CONSTRUCTION OF SUBCLASSES OF UNIVALENT HARMONIC MAPPINGS

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Abstract. Complex-valued harmonic functions that are univalent and sense-preserving in the open unit disk are widely studied. A new methodology is employed to construct subclasses of univalent harmonic mappings from a given subfamily of univalent analytic functions. The notions of harmonic Alexander operator and harmonic Libera operator are introduced and their properties are investigated.

1. Introduction

Let \( H \) denote the class of all complex-valued harmonic functions \( f \) in the open unit disk \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) normalized by \( f(0) = 0 = f_z(0) - 1 = f_{\bar{z}}(0) \). Such functions can be written in the form

\[
    h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n
\]

are analytic in \( \mathbb{D} \). In 1984, Clunie and Sheil-Small [7] investigated the subclass \( S_0^H \) of \( H \) consisting of univalent and sense-preserving functions. A function \( f = h + g \in H \) is sense-preserving if the Jacobian \( J_f(z) = |h'(z)|^2 - |g'(z)|^2 \) is positive or equivalently \( |g'(z)| < |h'(z)| \) in \( \mathbb{D} \). The class \( S_0^H \) is a compact family with respect to the topology of locally uniform convergence. The classical family \( S \) of normalized analytic univalent functions is a subclass of \( S_0^H \). Let \( S^{*0}_H, K^0_H \) and \( C^0_H \) be the subclasses of \( S_0^H \) consisting of functions mapping \( \mathbb{D} \) onto starlike, convex and close-to-convex domains, respectively, just as \( S^{*}, K \) and \( C \) are the subclasses of \( S \) mapping \( \mathbb{D} \) onto their respective domains.

In [26], we have investigated the properties of functions in the subclass \( F^0_H \subset C^0_H \) defined by the condition \( |f_z(z) - 1| < 1 - |f_z(z)| \) for all \( z \in \mathbb{D} \). This subclass was closely related to the class \( F \subset C \), introduced by MacGregor [20], consisting...

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of analytic functions satisfying $|f'(z) - 1| < 1$ for $z \in \mathbb{D}$. We proved that a harmonic function $f = h + \bar{g} \in F_0^H$ if and only if the analytic functions $h + \epsilon g$ belong to $F$ for each $|\epsilon| = 1$. Using this property, the coefficient estimates, growth results, boundary behavior, convolution properties and sharp bound for radius of convexity and starlikeness for the class $F_0^H$ were investigated. This connection between the classes $F$ and $F_0^H$ has motivated to give the following definition which turns out to be a simple but an effective method in construction of subclasses of univalent harmonic mappings from a given subfamily of $S$.

**Definition 1.1.** Suppose that $G$ is a subfamily of $S$. Denote by $G_0^H$ the class consisting of harmonic functions $f = h + \bar{g}$ for which $h + \epsilon g \in G$ for each $|\epsilon| = 1$, $h$ and $g$ being analytic functions in $\mathbb{D}$. We call $G_0^H$ the harmonic analogue of $G$ and write $G \triangleright G_0^H$.

By Definition 1.1, it readily follows that $F \triangleright F_0^H$. If $G_0^H$ is the harmonic analogue of $G \subset S$, then it is easy to see that $G \subset G_0^H$. Further properties of the harmonic analogue $G_0^H$ for a subfamily $G \subset S$ are investigated in Section 2.

In Section 3, the harmonic analogues of some well-known subclasses of $S$ are determined and their properties are discussed.

Let $A$ be the subclass of $H$ consisting of normalized analytic functions. Let $\Lambda : A \to A$ be the Alexander integral operator [1] defined by

\begin{equation}
\Lambda[f](z) = \int_0^z \frac{f(t)}{t} \, dt.
\end{equation}

Krzyż and Lewandowski [16] constructed an example to show that $\Lambda$ does not carry $S$ into $S$. Another familiar integral operator $\Theta : A \to A$ is the Libera operator [18] defined by

\begin{equation}
\Theta[f](z) = \frac{2}{z} \int_0^z f(t) \, dt.
\end{equation}

Even this operator does not preserve univalence. Campbell and Singh [4] gave examples of univalent functions which the operator $\Theta$ takes to non-univalent functions. However, these two operators preserve certain subclasses of univalent functions. In the last section of this paper, two notions of harmonic Alexander operators $\Lambda^+_H, \Lambda^-_H : H \to H$ and a notion of harmonic Libera operator $\Theta_H : H \to H$ are introduced and their properties are investigated.

### 2. Some properties of harmonic analogue $G_0^H$

In this section, we will investigate the properties of the harmonic analogue $G_0^H$ for subfamily $G \subset S$. For this, the notion of stable harmonic mappings introduced by Hernández and Martín in [15] is needed. A sense-preserving harmonic mapping $f = h + \bar{g}$ is said to be stable univalent (resp. stable starlike, stable convex and stable close-to-convex) if all the mappings $f_\lambda = h + \lambda \bar{g}$ with $|\lambda| = 1$ are univalent (resp. starlike, convex and close-to-convex) in $\mathbb{D}$. The following result was proved in [15].
Lemma 2.1. A sense-preserving harmonic mapping \( f = h + \bar{g} \) is stable univalent (resp. stable starlike, stable convex and stable close-to-convex) if and only if the analytic functions \( F_x = h + \lambda g \) are univalent (resp. starlike, convex and close-to-convex) in \( \mathbb{D} \) for each \( |\lambda| = 1 \).

Let \( SS_H^0, SS_H^0, SK_H^0 \) and \( SC_H^0 \) be subclasses of \( S_H^0 \) consisting of stable univalent, stable starlike, stable convex and stable close-to-convex mappings, respectively. Then \( S^* \subset SS_H^0 \subset S_H^0, K \subset SK_H^0 \subset K_H^0 \) and \( C \subset SC_H^0 \subset C_H^0 \). Moreover, \( SK_H^0 \subset SS_H^0 \subset SC_H^0 \subset SK_H^0 \). In view of Definition 1.1 and Lemma 2.1, it follows that \( SS_H^0, SS_H^0, SK_H^0 \) and \( SC_H^0 \) are harmonic analogues of \( S, S^*, K \) and \( C \), respectively.

The first theorem is quite simple but a useful tool in the investigation of results regarding the harmonic analogue \( G_H^0 \) for a subfamily \( G \subset S \).

Theorem 2.2. Suppose that \( G \subset S \) and \( G \vdash G_H^0 \). Then

(i) \( G_H^0 \subset SS_H^0 \),
(ii) \( \text{if } f \in S \cap G_H^0, \text{ then } f \in G \),
(iii) \( \text{if } f = h + \bar{g} \in G_H^0, \text{ then the harmonic mappings } f_\lambda = h + \lambda \bar{g} \in G_H^0 \text{ for each } |\lambda| = 1 \),
(iv) \( \text{if } J \subset G, \text{ then } J_H^0 \subset G_H^0 \text{ where } J_H^0 \text{ is the harmonic analogue of } J \).

