neg 1, 0) = 0; \quad \sup_{x} \sup_{y} d(x \wedge y, -y \wedge -x) = 0 \)

(P1): \( \sup_{x_{1}} \sup_{x_{2}} \sup_{y_{1}} \sup_{y_{2}} \left( d(x_{1} + y_{1}, x_{2} + y_{2}) \wedge (d(x_{1}, x_{2}) + d(y_{1}, y_{2})) \right) = 0 \)

(P2): \( \sup_{x} \sup_{y} \left( d(x + y, x \vee y) \wedge \max(I(x \wedge -x), I(y \wedge -y)) \right) = 0 \)
(P3): \( \sup_x \sup_y \sup_z d\left( (x+y) + z, x + (y+z) \right) = 0 \)
(S1): \( \sup_x \sup_y \sup_z d\left( (z \lor y) + x, (z \lor x) \lor y \right) - I(x \land y) = 0 \)
(S2): \( \sup_x \sup_y \sup_z I\left( \left( (x+y) \land z \right) \lor \left( (x \land z) \lor (y \land z) \right) \right) = 0 \)
(L1): \( \sup_x \sup_y d(x \lor y, y \lor x) = 0 \)
(L2): \( \sup_x \sup_y \sup_z d(x \lor (y \lor z), (x \lor y) \lor z) = 0 \)
(L3): \( \sup_x \sup_y \sup_z d(x \land (y \land z), (x \land y) \land z) = 0 \)
(L4): \( \sup_x \sup_y \sup_z d(x \lor (x \land y), x) = 0 \)
(L5): \( \sup_x \sup_y \sup_z d(x \land (y \land z), (x \lor y) \land (x \lor z)) = 0 \)
(L6): \( \sup_x \sup_y \sup_z d(x \land (y \land z), (x \lor y) \lor (x \lor z)) = 0 \)

Axioms (L1) to (L6) are the axioms for distributive lattices.

Let ARV be RV together with the following axiom:
(NA): \( \sup_x \inf_y (\max(I(y \land \neg y), |I(y \land x) - \frac{|I(x)|}{2}|)) = 0 \)

**Proposition 2.1.** Every random variable structure \( M = (M, 0, \land, \lor, \neg, \frac{1}{2}, I, d) \) is a model of RV. Further, if \( M \) is an atomless random variable structure, then it is a model of ARV.

**Proof.** Assume \( M = L^1((\Omega, F, \mu), [0, 1]) \) for some probability space \( (\Omega, F, \mu) \). By [3, Theorem 3.7], it suffices to consider axioms in the \( L_{RV} \)-prestructure \( M_0 = L^1((\Omega, F, \mu), [0, 1]) \). Note that \( f \land g = \min(f, g) \) and \( f \lor g = \max(f, g) \) for all \( f, g \in M_0 \). Most axioms are easy to verify and some of them are just arithmetic. We will check Axioms (E1), (E2), (APPR), and leave the rest to the readers.

(E1) and (E2): We consider
\[ X = \{ f \in M_0 \mid f \text{ is a characteristic function} \}. \]
For all \( f \in M_0 \), we have
\[ |f(\omega) - \chi_{\{f \geq \frac{1}{2}\}}(\omega)| \leq |f(\omega) - \chi_A(\omega)| \text{ for all } A \in F \text{ and all } \omega \in \Omega, \]
whereby \( \dist(f, X) = d(f, \chi_{\{f \geq \frac{1}{2}\}}) \). Also we note that
\[ d(f, \chi_{\{f \geq \frac{1}{2}\}}) = \int |f - \chi_{\{f \geq \frac{1}{2}\}}|d\mu = \int |f \land (1 - f)|d\mu = I^{M_0}(f \land \neg f), \]
whereby \( \dist(f, X) = I^M(f \land \neg f) \). Then to verify Axioms (E1) and (E2), we need only check that
\[ \sup_x \inf_y \max(\dist(y, X), |\dist(x, X) - d(x, y)|) = 0 \]
and
\[ \sup_x |\dist(x, X) - \inf_y (\dist(y, X) + d(x, y))| = 0. \]
Both are clear here.
(APPR): This axiom is an approximation result from real analysis. For all \( n \in \mathbb{N} \) and \( f \in M_0 \), let \( g_i = \chi_{\{f \geq \frac{i-1}{2^n}\}} \), for every \( 1 \leq i \leq 2^n \). Then
\[
\frac{1}{2^n} g_1 + \cdots + \frac{1}{2^n} g_{2^n} = \sum_{i=1}^{2^n} \frac{1}{2^n} g_i = \frac{1}{2^n} \chi_{\{0 \leq f < \frac{1}{2^n}\}} + \frac{2}{2^n} \chi_{\{\frac{1}{2^n} \leq f < \frac{2}{2^n}\}} + \cdots + \frac{2^n}{2^n} \chi_{\{\frac{2^n-1}{2^n} \leq f\}}.
\]
Thus
\[
d(f, \frac{1}{2^n} g_1 + \cdots + \frac{1}{2^n} g_{2^n}) = d(f, \sum_{i=1}^{2^n} \frac{i}{2^n} \chi_{\{\frac{i-1}{2^n} \leq f < \frac{i}{2^n}\}})
= \int_\Omega |f - \sum_{i=1}^{2^n} \frac{i}{2^n} \chi_{\{\frac{i-1}{2^n} \leq f < \frac{i}{2^n}\}}| \, d\mu
\leq \int_\Omega \sum_{i=1}^{2^n} \frac{1}{2^n} \chi_{\{\frac{i-1}{2^n} < f \leq \frac{i}{2^n}\}} \, d\mu = \frac{1}{2^n} \mu(\Omega) = \frac{1}{2^n}.
\]
Also note that \( I(g_i \land \neg g_i) = 0 \) for every \( 1 \leq i \leq 2^n \). Consequently, (APPR) is true in \( \mathcal{M} \).

When \( (\Omega, \mathcal{F}, \mu) \) is atomless, clearly \( (L^1((\Omega, \mathcal{F}, \mu), [0, 1]), 0, \neg, \frac{1}{2}, I, d) \) satisfies (NA). Hence it is a model of ARV. \( \square \)

Indeed, RV also axiomatizes the class \( \mathcal{RV} \) (see Theorem 2.10) and then ARV axiomatizes the class \( \mathcal{ARV} \) (see Corollary 2.11), which are the main results from this section. Toward the proof of Theorem 2.10, we prove the following results about models of RV. In the following arguments, we interpret symbols of \( L_{RV} \) in a given model \( \mathcal{M} \) of RV without putting \( \mathcal{M} \) explicitly into the notations, for easier readability.