Proof. Let \( f = h + \bar{g} \in G_H^0 \). Then \( h + \epsilon g \in G \) for each \( |\epsilon| = 1 \) which imply that \( h(0) = g(0) = h'(0) - 1 = g'(0) = 0 \) using the normalization of functions in \( G \). Also, since \( h + \epsilon g \) is univalent, \( (h + \epsilon g)' \neq 0 \) in \( \mathbb{D} \) for each \( |\epsilon| = 1 \). This imply that the Jacobian \( J_f(z) \neq 0 \) for all \( z \in \mathbb{D} \) and since \( J_f(0) = 1 > 0 \), \( f \) is sense-preserving in \( \mathbb{D} \). By Lemma 2.1, it follows that \( f \in SS_H^0 \). This proves (i). The part (ii) follows immediately from Definition 1.1. For the proof of (iii), let \( f = h + \bar{g} \in G_H^0 \) and \( |\lambda| = 1 \). Then it is easy to see that \( h + \lambda \bar{g} \in G \) for each \( |\epsilon| = 1 \) and so \( h + \lambda \bar{g} \in G_H^0 \). To prove (iv), let \( f = h + \bar{g} \in J_H^0 \). As \( J \vdash J_H^0 \), \( h + \epsilon g \in J \) for each \( |\epsilon| = 1 \). Since \( J \subset G \) we have \( h + \epsilon g \in G \) for each \( |\epsilon| = 1 \) which shows that \( f \in G_H^0 \). This completes the proof of the theorem. \( \square \)

Theorem 2.2(ii) conveys that every analytic univalent function in \( G_H^0 \) is a member of \( G \). Since the members of \( G_H^0 \) are stable univalent by Theorem 2.2(i), we have the following corollary which follows by [15, Theorem 7.1].

Corollary 2.3. Suppose that \( G \subset S \) and \( G \vdash G_H^0 \). If \( f = h + \bar{g} \in G_H^0 \), then the analytic mappings \( F_\mu = h + \mu g \) are univalent in \( \mathbb{D} \) for each \( |\mu| \leq 1 \). In particular, \( h \) is univalent.

Recall that convexity and starlikeness are hereditary properties for conformal mappings and they do not extend to harmonic mappings (see [10]). Chuaqui, Duren and Osgood [6] introduced the notion of fully starlike and fully convex functions that do inherit the properties of starlikeness and convexity, respectively (see also [27]). A harmonic mapping \( f \) of the unit disk \( \mathbb{D} \) is fully convex if it maps every circle \( |z| = r < 1 \) in a one-to-one manner onto a convex curve. Such a harmonic mapping \( f \) with \( f(0) = 0 \) is fully starlike if it maps every
circle \(|z| = r < 1\) in a one-to-one manner onto a curve that bounds a domain starlike with respect to the origin. Applying Theorem 2.2(iv) and using the fact that stable starlike (resp. stable convex) mappings are fully starlike (resp. fully convex) (see [15, 27]), we have:

**Corollary 2.4.** Suppose that \(G \subset \mathcal{S}\) and \(G \uparrow \mathcal{G}_H^0\). If \(G \subset \mathcal{S}^+\) (resp. \(G \subset \mathcal{K}\)), then members of \(\mathcal{G}_H^0\) are fully starlike (resp. fully convex) in \(D\).

The harmonic Koebe function

\[
(4) \quad K(z) = H(z) + \frac{\overline{G(z)}}{z}, \quad H(z) := \frac{z - \frac{1}{2}z^2 + \frac{1}{3}z^3}{(1 - z)^3}, \quad G(z) := \frac{z - \frac{1}{2}z^2 + \frac{1}{3}z^3}{(1 - z)^3}
\]

shows that the classes \(\mathcal{S}_H^0\), \(\mathcal{S}_{H}^0\) and \(\mathcal{C}_H^0\) are not harmonic analogues of any subfamily of \(\mathcal{S}\) since

\[
H(z) + G(z) = \frac{z + \frac{1}{3}z^3}{(1 - z)^3}, \quad z \in D,
\]

and \((H + G)(i/\sqrt{3}) = (H + G)(-i/\sqrt{3})\) which imply that \(H + G\) is not univalent in \(D\). Similarly, \(\mathcal{K}_H^0\) is not a harmonic analogue of any subfamily \(G \subset \mathcal{S}\). For if \(G \uparrow \mathcal{K}_H^0\), then \(G \subset \mathcal{K}\). The harmonic half-plane mapping

\[
(5) \quad L(z) = M(z) + \frac{\overline{N(z)}}{z}, \quad M(z) := \frac{z - \frac{1}{2}z^2}{(1 - z)^2}, \quad N(z) := \frac{-\frac{1}{2}z^2}{(1 - z)^2}
\]

belongs to \(\mathcal{K}_H^0\) and \(M(z) - N(z) = z/(1-z)^2 \notin \mathcal{K}\). These observations suggest that given a subfamily \(\mathcal{G}_H^0 \subset \mathcal{S}_H^0\), it is possible that \(\mathcal{G}_H^0\) is not a harmonic analogue of any subclass of \(\mathcal{S}\). This motivates us to determine a necessary and sufficient condition for a subfamily \(\mathcal{G}_H^0 \subset \mathcal{S}_H^0\) to be a harmonic analogue of some family \(G \subset \mathcal{S}\). This is contained in the following corollary.

**Corollary 2.5.** A subfamily \(\mathcal{G}_H^0 \subset \mathcal{S}_H^0\) is a harmonic analogue of some family \(G \subset \mathcal{S}\) if and only if \(\mathcal{G}_H^0 \subset \mathcal{S}_H^0\).

**Proof.** The necessary part follows by Theorem 2.2(i). For the sufficient part, suppose that \(\mathcal{G}_H^0 \subset \mathcal{S}_H^0\). Considering the set \(\mathcal{G} = \{h + \epsilon g : h + \epsilon g \in \mathcal{G}_H^0\}\) and \(\epsilon = 1\), it is evident that \(G \subset \mathcal{S}\) and \(G \uparrow \mathcal{G}_H^0\) by using Lemma 2.1. \(\square\)

It is easy to see that if \(\mathcal{I}\) and \(\mathcal{J}\) are subclasses of \(\mathcal{S}\) with \(\mathcal{I} \uparrow \mathcal{I}_H^0\) and \(\mathcal{J} \uparrow \mathcal{J}_H^0\), then \(\mathcal{I} \cap \mathcal{J} \uparrow \mathcal{I}_H^0 \cap \mathcal{J}_H^0\) and \(\mathcal{I} \cup \mathcal{J} \uparrow \mathcal{I}_H^0 \cup \mathcal{J}_H^0\). The next theorem determines the coefficient bounds for functions in the harmonic analogue \(\mathcal{G}_H^0\).

**Theorem 2.6.** Suppose that \(G \subset \mathcal{S}\) and \(G \uparrow \mathcal{G}_H^0\). Let the Taylor coefficients \(a_n(f)\) of the series of each \(f \in \mathcal{G}\) satisfies \(|a_n(f)| \leq p(n)\) for \(n = 2, 3, \ldots\) where \(p\) is a function of \(n\). Then the respective Taylor coefficients \(A_n(f)\) and \(B_n(f)\) of the series of \(h\) and \(g\) of each function \(f = h + \dot{\gamma} \in \mathcal{G}_H^0\) satisfies

\[
|A_n(f)| + |B_n(f)| \leq p(n) \quad \text{for} \quad n = 2, 3, \ldots.
\]

In particular, we have
(a) \(|A_n(f)| - |B_n(f)|| \leq p(n), \quad n = 2, 3, \ldots \)
(b) Let \(h_0 \in \mathcal{G}\) be such that its Taylor coefficients satisfy \(|a_n(h_0)| = p(n)\) for \(n = 2, 3, \ldots\). Then for an analytic function \(g_0\), the harmonic function \(f_0 = h_0 + \overline{g_0} \in \mathcal{G}_H^0\) if and only if \(g_0 \equiv 0\).

Proof. Let \(f = h + \overline{g} \in \mathcal{G}_H^0\). Then \(h + \epsilon g \in \mathcal{G}\) for each \(|\epsilon| = 1\) so that \(|a_n(h + \epsilon g)| \leq p(n)\) for \(n = 2, 3, \ldots\). But \(a_n(h + \epsilon g) = A_n(f) + \epsilon B_n(f)\) for \(n = 2, 3, \ldots\) so that (6) is satisfied with appropriate choice of \(\epsilon = \epsilon(n)\).

The part (a) is evident from (6). For (b), suppose that \(f_0 = h_0 + \overline{g_0} \in \mathcal{G}_H^0\). Then \(|A_n(f_0)| + |B_n(f_0)| \leq p(n)\) for \(n = 2, 3, \ldots\). But \(|A_n(f_0)| = |a_n(h_0)| = p(n)\) for \(n = 2, 3, \ldots\) so that \(B_n(f_0) = 0\) for \(n = 2, 3, \ldots\). Thus \(g_0 \equiv 0\). The converse part is obvious. \(\square\)

The next theorem determines the upper and lower bounds on the growth of a harmonic mapping in \(\mathcal{G}_H^0\).

**Theorem 2.7.** Suppose that \(\mathcal{G} \subset \mathcal{S}\) and \(\mathcal{G} \supset \mathcal{G}_H^0\). If

\[
P(|z|) \leq |f'(z)| \leq Q(|z|), \quad z \in \mathbb{D}
\]

for each \(f \in \mathcal{G}\) where \(P\) and \(Q\) are integrable functions of \(|z|\), then each \(f \in \mathcal{G}_H^0\) satisfies

\[
\int_0^{|z|} P(\rho) \, d\rho \leq |f(z)| \leq \int_0^{|z|} Q(\rho) \, d\rho, \quad z \in \mathbb{D}.
\]

In particular, we have the following.

(i) The range of every function \(f \in \mathcal{G}_H^0\) contains the disk

\[
\left\{ w \in \mathbb{C} : |w| < \lim_{|z| \to 1} \int_0^{|z|} P(\rho) \, d\rho \right\},
\]

provided the limit exists.

(ii) The Jacobian \(J_f\) of each function \(f \in \mathcal{G}_H^0\) satisfies

\[
P^2(|z|) \leq J_f(z) \leq Q^2(|z|), \quad z \in \mathbb{D}.
\]

Proof. Let \(f = h + \overline{g} \in \mathcal{G}_H^0\). Then \(h + \epsilon g \in \mathcal{G}\) for each \(|\epsilon| = 1\) so that

\[
P(|z|) \leq |h'(z) + \epsilon g'(z)| \leq Q(|z|), \quad z \in \mathbb{D}.
\]

In particular, this shows that

\[
P(|z|) \leq |h'(z)| - |g'(z)| \quad \text{and} \quad |h'(z)| + |g'(z)| \leq Q(|z|), \quad z \in \mathbb{D}.
\]

If \(\Gamma\) is the radial segment from 0 to \(z\), then

\[
|f(z)| = \left| \int_{\Gamma} \frac{\partial f}{\partial \zeta} \, d\zeta + \frac{\partial f}{\partial \bar{\zeta}} \, d\bar{\zeta} \right| \leq \int_{\Gamma} |(h'(\zeta)) + |g'(\zeta)||d\zeta| \leq \int_0^{|z|} Q(\rho) \, d\rho.
\]
Next, let $\Gamma$ be the pre-image under $f$ of the radial segment from 0 to $f(z)$. Then
\[
|f(z)| = \int_{\Gamma} \left| \frac{\partial f}{\partial \zeta} d\zeta + \frac{\partial f}{\partial \bar{\zeta}} d\bar{\zeta} \right| \geq \int_{\Gamma} (|h'(\zeta)| - |g'(\zeta)|) |d\zeta| \geq \int_{0}^{\Gamma} |z| P(\rho) d\rho.
\]
This proves (7).

The covering result in (i) follows from the left hand inequality of (7) by letting $|z| \to 1$. For the proof of (ii), let $f = h + \bar{g} \in \mathcal{G}_0^H$. Then (8) gives $|h'(z)| - |g'(z)| \leq Q(|z|)$ and $|h'(z)| + |g'(z)| \leq Q(|z|)$. Multiplying the corresponding sides of these two inequalities, we obtain $J_f(z) \leq Q^2(|z|)$ for $z \in \mathbb{D}$. The left hand inequality follows on similar lines. 

If a subfamily $\mathcal{G} \subset \mathcal{S}$ is compact with respect to the topology of locally uniform convergence, then so is its harmonic analogue $\mathcal{G}_0^H$. This is seen by the following theorem.

**Theorem 2.8.** Suppose that $\mathcal{G} \subset \mathcal{S}$ and $\mathcal{G} \triangleright \mathcal{G}_0^H$. Then $\mathcal{G}$ is compact if and only if $\mathcal{G}_0^H$ is compact.

**Proof.** For necessary part, suppose that $f_n = h_n + \bar{g}_n \in \mathcal{G}_0^H$ for $n = 1, 2, \ldots$ and that $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$. Then $f$ is harmonic and so $f = h + \bar{g}$. It is easy to see that $h_n \to h$ and $g_n \to g$ locally uniformly so that $h_n + \epsilon g_n \to h + \epsilon g$ for each $|\epsilon| = 1$. Since $h_n + \epsilon g_n \in \mathcal{G}$, it follows that $h + \epsilon g \in \mathcal{G}$ for each $|\epsilon| = 1$ using the compactness of $\mathcal{G}$. Thus $f = h + \bar{g} \in \mathcal{G}_0^H$.

For sufficient part, let $f_n \in \mathcal{G}$ for $n = 1, 2, \ldots$ such that $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$. Then $f$ is univalent. Since $\mathcal{G} \subset \mathcal{G}_0^H$ and $\mathcal{G}_0^H$ is compact, $f \in \mathcal{G}_0^H$. By Theorem 2.2(ii), $f \in \mathcal{G}$. \hfill \Box

The next theorem investigates the relation between the radius of starlikeness, convexity and close-to-convexity of the classes $\mathcal{G}$ and $\mathcal{G}_0^H$.

**Theorem 2.9.** Suppose that $\mathcal{G} \subset \mathcal{S}$ and $\mathcal{G} \triangleright \mathcal{G}_0^H$. Then the classes $\mathcal{G}$ and $\mathcal{G}_0^H$ have the same radius of starlikeness, convexity and close-to-convexity.