**Fact 2.2.** Let \( \mathcal{M} \) be a model of RV. For all \( x, y \in M \), we have the following properties:

(i) \( I(x) = 0 \) if and only if \( x = 0 \).
(ii) \( 0 \vdash x = 0 \) and \( x \vdash 1 = 0 \).
(iii) \( \neg x = 1 \vdash x \) and \( \neg \neg x = x \).
(iv) \( x \land 0 = 0 \) and \( x \land 1 = x \).
(v) \( x \lor 0 = x \) and \( x \lor 1 = 1 \).
(vi) \( x \vdash x = 0 \).
(vii) \( I(\neg x) = 1 - I(x) \).
(viii) \( I(x) = I(\frac{1}{2}) \).
(ix) \( x + y = y + x \).
(x) \( d(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2} d(x, y) \).
(xi) \( 1 + x = x + 1 = 1 \) and \( 0 + x = x + 0 = x \).
(xii) If \( \frac{1}{2} + \frac{1}{2} = \frac{1}{2} \), then \( x = \neg y \).
Proposition 2.3. Let $\mathcal{M}$ be a model of RV. Let $D = \{ x \in M \mid I(x \land \neg x) = 0 \}$. For all $x, y \in D$, define $\bar{x} := \neg x$, $\bar{x} \land y := x \land y$, $\bar{x} \land y := x \lor y$, and $\mu(x) := I(x)$. Then $D$ is a uniformly definable set in $M$ and $(D, 0, 1, \bar{x}, \land, \lor, \cap, \cup, \mu)$ is a model of RV.
Moreover, if \( M \) is of the form \( L^1((\Omega,\mathcal{F},\mu),[0,1]) \) for the probability space \((\Omega,\mathcal{F},\mu)\), then \( D \) is \( L^1((\Omega,\mathcal{F},\mu),[0,1]) \).

**Proof.** Let \( M \) be a model of \( \text{Pr} \) and let \( D = \{ y \in M \mid I(y \wedge \neg y) = 0 \} \). By Fact 2.2(i), we know \( D = \{ y \in M \mid y \wedge \neg y = 0 \} \). By [3, Theorem 9.12], (E1), and (E2), we know that \( I(x \wedge \neg x) = \text{dist}(x,D) \). Hence, \( D \) is a uniformly definable set in \( M \); that is, the defining formula for \( D \) does not depend on \( M \).

First, we want to show that \( D \) is closed under \( \neg, \land, \lor \) and also \( 0, 1 \in D \). For all \( x, y \in D \), we have \( x \land \neg x = y \land \neg y = 0 \). Since \( \neg \neg x = x \) and \( x \land \neg x = \neg x \land x \), we get \( \neg x \land \neg(\neg x) = 0 \), whence \( \neg x \in D \). To show \( x \land y \in D \), it suffices to show \( (x \land y) \land \neg(x \land y) = (x \land y) \land (\neg x \lor \neg y) = 0 \). By the fact that \( \lor \) and \( \land \) satisfy the distributive lattice axioms, we need only show that \((x \land y) \land \neg(x \land y) \lor (x \land y) \land \neg y) = 0 \), which is true since \( x \land \neg x = y \land \neg y = 0 \). Then since \( x \lor y = \neg(\neg x \land \neg y) \), we know \( x \lor y \in D \) as well. By Fact 2.2(iv), we know \( 0 \land \neg 0 = 0 \), and thus \( 0 \in D \). Hence, \( 1 = \neg 0 \in D \).

Second, for all \( x,y \in D \), define \( x \lor y := x \land y \), \( x \lor y := x \lor y \), and \( x^\mathcal{F} = \neg x \). We show that \((D,0,1,\mathcal{F},\lor,\land,\mu)\) is a model of \( \text{Pr} \). For all \( x \in D \), we have \( x \land \neg x = 0 \), and then \( \neg(x \land \neg x) = \neg 0 = 1 \). Then by Fact 2.2(iii) and (L1), we have \( 1 = \neg(x \land \neg x) = \neg x \lor \neg x = \neg x \lor x = x \lor \neg x \). Because \( \land, \lor \) also satisfy the axioms for distributive lattices, we see that \((D,0,1,\mathcal{F},\lor,\land,\mu)\) satisfies all boolean algebra axioms in \( \text{Pr} \).

For all \( x \in D \), define \( \mu(x) := I(x) \). By Fact 2.2(i) and (vii), we have \( \mu(0) = 0 \) and \( \mu(1) = 1 \). For all \( x, y \in D \), we have \( I(x \lor y) = I(\neg(\neg x \land \neg y)) = 1 - I(\neg x \land \neg y) \), by Fact 2.2(vii). By (ADD) and (N), we have

\[
I(y) = I(\neg y \lor \neg x) + I(\neg x \land \neg y) = I(\neg y \lor x) + I(\neg x \land \neg y),
\]

and thus \( I(\neg x \land \neg y) = I(\neg y) - I(x \lor y) \). Hence,

\[
I(x \lor y) = 1 - I(\neg x \land \neg y) = 1 - (I(\neg y) - I(x \lor y)),
\]

whence \( I(x \lor y) = I(y) + I(x \lor y) \) by Fact 2.2(vii). By (ADD), we have

\[
I(x) = I(x \lor y) + I(y \land x).
\]

Then by eliminating the term \( I(x \lor y) \), we get \( I(x \land y) + I(y \land x) = I(x) + I(y) \), whence \( I(x \land y) + I(x \land y) = I(x) + I(y) \). Therefore \( \mu(x \land y) + \mu(x \land y) = \mu(x) + \mu(y) \). Consequently, \((D,0,1,\mathcal{F},\lor,\land,\mu)\) satisfies the measure axioms in \( \text{Pr} \).

Next, for all \( x, y \in D \), by (P2) we know \( d(x + y, x \lor y) = 0 \), and thus \( x + y = x \lor y \). Since \( x + y = \neg(\neg x \lor \neg y) \), by Fact 2.2(iii) we have \( x + y = \neg(\neg x \lor \neg y) \). Then by Fact 2.2(iii), we have \( x + y = \neg(\neg x \lor \neg y) = \neg(\neg x \lor y) = \neg x \land x \land y = x \land y \).

By (MET), we have

\[
d(x,y) = I(x \lor y) + I(y \land x) = I(x \lor y) + I(y \land x) = \mu(x \land y) + \mu(y \land x) = \mu(x \lor y) + \mu(y \land x) = \mu(x \lor y) + \mu(y \land x) = \mu(x \lor y).
\]
Hence, \((D, 0, 1, \mathcal{L}, \cap, \cup, \mu)\) satisfies Axiom (iii) in \(Pr\). Since \(d\) is a complete metric on \(M\) and \(D\) is a zerost (thus it is closed), the metric \(d\) is complete on \(D\).

Since \(x \sim x\) for all \(x \in D\) and \(\sim\) is 1-Lipschitz, we get \(\cdot \mathcal{L}\) is 1-Lipschitz. By (P2) and Fact 2.2(iii), for all \(x, y \in D\), we have
\[x \wedge y = -(\neg x \wedge \neg y) = -(\neg x \vee \neg y) = \neg(x \vee y) = (\neg x)^{-} = (\neg y)^{-} = y \sim x\sim y.
\]
Since \(\sim\) is 1-Lipschitz and \(\sim\) is 2-Lipschitz, we have that \(\cap\) is 2-Lipschitz. Since \(x \cup y = (x \cap y)^{\sim}\) for all \(x, y \in D\), we know that \(\cup\) is 2-Lipschitz. Since \(\mu(x) = I(x)\) for all \(x \in D\) and \(I\) is 1-Lipschitz, we know that \(\mu\) is 1-Lipschitz. Hence, \((D, 0, \sim, \mathcal{L}, \cap, \cup, \mu)\) is an \(L_{\mu}\)-structure. Therefore, \((D, 0, \sim, \mathcal{L}, \cap, \cup, \mu)\) is a model of \(Pr\).

Suppose \(M\) is of the form \(L^{1}(\Omega, \mathcal{F}, \mu), [0, 1]\), where \((\Omega, \mathcal{F}, \mu)\) is a probability space. Then for every \(f \in L^{1}(\Omega, \mathcal{F}, \mu), \{0, 1\}\), there is \(A \in \mathcal{F}\), such that \(f = [\chi_{A}]_{a.s.}\). Thus \(f \wedge \neg f = 0\), whereby \(f \in D\). For the converse, take \(x \in D\) with \(I(x \wedge \neg x) = 0\). Suppose \(x = [f]_{a.s.}\) for an \(\mathcal{F}\)-measurable \(f: \Omega \rightarrow [0, 1]\). Then \(\int_{0}^{1} \min(f, 1 - f) d\mu = 0\), whereby \(f\) is \(a.s.\) a characteristic function. Hence \(x \in L^{1}(\Omega, \mathcal{F}, \mu), \{0, 1\}\), and thus \(D = L^{1}(\Omega, \mathcal{F}, \mu), \{0, 1\}\).

The following lemmas are used in the proofs of Proposition 2.7 and Theorem 2.10.

Lemma 2.4. Let \(M \models RV\). Then:

(i) For all \(m, n \in \mathbb{N}\) and all \(x_{1}, \ldots, x_{n} \in M\), we have
\[\frac{1}{2^{m}} \frac{x_{1}}{2^{m}} + \cdots + \frac{x_{n}}{2^{m}} = \frac{x_{1}}{2^{m+1}} + \cdots + \frac{x_{n}}{2^{m+1}}.
\]

(ii) For all \(m, n \in \mathbb{N}\) and all \(x_{1}, \ldots, x_{m}, y_{1}, \ldots, y_{n} \in M\), we have
\[(x_{1} + \cdots + x_{m}) + (y_{1} + \cdots + y_{n}) = x_{1} + \cdots + x_{m} + y_{1} + \cdots + y_{n}.
\]

(iii) For all \(n \in \mathbb{N}\) and all \(x \in M\), we have
\[\frac{x}{2^{m}} + \cdots + \frac{x}{2^{m}} = x.
\]

(iv) For all \(m, n \in \mathbb{N}\) and all \(x_{1}, \ldots, x_{n} \in M\), we have
\[\frac{1}{2^{m}}(x_{1} \vee \cdots \vee x_{n}) = \frac{x_{1}}{2^{m}} \vee \cdots \vee \frac{x_{n}}{2^{m}}.
\]

Proof. (i): Use induction on \(n\) and (H5).
(ii): Use induction on \(n\) and (P3).
(iii): Use induction on \(n\), (ii), and (H2).
(iv): Use induction on \(n\) and (H3).

□

Lemma 2.5. Let \(M \models RV\) and let \(x, y, z \in M\) be such that \(x \wedge y = y \wedge z = z \wedge x = 0\). Then for all \(n \in \mathbb{N}\), all \(x_{0}, x_{1}, \ldots, x_{n} \in M\) such that \(x_{i} \wedge x_{j} = 0\) if \(i \neq j\), and all \(r, s, t, r_{0}, r_{1}, \ldots, r_{n} \in D\), we have:
(i) \( rx \land sy = 0 \).
(ii) \( rx \lor sy = rx \).
(iii) \( rx + sy = rx \lor sy \).
(iv) \( (r_0x_0 + \cdots + r_{n-1}x_{n-1}) \land r_nx_n = 0 \).
(v) \( r_1x_1 + \cdots + r_nx_n = r_1x_1 \lor \cdots \lor r_nx_n \).
(vi) \( I(r_1x_1 + \cdots + r_nx_n) = I(r_1x_1) + \cdots + I(r_nx_n) \).

Proof. We leave the proofs of (i), (ii), and (iii) to the readers.

(iv): We use induction on \( n \).
(v): We use induction on \( n \).
(vi): We use induction on \( n \). \(\square\)

Lemma 2.6. Let \( M \models RV \) and let \( D = \{ x \in M \mid I(x \land \lnot x) = 0 \} \). Then for all \( r, s \in D \), all \( x \in M \), and all \( a, a_1, \ldots, a_k \in D \), where \( k \in \mathbb{N} \) and \( a_i \land a_j = 0 \) if \( i \neq j \), we have

(i) \( \frac{r}{2^k} = \frac{x}{2^k} \).
(ii) \( ra + sa = (r + s)a \).
(iii) \( \lnot(ra) = (\lnot r)a + \lnot a \).
(iv) \( ra + sa = (r \lor s)a \).
(v) \( r(a_1 + \cdots + a_k) = ra_1 + \cdots + ra_k \).
(vi) \( ra \land sa = (r \land s)a \).
(vii) \( I(ra) = rI(a) \), and thus \( ra = 0 \) if and only if \( r = 0 \) or \( a = 0 \).