**Proof.** Since $\mathcal{G} \subset \mathcal{G}_0^H$, it suffices to show that if $r_0$ is the radius of starlikeness (resp. convexity and close-to-convexity) of $\mathcal{G}$, then $f$ is starlike (resp. convex and close-to-convex) in $|z| < r_0$ for each $f \in \mathcal{G}_0^H$. To see this, suppose that $f = h + \bar{g} \in \mathcal{G}_0^H$. Then the analytic functions $h + \epsilon g$ belong to the class $\mathcal{G}$. Consequently the functions $h + \epsilon g$ are starlike (resp. convex and close-to-convex) in $|z| < r_0$. In view of Lemma 2.1, it follows that $f$ is starlike (resp. convex and close-to-convex) in $|z| < r_0$. \hfill \Box

For analytic functions
\begin{equation}
(9) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad F(z) = z + \sum_{n=2}^{\infty} A_n z^n
\end{equation}
belonging to $\mathcal{A}$, their convolution (or Hadamard product) is defined as

$$(f \ast F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n, \quad z \in \mathbb{D}.$$ 

In the harmonic case, with $f = h + \bar{g}$ and $F = H + \bar{G}$ belonging to $\mathcal{H}$, their harmonic convolution is defined as $f \ast F = h \ast H + \bar{g} \ast \bar{G}$. Harmonic convolutions are investigated in [7, 8, 9, 12, 33].

Suppose that $\mathcal{I}$ and $\mathcal{J}$ are subclasses of $\mathcal{H}$. We say that a class $\mathcal{I}$ is closed under convolution if $\mathcal{I} \ast \mathcal{I} \subset \mathcal{I}$, that is, if $f, g \in \mathcal{I}$ then $f \ast g \in \mathcal{I}$. Similarly, the class $\mathcal{I}$ is closed under convolution with members of $\mathcal{J}$ if $\mathcal{I} \ast \mathcal{J} \subset \mathcal{I}$.

Given a subfamily $\mathcal{G} \subset \mathcal{S}$, the next theorem discusses the convolution properties of its harmonic analogue $\mathcal{G}_H^0$.

**Theorem 2.10.** Suppose that $\mathcal{G} \subset \mathcal{S}$ is closed under convolution and $\mathcal{G} \triangleright \mathcal{G}_H^0$. Then

(i) The convolution of each member of $\mathcal{G}_H^0$ with itself is again a member of $\mathcal{G}_H^0$;

(ii) If $(f + g)/2 \in \mathcal{G}$ for all $f, g \in \mathcal{G}$, then $\mathcal{G}_H^0$ is closed under convolution.

**Proof.** Let $f = h + \bar{g} \in \mathcal{G}_H^0$. To prove (i), it suffices to show that $(h \ast h) + \epsilon (g \ast g) \in \mathcal{G}$ for each $|\epsilon| = 1$. For $|\epsilon| = 1$, note that

$$(h \ast h) + \epsilon (g \ast g) = (h + \nu g) \ast (h - \nu g)$$

where $\pm \nu$ are square roots of $\epsilon$. Since $\mathcal{G}$ is closed under convolution, it follows that $(h \ast h) + \epsilon (g \ast g) \in \mathcal{G}$ so that $f \ast f \in \mathcal{G}_H^0$. This proves (i).

For the proof of (ii), let $f_i = h_i + g_i \in \mathcal{G}_H^0$ ($i = 1, 2$). Considering the analytic functions

$$F_1 = (h_1 - g_1) \ast (h_2 - g_2) = (h_1 \ast h_2) - \epsilon (h_1 \ast g_2) - (h_2 \ast g_1) + \epsilon (g_1 \ast g_2)$$

and

$$F_2 = (h_1 + g_1) \ast (h_2 + g_2) = (h_1 \ast h_2) + \epsilon (h_1 \ast g_2) + (h_2 \ast g_1) + \epsilon (g_1 \ast g_2)$$

for $|\epsilon| = 1$, we see that

$$\frac{1}{2} (F_1 + F_2) = (h_1 \ast h_2) + \epsilon (g_1 \ast g_2).$$

Since $F_1, F_2 \in \mathcal{G}$ and using the hypothesis, it is easy to deduce that $f_1 \ast f_2 \in \mathcal{G}_H^0$. \hfill $\square$

If $\mathcal{G}$ is a convex subset of $\mathcal{S}$, then $(1 - t)f + tg \in \mathcal{G}$ for all $f, g \in \mathcal{G}$ and $t \in [0, 1]$. As a result, Theorem 2.10(ii) gives the following corollary.

**Corollary 2.11.** Suppose that $\mathcal{G} \subset \mathcal{S}$ is a convex set and is closed under convolution. If $\mathcal{G} \triangleright \mathcal{G}_H^0$, then $\mathcal{G}_H^0$ is closed under convolution.
In [12], Goodloe considered the Hadamard product $\tilde{\ast}$ of a harmonic function with an analytic function defined as follows:

\begin{equation}
(10) \quad f \tilde{\ast} \varphi = \varphi \tilde{\ast} f = h \ast \varphi + \bar{g} \ast \varphi,
\end{equation}

where $f = h + \bar{g}$ is harmonic and $\varphi$ is analytic in $D$. The next theorem investigates the properties of the product $\tilde{\ast}$.

**Theorem 2.12.** Suppose that $G \subset S$ and $G \triangleright G_0^H$. Let $O$ be a subfamily of $A$ such that $G$ is closed under convolution with members of $O$. Then $\varphi \tilde{\ast} f \in G_0^H$ for all $\varphi \in O$ and $f \in G^H$.

**Proof.** Let $f = h + \bar{g} \in G_0^H$ and $\varphi \in O$. Then

$$\varphi \tilde{\ast} f = \varphi \ast h + \bar{\varphi} \ast \bar{g} = H + \bar{G},$$

where $H = \varphi \ast h$ and $G = \varphi \ast g$ are analytic in $D$. Setting $F = H + \epsilon G = \varphi \ast (h + \epsilon g)$ where $|\epsilon| = 1$, we note that $F \in G$ since $G \ast O \subset G$. Thus $H + \bar{G} \in G_0^H$ as desired.

The next theorem indicates that the classes $G$ and $G_0^H$ have similar convex combination properties.

**Theorem 2.13.** Suppose that $G \subset S$ and $G \triangleright G_0^H$. Then $G$ is closed under convex combinations if and only if $G_0^H$ is closed under convex combinations.

**Proof.** Firstly we will prove the necessary part. For $n = 1, 2, \ldots$, suppose that $f_n \in G_0^H$ where $f_n = h_n + \bar{g}_n$. For $\sum_{n=1}^{\infty} t_n = 1$, $0 \leq t_n \leq 1$, the convex combination of $f_n$’s may be written as

$$f(z) = \sum_{n=1}^{\infty} t_n f_n(z) = h(z) + \bar{g}(z),$$

where

$$h(z) = \sum_{n=1}^{\infty} t_n h_n(z) \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} t_n g_n(z).$$

are analytic in $D$ with $h(0) = g(0) = h'(0) - 1 = g'(0) = 0$. For $|\epsilon| = 1$, we have

$$(h + \epsilon g)(z) = \sum_{n=1}^{\infty} t_n (h_n + \epsilon g_n)(z), \quad z \in D.$$

Since the class $G$ is closed under convex combination and $h_n + \epsilon g_n \in G$ for $n = 1, 2, \ldots$, it follows that $h + \epsilon g \in G$. Thus $f = h + \bar{g} \in G_0^H$. This proves the necessary part.

The sufficient part follows by using the fact that $G \subset G_0^H$ and applying Theorem 2.2(ii).