Proof. We assume familiarity with Fact 2.2. Suppose \( r \) or \( s \) is neither 0 nor 1, otherwise this is trivial.

(i): This follows from (H5) and Lemma 2.4(i).
(ii): Suppose \( r = \frac{m_1}{2^{n_1}}, s = \frac{m_2}{2^{n_2}} \), where \( n_1, n_2 \in \mathbb{N} \), \( 0 < m_1 < 2^{n_1} \), \( 0 < m_2 < 2^{n_2} \), and \( n_1 \leq n_2 \). By Lemma 2.4(iii), we have

\[
\frac{a}{2^{n_2}} = \frac{a}{2^{n_2}} + \cdots + \frac{a}{2^{n_2}} \text{ \( k \) times}
\]

Then by Lemma 2.4(ii) and induction, we have \( ra = \frac{a}{2^{n_2}} + \cdots + \frac{a}{2^{n_2}} \), and thus

\[
ra + sa = \frac{a}{2^{n_2}} + \cdots + \frac{a}{2^{n_2}} \text{ \( \frac{m_1}{2^{n_2}} + \cdots + \frac{m_2}{2^{n_2}} \) times}
\]

Suppose \( r + s < 1 \). If \( 2 \mid m_12^{n_2-n_1} + m_2 \), then

\[
\frac{a}{2^{n_2}} + \cdots + \frac{a}{2^{n_2}} = \frac{m_12^{n_2-n_1} + m_2}{2^{n_2}} a = (r + s)a.
\]
Otherwise, say \( \frac{m_2^{2^{n_2-n_1}+m_2}}{2^{n_2}} = \frac{m}{2^{n_3}} \), where \( n_3 \in \mathbb{N} \) and \( 0 < m_3 < 2^{n_3} \). Then by Lemma 2.4(ii, iii) and induction, we have

\[
\frac{a}{2^{n_2}} + \cdots + \frac{a}{2^{n_3}} = \frac{a}{2^{n_3}} + \cdots + \frac{a}{2^{n_3}} = (r + s)a.
\]

Hence \( ra + sa = (r + s)a \) if \( r + s < 1 \).

Suppose \( r + s \geq 1 \). By Lemma 2.4(ii, iii) and induction, it suffices to prove \( a + ta = a \) for all \( t \in D \), which follows from

\[
a + ta = \neg(-a \land ta) = \neg((0 \lor \neg a) \lor ta) = \neg((0 \land ta) \lor \neg a) = \neg a = a.
\]

(iii): Since \( a \in D \), by Proposition 2.3 we have \( a \land \neg a = 0 \) and \( a \lor \neg a = 1 \).

It is easy to verify that

\[
\frac{ra}{2} \lor \frac{(-r)a + \neg a}{2} = \frac{1}{2}(a \lor \neg a) = \frac{1}{2}.
\]

Then by Fact 2.2(xii), we have \( \neg(ra) = (-r)a + \neg a \).

(iv): Note that \( \neg(t_1 \lor t_2) = \neg t_1 + \neg t_2 \) for all \( t_1, t_2 \in [0, 1] \). Then by (iii), (ii), (P3), and Fact 2.2, we have

\[
\neg((r + s)a) = \neg((-r)a + sa) + \neg a = \neg(ra \lor sa).
\]

Hence, \( ra \lor sa = (r + s)a \).

(v): Suppose \( r = \frac{m}{2^n} \), where \( n \in \mathbb{N} \), \( 0 < m < 2^n \), and \( 2 \nmid m \). Then by Lemma 2.5(v), Lemma 2.4(iv), (P3), Fact 2.2(ix), and induction, we have

\[
r(a_1 + \cdots + a_k) = ra_1 + \cdots + ra_k.
\]

(vi): Since \( ra \land sa = ra \land (ra \lor sa) \), this follows from (iv).

(vii): Suppose \( r = \frac{m}{2^n} \), where \( m, n \in \mathbb{N} \), \( 0 < m < 2^n \), and \( 2 \nmid m \).

Using induction on \( k \), we have that \( I(\frac{m}{2^n}a) = \frac{m}{2^n}I(a) \) for all \( 1 \leq k \leq 2^n \), and thus \( I(ra) = rI(a) \). \( \square \)

**Proposition 2.7.** Let \( \mathcal{M} \) be a model of RV and let \( D = \{ x \in \mathcal{M} \mid x \land \neg x = 0 \} \). Let \( \mathcal{M}_0 \subseteq \mathcal{M} \) be the smallest \( L_{RV} \)-prestructure containing \( D \). Then \( \mathcal{M}_0 \) is the set \( \{ r_1a_1 + \cdots + r_ka_k \mid k \in \mathbb{N}, r_1, \ldots, r_k \in D, a_1, \ldots, a_k \in \mathcal{M}, \text{ and } a_i \land a_j = 0 \text{ if } i \neq j \} \). Moreover, every nonzero element in \( \mathcal{M}_0 \) has a unique decomposition \( r_1a_1 + \cdots + r_ka_k \), where

1. \( k \in \mathbb{N} \);
2. \( r_1, \ldots, r_k \in D \) with \( 0 < r_1 < \cdots < r_k \);
3. \( a_1, \ldots, a_k \in \mathcal{M} \) such that \( a_i \neq 0 \) for each \( i \), and \( a_i \land a_j = 0 \) whenever \( i \neq j \).

**Proof.** By Proposition 2.3, \( (D, 0, 1, \neg, \land, \lor) \) is a boolean algebra. Let \( S \) denote the set \( \{ r_1a_1 + \cdots + r_ka_k \mid k \in \mathbb{N}, r_1, \ldots, r_k \in D, a_1, \ldots, a_k \in \mathcal{M}, \text{ and } a_i \land a_j = 0 \text{ if } i \neq j \} \). Clearly, \( S \subseteq \mathcal{M}_0 \). We will show that \( (S, 0, \neg, \frac{1}{2}, I, d) \) is an \( L_{RV} \)-prestructure. Taking \( k = 1, r_1 = 0 \), and \( a_1 = 0 \) in the definition of membership
shows that \( 0 \in S \). By Fact 2.2(iii), for all \( x, y \in M \) we have \( x + y = \neg(\neg x + y) \).

Hence, we need only show that \( S \) is closed under \( \neg, +, \) and \( \frac{1}{2} \).

Take \( x = r_1a_1 + \cdots + r_ka_k \in S \), where \( r_1, \ldots, r_k \in D \), \( a_1, \ldots, a_k \in D \), and \( a_i \land a_j = 0 \) if \( i \neq j \). By Lemma 2.5(v), we have \( x = r_1a_1 \lor \cdots \lor r_ka_k \). Then by Lemma 2.4(iv), Lemma 2.6(i), and Lemma 2.5(v), we have

\[
\frac{x}{2} = \frac{r_1a_1 + \cdots + r_ka_k}{2} = \frac{r_1}{2}a_1 \lor \cdots \lor \frac{r_k}{2}a_k = \frac{r_1}{2}a_1 \lor \cdots \lor \frac{r_k}{2}a_k \in S.
\]

Hence, \( S \) is closed under \( \frac{1}{2} \). Let \( y = (\neg r_1)a_1 + \cdots + (\neg r_k)a_k + (a_1 \lor \cdots \lor a_k) \).

Because \( D \) is a boolean algebra and \( a_i \land a_j = 0 \) if \( i \neq j \), we know \( y \in S \). We will show \( y = x \). Similar to the calculation of \( \frac{1}{2} \), we have \( \frac{y}{2} = \frac{r_1}{2}a_1 \lor \cdots \lor \frac{r_k}{2}a_k + \frac{1}{2}(a_1 \lor \cdots \lor a_k) \).

Then by (P3), Fact 2.2(ix), induction, Lemma 2.6(ii), Lemma 2.5(v), and the fact that \((D, 0, 1, \neg, \land, \lor)\) is a boolean algebra, we have

\[
\frac{x}{2} + \frac{y}{2} = \left(\frac{r_1}{2}a_1 + \cdots + \frac{r_k}{2}a_k\right) + \left(\frac{r_1}{2}a_1 + \cdots + \frac{r_k}{2}a_k + \frac{r_1}{2}a_1 + \cdots + \frac{r_k}{2}a_k\right) = \frac{1}{2}a_1 + \cdots + \frac{1}{2}a_k + \frac{1}{2}(a_1 \lor \cdots \lor a_k)
\]

\[
= \frac{1}{2}(a_1 \lor \cdots \lor a_k) = \frac{1}{2} = \frac{1}{2}.
\]

Hence by Fact 2.2(xii), we have \( x = y \in S \). That is, \( S \) is closed under \( \neg \).

Take \( x = r_1a_1 + \cdots + r_ka_k \) and \( y = s_1b_1 + \cdots + s_sb_l \in S \), where \( r_1, \ldots, r_k \), \( s_1, \ldots, s_l \in D \), \( a_1, \ldots, a_k \), \( b_1, \ldots, b_l \in D \), \( a_i \land a_j \land b_i \land b_j = 0 \) if \( 1 \leq i \neq j \leq k \), and \( b_j \land b_j = 0 \) if \( 1 \leq j \leq l \). Let \( a_0 \) be \( (a_1 \lor \cdots \lor a_k) \) and let \( b_0 \) be \( (b_1 \lor \cdots \lor b_l) \). Since \((D, 0, 1, \neg, \land, \lor)\) is a boolean algebra, we have that \( \{a_0, \ldots, a_k\} \) and \( \{b_0, \ldots, b_l\} \) are two partitions of \( 1 \). Let \( \{c_1, \ldots, c_m\} \) is the partition generated by partitions \( \{a_0, \ldots, a_k\} \) and \( \{b_0, \ldots, b_l\} \). Then by Lemma 2.6(v), (P3), Fact 2.2(ix), and induction, we may assume that \( x = r'_1c_1 + \cdots + r'_mc_m \) and \( y = s'_1c_1 + \cdots + s'_mc_m \), where \( r'_1, \ldots, r'_m, s'_1, \ldots, s'_m \in D \) (could be 0), and \( \{c_1, \ldots, c_m\} \) is a partition of \( 1 \). Then by Lemma 2.6, we have \( x + y = (r'_1 + s'_1)c_1 + \cdots + (r'_m + s'_m)c_m \in S \). Thus \( S \) is closed under \( + \).

Therefore, \( S \) is an \( L_{RV} \)-prestructure, and thus \( S \supseteq M_0 \). Hence, \( S_0 = M_0 \), the smallest \( L_{RV} \)-prestructure containing \( D \) in \( M \).

Consider a nonzero element \( x \in M_0 \). Suppose \( x \) is of the form \( x = t_1c_1 + \cdots + t_kc_k \), where \( t_1, \ldots, t_k \in D \), \( c_1, \ldots, c_k \in D \), and \( c_i \land c_j = 0 \) if \( i \neq j \). Suppose \( k \) is the smallest integer for such decomposition. By (P3), Fact 2.2(ix), and induction, we may reorder those terms such that \( x = r_1a_1 + \cdots + r_ka_k \), where \( r_1 \leq r_2 \leq \cdots \leq r_k \in D \) and \( a_1, \ldots, a_k \in D \), and \( a_i \land a_j = 0 \) if \( i \neq j \). By the fact that \( k \) is chosen to be smallest, we have \( 0 < r_1 < \cdots < r_k \) and \( a_i \neq 0 \) for each \( 1 \leq i \leq k \). Then a standard manipulation of lattices yields that this decomposition is unique. We leave it to the readers. \( \Box \)
Proposition 2.8. Let $\mathcal{M}$ be a model of RV and let $D = \{x \in M \mid x \wedge \neg x = 0\}$. Let $M_0 \subseteq \mathcal{M}$ be the smallest $L_{RV}$-prestructure containing $D$. Then $M_0$ is the set $\{r_1a_1 + \cdots + r_na_n \mid n \in \mathbb{N}, r_1, \ldots, r_n \in \mathbb{D}, a_1, \ldots, a_n \in D, \text{ and } a_i \land a_j = 0 \text{ if } i \neq j\}$. Moreover, every $L_{Pr}$-isomorphism $\phi : D \to L^1((\Omega, F, \mu), \{0, 1\})$, where $(\Omega, F, \mu)$ is a probability space, will be uniquely extended to an $L_{RV}$-embedding

$$
\Phi : M_0 \to L^1((\Omega, F, \mu), \mathbb{D})
$$

which is defined by $\Phi(r_1a_1 + \cdots + r_na_n) = r_1\phi(a_1) + \cdots + r_n\phi(a_n)$, where $n \in \mathbb{N}$, $r_i \in \mathbb{D}$ and $a_i \in D$ for all $1 \leq i \leq n$, and $a_i \land a_j = 0$ whenever $i \neq j$.