Theorem 2.13 immediately yields:

**Corollary 2.14.** Suppose that $G \subset S$ and $G \triangleright G_0^H$. Then $G$ is a convex set if and only if $G_0^H$ is a convex set.
Keeping in mind that $S \triangleright SS^0_H$, $S^* \triangleright SS^0_H$, $K \triangleright SK^0_H$ and $C \triangleright SC^0_H$, we determine the coefficient estimates, growth results, convolution properties and sharp bound for radius of starlikeness, convexity and close-to-convexity for the classes $SS^0_H$, $SS^0_H$, $SK^0_H$ and $SC^0_H$, using the results proved in this section. Note that parts (i) and (ii) of the following theorem have been independently proved in [15, Section 8].

**Theorem 2.15.** Let $f = h + g \in SS^0_H$ where $h$ and $g$ are given by (1).

(i) (Coefficient estimates) If $f \in SS^0_H, SS^0_H$ or $SC^0_H$, then the sharp inequality $|a_n| - |b_n| \leq n$ holds for $n = 2, 3, \ldots$ Equality occurs for the analytic Koebe function $k(z) = z/(1 - z)^2$. In case, $f \in SK^0_H$ then $|a_n| - |b_n| \leq 1$ for $n = 2, 3, \ldots$, with the equality occurring for the analytic half-plane mapping $l(z) = z/(1 - z)$.

(ii) (Growth estimates and covering theorem) If $f \in SS^0_H, SS^0_H$ or $SC^0_H$, then we have

$$|z| (1 + |z|)^2 \leq |f(z)| (1 - |z|)^2, \quad z \in \mathbb{D}.$$  

In particular, the range $f(\mathbb{D})$ contains the disk $|w| < 1/4$. These results are sharp for the analytic Koebe function $k$. If $f \in SK^0_H$, then

$$|z| (1 + |z|)^2 \leq |f(z)| (1 - |z|)^2, \quad z \in \mathbb{D},$$

and therefore the range $f(\mathbb{D})$ contains the disk $|w| < 1/2$. The analytic half-plane mapping $l$ shows that these results are best possible.

(iii) (Compactness) The classes $SS^0_H$, $SS^0_H$, $SK^0_H$ and $SC^0_H$ are compact with respect to the topology of locally uniform convergence.

(iv) (Radii of starlikeness, convexity and close-to-convexity) Let $r_S(G^0_H)$, $r_C(G^0_H)$ and $r_{CC}(G^0_H)$ denote the radius of starlikeness, convexity and close-to-convexity, respectively of a subclass $G^0_H \subset SS^0_H$. Then

$$r_S(SS^0_H) = r_S(SK^0_H) = r_C(SK^0_H) = r_{CC}(SK^0_H) = r_{CC}(SC^0_H) = 1;$$

$$r_C(SS^0_H) = r_C(SS^0_H) = r_C(SC^0_H) = 2 - \sqrt{3};$$

$$r_S(SS^0_H) = \tanh(\pi/4), \quad \text{and} \quad r_S(SC^0_H) = 4\sqrt{2} - 5.$$  

For $r_{CC}(SS^0_H)$, refer to [16].

(v) (Convolution properties)

(a) If $f \in SK^0_H$, then $f * f \in SK^0_H$.

(b) If $\varphi \in K$ and $f \in SS^0_H$ (resp. $f \in SK^0_H$ and $f \in SC^0_H$), then $f * \varphi \in SS^0_H$ (resp. $f * \varphi \in SK^0_H$ and $f * \varphi \in SC^0_H$).

(vi) If $f = h + g \in SK^0_H$, then

$$\frac{Re h(z)}{z} > \frac{1}{2} + \frac{|g(z)|}{z}$$

for all $z \in \mathbb{D}$. The analytic half-plane mapping $l$ shows that the constant $1/2$ is best possible.
Proof. Making use of the well-known coefficient estimates and distortion theorems for functions in the class $S$ (see [14]), parts (i) and (ii) follow by applying Theorems 2.6 and 2.7, respectively. Theorem 2.8 gives (iii), while (iv) follows by using [14, Chapter 13] and Theorem 2.9. Since $K \ast S^* \subset S^*$, $K \ast K \subset K$ and $K \ast C \subset C$, the convolution properties are easy to deduce from Theorems 2.10(i) and 2.12. For (vi), let $f = h + \bar{g} \in SK_0^H$. Then $h + \epsilon g \in K$ for each $|\epsilon| = 1$. By the well-known Marx Strohhäcker theorem [22, Theorem 2.6(a), p. 57], it follows that $\text{Re}(h + \epsilon g) (z)/z > 1/2$ for $z \in D$. By picking $\epsilon$ wisely, we obtain the desired result.

We close this section with the following remark.

**Remark 2.16.** It is clear that the classes $SS_0^H$, $SS_*^0$ and $SC_0^H$ are not closed under convolution. However, since $K \ast K \subset K$, $K \triangleright SK_0^H$ and $K$ is a non-convex set, it is expected that $SK_0^H$ is also not closed under convolution in view of Corollary 2.11. It will be an interesting open problem to determine whether $SK_0^H$ is closed under convolution.

### 3. Harmonic analogues of subclasses of $S$

In this section, we will determine the harmonic analogues of certain subclasses of $S$. Apart from results of Section 2, we will make use of the following two lemmas which are the generalization of Theorems 2.10 and 2.12. Their proof being similar are omitted.

**Lemma 3.1.** Let $I$ and $J$ be subfamilies of $S$ such that $I \ast I \subset J$. If $I_0^H$ and $J_0^H$ denote the harmonic analogues of $I$ and $J$, respectively, then

(a) If $f \in I_0^H$, then $f \ast f \in J_0^H$;
(b) If $(f + g)/2 \in J$ for all $f, g \in J$, then $I_0^H \ast I_0^H \subset J_0^H$.

**Lemma 3.2.** Suppose that $I$ and $J$ are subfamilies of $S$. Let $O \subset A$ be such that $f \ast g \in J$ for all $f \in I$ and $g \in O$. Then $\varphi \ast f \in J_0^H$ for all $\varphi \in O$ and $f \in I_0^H$, where $I \triangleright I_0^H$, $J \triangleright J_0^H$ and $\ast$ is defined by (10).

#### 3.1. Class $R$

Denote by $R$ the class consisting of functions $f \in A$ which satisfy $\text{Re} f'(z) > 0$ for $z \in D$. By well-known Noshiro-Warschawski Theorem (see [14, Chapter 7, p. 88]), $R \subset S$. In [19], MacGregor investigated the properties of functions in the class $R$. Also, it is easy to see that $R$ is a compact family and is closed under convex combinations. However, the class $R$ is not closed under convolutions. The analytic function

$$f(z) = -z - 2 \log(1 - z) = z + \sum_{n=2}^{\infty} \frac{2}{n} z^n$$

belongs to $R$ but $f \ast f \notin R$. The first theorem of this section determines the harmonic analogue of the class $R$ and discusses its properties.
The harmonic analogue of $\mathcal{R}$ is the class $\mathcal{R}_H^0$ defined by

$$\mathcal{R}_H^0 = \{ f = h + \tilde{g} \in \mathcal{H} : \Re h'(z) > |g'(z)| \text{ for all } z \in \mathbb{D} \}. $$

In particular, $\mathcal{R}_H^0 \subset \mathcal{S}C_0^\alpha$. Moreover, we have the following:

(i) If $f = h + \tilde{g} \in \mathcal{R}_H^0$ where $h$ and $g$ are given by (1), then $|a_n| + |b_n| \leq 2/n$ for $n = 2, 3, \ldots$. Equality holds for the function $f$ given by (11).

(ii) Every function $f \in \mathcal{R}_H^0$ satisfies

$$-|z| + \frac{2}{2 - 2\log(1 + |z|)} \leq \frac{|f(z)|}{|f'(z)|} \leq -|z| - 2\log(1 - |z|), \quad z \in \mathbb{D},$$

and hence the range of each function $f \in \mathcal{R}_H^0$ contains the disk $|w| < 2\log 2 - 1$. These results are sharp for the function $f$ given by (11).

(iii) The class $\mathcal{R}_H^0$ is compact with respect to the topology of locally uniform convergence.

(iv) $r_C(\mathcal{R}_H^0) = \sqrt{2} - 1$ and $r_{CC}(\mathcal{R}_H^0) = 1$.

(v) If $\varphi \in \mathcal{K}$ and $f \in \mathcal{R}_H^0$, then $f \ast \varphi \in \mathcal{R}_H^0$. Also, if $f \in \mathcal{A}$ with $\Re \varphi(z)/z > 1/2$ for $z \in \mathbb{D}$ and $f \in \mathcal{R}_H^0$, then $f \ast \varphi \in \mathcal{R}_H^0$.

(vi) The class $\mathcal{R}_H^0$ is closed under convex combinations of its members.

Proof. Suppose that $\mathcal{R} \triangleright \mathcal{G}_H^0$. If $f = h + \tilde{g} \in \mathcal{G}_H^0$, then the inequality $\Re(h'(z) + \epsilon g'(z)) > 0$ holds for each $z \in \mathbb{D}$ and $|\epsilon| = 1$. With appropriate choice of $\epsilon = \epsilon(z)$, it follows that

$$\Re h'(z) > |g'(z)|, \quad z \in \mathbb{D}$$

so that $f \in \mathcal{R}_H^0$. To prove the reverse inclusion, let $f = h + \tilde{g} \in \mathcal{R}_H^0$. Then for $|\epsilon| = 1$ we have

$$\Re(h'(z) + \epsilon g'(z)) \geq \Re h'(z) - |g'(z)| > 0, \quad z \in \mathbb{D}$$

which imply that $h + \epsilon g \in \mathcal{R}$ and hence $f \in \mathcal{G}_H^0$. This shows that $\mathcal{R} \triangleright \mathcal{R}_H^0$.

Since $\mathcal{R} \subset \mathcal{C}$, $\mathcal{R}_H^0 \subset \mathcal{S}C_0^\alpha$ by Theorem 2.2(iv). In view of [19, Theorems 1 and 2, p. 533], the proof of parts (i), (ii) and (iv) follow by applying Theorems 2.6, 2.7 and 2.9, respectively. Theorems 2.8 and 2.13 verify the validity of (iii) and (vi), respectively. Since $\mathcal{K} \ast \mathcal{R} \subset \mathcal{R}$ (by [2, Corollary 3.10]), Theorem 2.12 shows that $f \ast \varphi \in \mathcal{R}_H^0$ if $\varphi \in \mathcal{K}$ and $f \in \mathcal{R}_H^0$. For the proof of the other part of (v), it suffices to show that if $\varphi \in \mathcal{A}$ with $\Re \varphi(z)/z > 1/2$ and $f \in \mathcal{R}$, then $f \ast \varphi \in \mathcal{R}$. To see this, note that $(f \ast \varphi)'(z) = f'(z) \ast \varphi(z)/z$ for $z \in \mathbb{D}$. By [37, Lemma 4, p. 146], it follows that $\Re(f \ast \varphi)' > 0$ so that $f \ast \varphi \in \mathcal{R}$. This concludes the proof of the theorem. \qed

Note that Mocanu [23] independently proved that if $f$ is a harmonic mapping in a convex domain $\Omega$ such that $\Re f_z(z) > |f_z(z)|$ for $z \in \Omega$, then $f$ is univalent and sense-preserving in $\Omega$ while Ponnusamy et al. [30] showed that members of $\mathcal{R}_H^0$ are close-to-convex in $\mathbb{D}$.
Now we will determine the radius of convexity for a certain family of harmonic functions. For $G \in \mathcal{A}$, consider the family
\[ \mathcal{R}^0_H(G) = \left\{ f = h + \bar{g} \in \mathcal{H} : \text{Re} \frac{h'(z)}{G'(z)} > \text{Re} \frac{g'(z)}{G'(z)} \text{ for all } z \in \mathbb{D} \right\}. \]
If $G(z) = z$, then $\mathcal{R}^0_H(G)$ reduces to $\mathcal{R}^0_H$. In [29], it has been proved that if $G \in \mathcal{K}$, then $\mathcal{R}^0_H(G) \subset \mathcal{SC}^0_H$ (see also [23, 25]). The next theorem determines the radius of convexity of the class $\mathcal{R}^0_H(G)$ for specific choices of the function $G$.

**Theorem 3.4.** Let $r_C$ denote the radius of convexity of the class $\mathcal{R}^0_H(G)$ for $G \in \mathcal{A}$.

(i) If $G \in \mathcal{S}$, then $r_C = 3 - 2\sqrt{2}$;
(ii) If $G \in \mathcal{S}^*$, then $r_C = 3 - 2\sqrt{2}$;
(iii) If $G \in \mathcal{K}$, then $r_C = 2 - \sqrt{3}$;
(iv) If $G \in \mathcal{R}$, then $r_C = \sqrt{3} - 2$;
(v) If $G \in \mathcal{A}$ with $\text{Re} G'(z) > 1/2$, then $r_C = 3 - 2\sqrt{2}$.
Moreover, all these results are sharp.

**Proof.** Let $f = h + \bar{g} \in \mathcal{R}^0_H(G)$. Setting $F_\epsilon = h + \epsilon g$ for $|\epsilon| = 1$, note that
\[ \text{Re} \frac{F_\epsilon'(z)}{G'(z)} = \text{Re} \left( \frac{h'(z)}{G'(z)} + \epsilon \frac{g'(z)}{G'(z)} \right) \geq \text{Re} \frac{h'(z)}{G'(z)} - \text{Re} \frac{g'(z)}{G'(z)} > 0, \quad z \in \mathbb{D}. \]
If $G \in \mathcal{S}$, then $F_\epsilon$ is convex in $|z| < 3 - 2\sqrt{2}$ by [31, Theorem 1, p. 32] for each $|\epsilon| = 1$. By Lemma 2.1, $f$ is convex in $|z| < 3 - 2\sqrt{2}$. This proves (i). The proof of the other parts is similar. □

### 3.2. Class $\mathcal{W}$

In [5], Chichra introduced the class $\mathcal{W}$ of analytic functions $f \in \mathcal{A}$ which satisfy $\text{Re}(f'(z) + z f''(z)) > 0$ for $z \in \mathbb{D}$. He proved that the members of $\mathcal{W}$ are univalent in $\mathbb{D}$ by showing that $\mathcal{W} \subset \mathcal{R}$. Later Singh and Singh [36] proved that $\mathcal{W} \subset \mathcal{S}^*$. The class $\mathcal{W}$ is compact and is closed under convex combination of its members. Similar to the proof of Theorem 3.3, it can be shown that the set
\[ \mathcal{W}_H^0 = \left\{ f = h + \bar{g} \in \mathcal{H} : \text{Re}(h'(z) + z h''(z)) > |g'(z) + z g''(z)| \text{ for all } z \in \mathbb{D} \right\}. \]
is the harmonic analogue of $\mathcal{W}$. By Theorem 2.2(iv), $\mathcal{W}_H^0 \subset \mathcal{R}_H^0 \cap \text{SS}_H^0$. In particular, the members of $\mathcal{W}_H^0$ are fully starlike in $\mathbb{D}$ by Corollary 2.4. To determine the coefficient and growth estimates for functions in the class $\mathcal{W}_H^0$, we need to prove the following simple lemma.