Proof. Suppose $\phi$ can be extended to an $L_{RV}$-embedding

$$
\Phi : M_0 \to L^1((\Omega, F, \mu), \mathbb{D}).
$$

Then $\Phi(r_1a_1 + \cdots + r_na_n) = r_1\Phi(a_1) + \cdots + r_n\Phi(a_n)$, where $n \in \mathbb{N}$, $r_i \in \mathbb{D}$ and $a_i \in D$ for all $1 \leq i \leq n$, and $a_i \land a_j = 0$ for all $1 \leq i \neq j \leq n$. Since $\Phi$ is an extension of $\phi$, we have $\Phi(r_1a_1 + \cdots + r_na_n) = r_1\phi(a_1) + \cdots + r_n\phi(a_n)$. Hence such an extension $\Phi$ is uniquely determined by $\phi$.

Let $D = \{x \in M \mid I(x \wedge \neg x) = 0\}$. By Proposition 2.3, we know that the $L_{Pr}$-structure $(D, 0, \neg, \cap, \cup, \mu)$ is a model of $Pr$. By [5, Theorem 5.2], there is a probability space $(\Omega, F, \mu)$ such that $D$ as an $L_{Pr}$-structure is isomorphic to $\hat{F}$. Let $N$ denote the $L_{RV}$-structure $(L^1((\Omega, F, \mu), \{0, 1\}), 0, \neg, \land, \lor, \mu, \phi)$. By Proposition 2.1, we have $N \models RV$. Let $X$ denote $L^1((\Omega, F, \mu), \{0, 1\})$. By Proposition 2.3, we have $(X, 0, 1, \neg, \land, \lor, \mu) \models Pr$ and it is isomorphic to $\hat{F}$. Hence, $D$ is $L_{RV}$-isomorphic to $X$. We call this isomorphism $\phi : D \to X$. Then for all $x, y \in D$, we have that $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$, $\phi(0) = 0$, $\phi(1) = 1$, $\int \phi(x)d\mu = I^\mathbb{N}(\phi(x)) = I(x)$, and $d(x, y) = d^\mathbb{N}(\phi(x), \phi(y))$. Let $M_0$ be the smallest $L_{RV}$-prestructure containing $D$. By Proposition 2.7, we know that every nonzero element $x \in M_0$ has a unique decomposition of the form $x = r_1a_1 + \cdots + r_na_n$ where $n \in \mathbb{N}$, $0 < r_1 < \cdots < r_n \in \mathbb{D}$, $a_1, \ldots, a_n \in D$, $a_i \neq 0$ for each $i$, and $a_i \land a_j = 0$ whenever $i \neq j$. We extend $\phi : D \to X$ to a mapping $\Phi : M_0 \to N$, by defining

$$
\Phi(r_1a_1 + \cdots + r_na_n) := r_1\phi(a_1) + \cdots + r_n\phi(a_n),
$$

where $n \in \mathbb{N}$, $0 < r_1 < \cdots < r_n \in \mathbb{D}$, $a_1, \ldots, a_n \in D$, $a_i \neq 0$ for each $i$, and $a_i \land a_j = 0$ whenever $i \neq j$. Clearly, $\Phi$ is uniquely determined by $\phi$.

Next, we will check that $\Phi$ preserves $0, 1, \neg, \frac{1}{2}, \lor, I$ and $d$. We already know that $\Phi(0) = 0$ and $\Phi(1) = 1$. To show that $\Phi$ preserves $\neg, \frac{1}{2}, \lor, I$ and $d$, we need the following claim:

Claim 2.9. Take a nonzero $x \in M_0$. Suppose $x$ has the form $r_1a_1 + \cdots + r_na_n$, where $n \in \mathbb{N}$, $0 < r_1 < \cdots < r_n \in \mathbb{D}$, $a_1, \ldots, a_n \in D$, $a_i \neq 0$ for each $i$, and $a_i \land a_j = 0$ whenever $i \neq j$. Suppose $x$ has another form $s_1b_1 + \cdots + s_mb_m$, where $m \in \mathbb{N}$, $s_1, \ldots, s_m \in \mathbb{D}$, $b_1, \ldots, b_m \in D$, and $b_k \land b_l = 0$ whenever $k \neq l$.

Then $\Phi(x) = r_1\Phi(a_1) + \cdots + r_n\Phi(a_n) = s_1\Phi(b_1) + \cdots + s_m\Phi(b_m)$. 

Proof of Claim 2.9. Suppose $x$ has the form $s_1b_1 + \cdots + s_nb_n$, where $m \in \mathbb{N}$, $s_1, \ldots, s_m \in D$, $b_1, \ldots, b_m \in D$, and $b_k \land b_l = 0$ whenever $k \neq l$. Then after a standard procedure to reorder terms, delete 0 terms, and combine the terms with the same coefficients, the form of $x$ becomes $r_1a_1 + \cdots + r_na_n$, where $n \in \mathbb{N}$, $0 < r_1 < \cdots < r_n \in \mathbb{D}$, $a_1, \ldots, a_n \in D$, $a_i \neq 0$ for each $i$, and $a_i \land a_j = 0$ whenever $i \neq j$, which is the unique decomposition shown in Proposition 2.7. It is easy to verify that during the reordering, deleting, and combining processes, although the form of $x$ has changed, the sum $s_1\Phi(b_1) + \cdots + s_n\Phi(b_n)$ remains the same. This completes the proof of Claim 2.9.