**Lemma 3.5.** If $f \in \mathcal{W}$ is given by (9), then $|a_n| \leq 2/n^2$ for $n = 2, 3, \ldots$ and
\[ -1 + \frac{2}{|z|} \log(1 + |z|) \leq |f'(z)| \leq -1 - \frac{2}{|z|} \log(1 - |z|), \quad z \in \mathbb{D}. \]
The function

$$f(z) = -z - 2 \int_0^{|z|} \frac{1}{t} \log(1 - t) \, dt = z + \sum_{n=2}^{\infty} \frac{2}{n^2} z^n$$

shows that all these results are sharp.

**Proof.** Observe that \(f \in W\) if and only if \(zf' \in R\). The proof now follows by applying \([19, \text{Theorem 1, p. 533}]\).

**Theorem 3.6.** Let \(f = h + \bar{g} \in W_0^H\) where \(h\) and \(g\) are given by (1). Then

\[-|z| + 2 \int_0^{|z|} \frac{1}{t} \log(1 + t) \, dt \leq |f(z)| \leq -|z| - 2 \int_0^{|z|} \frac{1}{t} \log(1 - t) \, dt, \quad z \in \mathbb{D}.

In particular, the range \(f(\mathbb{D})\) contains the disk \(|w| < \pi^2/6 - 1\). All these results are sharp for the function \(f\) given by (12). Moreover, the following statements regarding the class \(W_0^H\) hold.

(i) The class \(W_0^H\) is compact with respect to the topology of locally uniform convergence.

(ii) \(r_S(W_0^H) = 1 = r_{CC}(W_0^H)\).

(iii) The class \(W_0^H\) is closed under convolutions.

(iv) (a) If \(\varphi \in K\) and \(f \in W_0^H\), then \(\varphi \ast f \in W_0^H\);

(b) If \(\varphi \in A\) with \(\Re \varphi(z)/z > 1/2\) and \(f \in W_0^H\), then \(\varphi \ast f \in W_0^H\);

(c) If \(\varphi \in W\) and \(f \in W_0^H\), then \(\varphi \ast f \in W_0^H \cap SK_0^H\).

(v) The class \(W_0^H\) is closed under convex combinations.

(vi) If \(f = h + \bar{g} \in W_0^H\), then

\[
\Re h'(z) > -1 + 2 \log 2 + |g'(z)|
\]

for all \(z \in \mathbb{D}\). The function \(f\) given by (12) shows that the constant \(-1 + 2 \log 2\) cannot be replaced by any larger one.

**Proof.** The growth and coefficient estimates for the class \(W_0^H\) follow by Lemma 3.5. Since \(W\) is a convex set and closed under convolutions (see \([37, \text{Theorem 3}, p. 150]\)), the class \(W_0^H\) is closed under convolutions by Corollary 2.11. This proves (iii). Since \(W_0^H \subset SS_0^H\), (ii) is obviously true. To prove (iv), note that \(K \ast W \subset W\) (by \([2, \text{Corollary 3.10}]\)) and if \(\varphi \in A\) with \(\Re \varphi(z)/z > 1/2\) and \(f \in W\), then \(f \ast \varphi \in W\) (by \([37, \text{Theorem 3'}, p. 150]\)). These observations lead to (a) and (b) by applying Theorem 2.12. Since \(W \ast W \subset W \cap K\) (by \([37, \text{Theorems 3 and 4}]\)), part (c) follows by Lemma 3.2. Theorems 2.8 and 2.13 verify the validity of the parts (i) and (v), respectively. For the proof of (vi), let \(f = h + \bar{g} \in W_0^H\). Then \(h + \epsilon g \in W\) for each \(|\epsilon| = 1\). Consequently, \(\Re(h + \epsilon g)' > -1 + 2 \log 2\) in \(\mathbb{D}\) by \([37, \text{Theorem 1(a)}, p. 146]\). In particular, we obtain the required result. \(\square\)
Remark 3.7. Since \( \mathcal{W} \ast \mathcal{W} \subset \mathcal{K} \), the convolution of each member of \( \mathcal{W}_H^0 \) with itself is convex in \( \mathbb{D} \) by Lemma 3.1(a). However, since \( \mathcal{K} \) is a non-convex set, it is not known whether \( \mathcal{W}_H^0 \ast \mathcal{W}_H^0 \subset S\mathcal{K}_H^0 \) in view of Lemma 3.1(b).

3.3. Classes \( \mathcal{U} \) and \( \mathcal{V} \)

Let \( \mathcal{U} \) and \( \mathcal{V} \) be subclasses of \( \mathcal{A} \) consisting of functions \( f \) of the form (9) that satisfy

\[
\sum_{n=2}^{\infty} n|a_n| \leq 1 \quad \text{and} \quad \sum_{n=2}^{\infty} n^2|a_n| \leq 1,
\]

respectively. Clearly \( \mathcal{V} \subset \mathcal{U} \). In [13], Goodman proved that \( \mathcal{U} \subset S^* \) and \( \mathcal{V} \subset \mathcal{K} \).

It is easy to see that \( \mathcal{U} \subset \mathcal{R} \) and \( \mathcal{V} \subset \mathcal{W} \). In fact, if \( f \in \mathcal{U} \) is given by (9), then

\[
\text{Re} f'(z) = 1 + \text{Re} \sum_{n=2}^{\infty} na_n z^{n-1} > 1 - \sum_{n=2}^{\infty} n|a_n| > 0.
\]

Similarly, if \( f \in \mathcal{V} \) is given by (9), then

\[
\text{Re}(f'(z) + zf''(z)) = 1 + \text{Re} \sum_{n=2}^{\infty} n^2 a_n z^{n-1} > 1 - \sum_{n=2}^{\infty} n^2|a_n| > 0.
\]

The next theorem determines the harmonic analogue of the classes \( \mathcal{U} \) and \( \mathcal{V} \).

**Theorem 3.8.** The harmonic analogues of the classes \( \mathcal{U} \) and \( \mathcal{V} \) are given by

\[
\mathcal{U}_0^H = \left\{ f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{H} : \sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq 1 \right\}
\]

and

\[
\mathcal{V}_0^H = \left\{ f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{H} : \sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) \leq 1 \right\},
\]

respectively.