Next, we will show that $\Phi$ preserves $\frac{1}{2}$, $\neg$, and $\lor$. Take a nonzero $x \in M_0$. Suppose $x$ has the form $r_1a_1 + \cdots + r_na_n$, where $n \in \mathbb{N}$, $0 < r_1 < \cdots < r_n \in \mathbb{D}$, $a_1, \ldots, a_n \in D$, $a_i \neq 0$ for each $i$, and $a_i \land a_j = 0$ whenever $i \neq j$. As shown in the proof of Proposition 2.7, we know that $\frac{1}{2} = \frac{1}{2}r_1 + \frac{1}{2}a_1 + \frac{1}{2}a_2 + \cdots + \frac{1}{2}a_n$, and 

$$-x = -r_1a_1 + \cdots + -r_na_n + -(a_1 \lor \cdots \lor a_n).$$

By Claim 2.9, we have 

$$\Phi\left(\frac{x}{2}\right) = \frac{r_1}{2}\Phi(a_1) + \cdots + \frac{r_n}{2}\Phi(a_n) = \frac{1}{2}(r_1\Phi(a_1) + \cdots + r_n\Phi(a_n)) = \frac{1}{2}\Phi(x).$$

Hence, $\Phi$ preserves $\frac{1}{2}$. Also 

$\Phi(\neg x) = \Phi(-r_1a_1 + \cdots + -r_na_n + -(a_1 \lor \cdots \lor a_n))$

$$= -r_1\Phi(a_1) + \cdots + -r_n\Phi(a_n) + \Phi(-(a_1 \lor \cdots \lor a_n)).$$

Then because $\Phi(a_1), \ldots, \Phi(a_n)$ are in $L^1(\mu, [0,1])$, $a_i \land a_j = 0$ if $i \neq j$, and $\Phi$: $D \rightarrow \mathcal{X}$ is an $L_{\nu}$-isomorphism, we have that 

$$-r_1\Phi(a_1) + \cdots + -r_n\Phi(a_n) + \Phi(-(a_1 \lor \cdots \lor a_n))$$

$$= (1 - r_1)\Phi(a_1) + \cdots + (1 - r_n)\Phi(a_n) + (1 - \Phi(a_1) - \cdots - \Phi(a_n))$$

$$= 1 - r_1\Phi(a_1) + \cdots + r_n\Phi(a_n) = -(r_1\Phi(a_1) + \cdots + r_n\Phi(a_n))$$

$$= -(r_1\Phi(a_1) + \cdots + r_n\Phi(a_n)).$$

Hence, $\Phi(\neg x) = \neg \Phi(x)$; that is, $\Phi$ preserves $\neg$.

Take nonzero $x, y \in M_0$. Suppose they have the form $x = r_1a_1 + \cdots + r_ka_k$ and $y = s_1b_1 + \cdots + s_kb_k$, where $k, l \in \mathbb{N}$, $0 < r_1 < \cdots < r_k \in \mathbb{D}$, $0 < s_1 < \cdots < s_l \in \mathbb{D}$, $a_1, \ldots, a_k, b_1, \ldots, b_l \in D$, $a_i \neq 0$ for all $1 \leq i \leq k$, $b_j \neq 0$ for all $1 \leq j \leq l$, $a_i \land a_i' = 0$ whenever $1 \leq i \neq i' \leq k$, and $b_j \land b_j' = 0$ whenever $1 \leq j \neq j' \leq l$. Let $a_0$ be $\neg(a_1 \lor \cdots \lor a_k)$ and let $b_0$ be $\neg(b_1 \lor \cdots \lor b_l)$. Since $(D, 0, 1, \land, \lor)$ is a boolean algebra, we have $a_0 \lor a_i \lor \cdots \lor a_k = b_0 \lor b_1 \lor \cdots \lor b_l = 1$, $a_i \land a_i' = 0$ for all $0 \leq i < i' \leq k$, and $b_j \land b_j' = 0$ for all $0 \leq j < j' \leq l$. That is, $\{a_0, \ldots, a_k\}$ and $\{b_0, \ldots, b_l\}$ are two partitions of 1. Then let $\{c_1, \ldots, c_m\}$ be the partition generated by partitions $\{a_0, \ldots, a_k\}$ and $\{b_0, \ldots, b_l\}$. Then by Lemma 2.6(v), (P3), Fact 2.2(ix), and induction, we may assume that $x = r'_1c_1 + \cdots + r'_mc_m$ and $y = s'_1c_1 + \cdots + s'_m$, where
$r_1', \ldots, r_m', s_1', \ldots, s_m' \in \mathbb{D}$ (could be 0), and \{\(c_1, \ldots, c_m\)\} is a partition of 1. Then by Lemma 2.6, we have \(x + y = (r_1' + s_1')c_1 + \cdots + (r_m' + s_m')c_m\). Hence
\[
\Phi(x + y) = (r_1' + s_1')\Phi(c_1) + \cdots + (r_m' + s_m')\Phi(c_m)
\]
\[
= (r_1'\Phi(c_1) + \cdots + r_m'\Phi(c_m)) + (s_1'\Phi(c_1) + \cdots + s_m'\Phi(c_m))
\]
\[
= \Phi(x) + \Phi(y).
\]
Thus, \(\Phi\) preserves \(+\).

Now, we will prove \(\Phi\) preserves \(I\) and \(d\). Take a nonzero \(x \in M_0\). Suppose \(x\) has the form \(r_1a_1 + \cdots + r_na_n\), where \(n \in \mathbb{N}\), \(0 < r_1 < \cdots < r_n \in \mathbb{D}\), \(a_1, \ldots, a_n \in D\), \(a_i \neq 0\) for each \(i\), and \(a_i \land a_j = 0\) whenever \(i \neq j\). Since \(\Phi: D \to X\) is an isomorphism, we have \(\Phi(a_i) \land \Phi(a_j) = 0\) whenever \(i \neq j\).

For all \(f \in N\), we have \(I^N(f) = \int f d\mu\). Therefore,
\[
I^N(\Phi(x)) = \int \Phi(x) d\mu = \int (r_1\Phi(a_1) + \cdots + r_n\Phi(a_n)) d\mu
\]
\[
= r_1 \int \Phi(a_1) d\mu + \cdots + r_n \int \Phi(a_n) d\mu
\]
\[
= r_1 I(\Phi(a_1)) + \cdots + r_n I(\Phi(a_n))
\]
By Lemma 2.6(vii), we have \(I^N(\Phi(x)) = I(r_1a_1) + \cdots + I(r_na_n)\). Then by Lemma 2.5, we have \(I(r_1a_1) + \cdots + I(r_na_n) = I(r_1a_1 + \cdots + r_na_n) = I(x)\), and thus \(I(x) = I^N(\Phi(x))\). That is, \(\Phi\) preserves \(I\). Since \(d(x, y) = I(x \lor y) + I(\neg x)\) for all \(x, y \in M\) and \(\Phi\) preserves \(I\) and \(\land\), it follows that \(\Phi\) preserves \(d\).

Therefore, \(\Phi\) is an \(L_{RV}\)-embedding from \(M_0\) to \(X\). \(\Box\)

**Theorem 2.10.** Let \(M\) be a model of \(RV\). Then \(M\) is isomorphic to the \(L_{RV}\)-structure \(\left(L^1((\Omega, \mathcal{F}, \mu), [0, 1]), 0, \land, \lor, \neg, \frac{1}{2}, I, d\right)\) for some probability space \((\Omega, \mathcal{F}, \mu)\).

**Proof.** Let \(D = \{x \in M \mid I(x \land \neg x)\} = 0\). By Proposition 2.3, we know that \((\Omega, \mathcal{F}, \mu)\) is a model of \(Pr\). By [5, Theorem 5.2], we know that there is a probability space \((\Omega, \mathcal{F}, \mu)\) such that \(D\) as an \(L_{Pr}\)-structure is isomorphic to \(\tilde{F}\). Let \(X = (L^1((\Omega, \mathcal{F}, \mu), [0, 1]), 0, \land, \lor, \neg, \frac{1}{2}, I, d)\) be a model of \(RV\). By Proposition 2.1, we have \(N \models RV\). Let \(X\) denote \(L^1((\Omega, \mathcal{F}, \mu), [0, 1])\). By Proposition 2.3, we have \((X, 0, 1, \land, \lor, \neg, \mu) \models Pr\) and it is isomorphic to \(\tilde{F}\). Hence, \(D\) as an \(L_{Pr}\)-structure is isomorphic to \(X\). We call this isomorphism \(\phi: D \to X\). Then by Proposition 2.8, we extend \(\phi: D \to X\) to an \(L_{RV}\)-embedding \(\Phi: M_0 \to N\).

Let \((M', d)\) be the completion of \((M_0, d)\) in \(M\). Because \(\Phi\) is isometric, we know that \(\Phi\) is extended uniquely to an embedding \(\overline{\Phi}\) from \(M'\) to \(N\). Note that dyadic number valued simple functions are dense in \(N\). Hence \(\overline{\Phi}\) is a surjective embedding; that is, \(\overline{\Phi}\) is an isomorphism between \(L_{RV}\)-structures \(M'\) and \(N\). Then we will show \(M' = M\).
For every $x \in M$ and $n \in \mathbb{N}$, using (APPR), there are elements $y_1, \ldots, y_{2^n} \in M$ such that
\[
\max I(y_i \wedge \neg y_i) \leq \frac{1}{n} \text{ and } d(x, \frac{1}{2^n} y_1 + \cdots + \frac{1}{2^n} y_{2^n}) \leq \frac{1}{2^n}. \]
For every $1 \leq i \leq 2^n$, since $I(y_i \wedge \neg y_i) = \text{dist}(y_i, D)$, there is $z_i \in D$ such that $d(y_i, z_i) \leq \frac{1}{2^n}$. Then by (P1),
\[
d\left(\frac{1}{2^n} y_1 + \cdots + \frac{1}{2^n} y_{2^n}, \frac{1}{2^n} z_1 + \cdots + \frac{1}{2^n} z_{2^n}\right) \\
\leq d\left(\frac{1}{2^n} y_1, \frac{1}{2^n} z_1\right) + \cdots + d\left(\frac{1}{2^n} y_{2^n}, \frac{1}{2^n} z_{2^n}\right) \\
= \frac{1}{2^n} \left(d(y_1, z_1) + \cdots + d(y_{2^n}, z_{2^n})\right) \\
\leq \max_{1 \leq i \leq 2^n} d(y_i, z_i) \leq \frac{1}{2^n}
\]
where the equality follows from Fact 2.2(x). Then
\[
d(x, \frac{1}{2^n} z_1 + \cdots + \frac{1}{2^n} z_{2^n}) \\
\leq d(x, \frac{1}{2^n} y_1 + \cdots + \frac{1}{2^n} y_{2^n}) + d\left(\frac{1}{2^n} y_1 + \cdots + \frac{1}{2^n} y_{2^n}, \frac{1}{2^n} z_1 + \cdots + \frac{1}{2^n} z_{2^n}\right) \\
\leq \frac{1}{2^n} + \frac{1}{2^n} = \frac{1}{2^{n-1}}.
\]
Since $\frac{1}{2^n} z_1 + \cdots + \frac{1}{2^n} z_{2^n} \in M_0$, it follows that $M_0$ is dense in $M$, whereby $M' = M$. Therefore $\Phi$ is an isomorphism from $M$ to $N$. \qed

Corollary 2.11. Let $M$ be an $L_{RV}$-structure. Then $M$ is a model of ARV if and only if $M$ is isomorphic to $\left(L^1((\Omega, F, \mu), [0, 1]), 0, \wedge, 1, I, d\right)$ for some atomless probability space $(\Omega, F, \mu)$.

Proof. By Proposition 2.1, we know that $\left(L^1((\Omega, F, \mu), [0, 1]), 0, \wedge, 1, I, d\right)$, where $(\Omega, F, \mu)$ is an atomless probability space, is a model of ARV.

For the other direction, by Theorem 2.10, $M$ as a model of RV is isomorphic to the $L_{RV}$-structure $\left(L^1((\Omega, F, \mu), [0, 1]), 0, \wedge, 1, I, d\right)$ for a probability space $(\Omega, F, \mu)$. Let $D$ denote $L^1((\Omega, F, \mu), [0, 1])$. By Proposition 2.3, $D$ is a model of $Pr$ and is $L_{Pr}$-isomorphic to $\bar{F}$. For all $x \in D$, using (NA) we know that for every $\epsilon > 0$, there is $y_\epsilon$ such that $\text{dist}(y_\epsilon, D) \leq \epsilon$ and $|I(y_\epsilon \wedge x) - \frac{f(x)}{2^n}| \leq \epsilon$. Then there is $y \in D$ such that $|I(y \wedge x) - \frac{f(x)}{2^n}| \leq 2\epsilon$. Thus $D$ as a $L_{Pr}$-structure satisfies Axiom (iv) in $AP_{Pr}$, whereby $D$ is a model of $AP_{Pr}$. By [5, Corollary 6.1], we have that $(\Omega, F, \mu)$ is atomless. \qed

Remark 2.12. In [2], Ben Yaacov showed that the class $\mathcal{R}V$ of random variable structures and the class $\mathcal{P}R$ of probability algebras are bi-interpretable. Then by [1, Theorem A.9], the class $\mathcal{R}V$ is elementary if and only if the class $\mathcal{P}R$ is,
which is clear. In the proof of Theorem A.9 there is a way to give axioms for RV, albeit not in a very intuitive form.

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