**Proof.** Suppose that \( \mathcal{U} \triangleright \mathcal{G}_H^0 \). If \( f = h + \bar{g} \in \mathcal{G}_H^0 \) where \( h \) and \( g \) are given by (1), then \( h + \epsilon g \in \mathcal{U} \) for each \( |\epsilon| = 1 \) so that

\[
\sum_{n=2}^{\infty} n|a_n + \epsilon b_n| \leq 1.
\]

On choosing \( \epsilon = \epsilon(n) \) wisely we deduce that \( f \in \mathcal{U}_0^H \). Conversely if \( f = h + \bar{g} \in \mathcal{U}_0^H \) where \( h \) and \( g \) are given by (1), then for \( |\epsilon| = 1 \) we have

\[
\sum_{n=2}^{\infty} n|a_n + \epsilon b_n| \leq \sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq 1
\]

so that \( h + \epsilon g \in \mathcal{U} \) and hence \( f \in \mathcal{G}_H^0 \). Thus \( \mathcal{U} \triangleright \mathcal{U}_0^H \). Similarly it can be shown that \( \mathcal{V} \triangleright \mathcal{V}_0^H \). \( \square \)
In view of Theorem 2.2(iv), $U^0_H \subset SS^0_H$ and $V^0_H \subset SK^0_H$. In particular, the members of $U^0_H$ (resp. $V^0_H$) are fully starlike (resp. fully convex) in $D$ by Corollary 2.4. Also since $U \subset R$ and $V \subset W$, therefore $U^0_H \subset R^0_H$ (see also [30, Corollary 1.4, p. 25]) and $V^0_H \subset W^0_H$. Using the results of [34] and applying the theorems of Section 2, we have

**Corollary 3.9.** Let $f = h + \bar{g} \in S^0_H$ where $h$ and $g$ are given by (1).

(a) If $f \in U^0_H$, then

$$|a_n| \leq 1/n, \quad |b_n| \leq 1/n \quad \text{and} \quad |a_n| - |b_n| \leq 1/n \quad \text{for} \quad n = 2, 3, \ldots.$$ 

Equality occurs for the functions $z + z^n/n$ and $z + \bar{z}^n/n$. If $f \in V^0_H$, then the sharp inequalities

$$|a_n| \leq 1/n^2, \quad |b_n| \leq 1/n^2 \quad \text{and} \quad |a_n| - |b_n| \leq 1/n^2 \quad \text{for} \quad n = 2, 3, \ldots.$$ 

hold with the equality occurring for the functions $z + z^n/n^2$ and $z + \bar{z}^n/n^2$.

(b) If $f \in U^0_H$, then

$$|z| - \frac{1}{2}|z|^2 \leq |f(z)| \leq |z| + \frac{1}{2}|z|^2, \quad z \in D.$$ 

In particular, the range $f(\mathbb{D})$ contains the disc $|w| < 1/2$. If $f \in V^0_H$, then

$$|z| - \frac{1}{4}|z|^2 \leq |f(z)| \leq |z| + \frac{1}{4}|z|^2, \quad z \in D,$$

and therefore $f(\mathbb{D})$ contains the disk $|w| < 3/4$.

(c) The classes $U^0_H$ and $V^0_H$ are compact with respect to the topology of locally uniform convergence.

(d) $r_s(U^0_H) = r_{CC}(U^0_H) = r_s(V^0_H) = r_{CC}(V^0_H) = 1$ and $r_C(U^0_H) = 1/2$.

(e) The classes $U^0_H$ and $V^0_H$ are closed under convex combinations.

Avci and Zlotkiewicz [3] investigated certain properties of the classes $U^0_H$ and $V^0_H$ (see also [35]). The next theorem investigates the convolution properties of the classes $U^0_H$ and $V^0_H$.

**Theorem 3.10.** The classes $U^0_H$ and $V^0_H$ are closed under convolutions. Moreover, we have

(i) $U^0_H \ast U^0_H \subset SK^0_H$;

(ii) If $\varphi \in K$ and $f \in U^0_H$, then $\varphi \ast f \in U^0_H$;

(iii) If $\varphi \in K$ and $f \in V^0_H$, then $\varphi \ast f \in V^0_H$.

**Proof.** The main crux of the proof relies on the observation that if $f \in V$ is given by (9), then $\sum_{n=2}^{\infty} n^2|a_n|^2 \leq 1$. Since $U$ and $V$ are convex sets, therefore it suffices to show that the classes $U$ and $V$ are closed under convolution in view of Corollary 2.11. Let $f, F \in V$ be given by (9). Then

$$\sum_{n=2}^{\infty} n^2|a_nA_n| \leq \frac{1}{2} \sum_{n=2}^{\infty} n^2|a_n|^2 + \frac{1}{2} \sum_{n=2}^{\infty} n^2|A_n|^2 \leq 1$$
using the fact that the geometric mean is less than or equal to the arithmetic mean. This shows that \( f \ast F \in V \). The same calculation shows that if \( f, F \in U \), then \( f + F \in V \subseteq U \).

The proof of part (i) follows by Lemma 3.1(b) since \( U \ast U \subseteq V, V \) is a convex set and \( V \subseteq \mathcal{K} \). Since the classes \( U \) and \( V \) are closed under convolution with convex functions, (ii) and (iii) follows immediately from Theorem 2.12.

\[ \square \]

3.4. Class \( S_R \)

Let \( S_R \) be the subclass of \( S \) consisting of functions \( f \) of the form (9) whose coefficients \( a_n \) are all real. The following theorem determines its harmonic analogue.

**Theorem 3.11.** The harmonic analogue of \( S_R \) is itself.

**Proof.** Suppose that \( S_R \triangleright G^0_H \). Then \( S_R \subseteq G^0_H \). To prove the reverse inclusion, let \( f = h + \bar{g} \in G^0_H \) where \( h \) and \( g \) are given by (1). Then \( h + \epsilon g \in S_R \) for each \( |\epsilon| = 1 \) which imply that all the coefficients \( a_n + \epsilon b_n \) are real for each \( |\epsilon| = 1 \). But this is possible only if \( a_n \) are real and \( b_n = 0 \) for \( n = 2, 3, \ldots \). Thus \( g \equiv 0 \) and \( f \in S_R \). Hence \( S_R \triangleright S_R \). \[ \square \]

4. Harmonic integral operators

In the theory of analytic univalent functions, Alexander operator \( \Lambda \) given by (2) and Libera operator \( \Theta \) defined by (3) play a crucial role. In this section, we will introduce and investigate the properties of harmonic Alexander operator and harmonic Libera operator.

4.1. Harmonic Alexander operator

**Definition 4.1.** Define an integral operator \( \Lambda^+_H : \mathcal{H} \to \mathcal{H} \) by

\[
\Lambda^+_H[f] = \Lambda[h] + \Lambda[\bar{g}], \quad f = h + \bar{g} \in \mathcal{H},
\]

where \( \Lambda \) is the Alexander operator defined by (2). We call \( \Lambda^+_H \) the positive harmonic Alexander operator.

Since \( \Lambda \) is linear, therefore so is the operator \( \Lambda^+_H \), that is, \( \Lambda^+_H[f_1 + f_2] = \Lambda^+_H[f_1] + \Lambda^+_H[f_2] \) for all \( f_1, f_2 \in \mathcal{H} \). The first theorem shows that if a subfamily \( \mathcal{G} \subseteq \mathcal{S} \) is preserved under \( \Lambda \), then its harmonic analogue \( \mathcal{G}^0_H \) is preserved under \( \Lambda^+_H \).

**Theorem 4.2.** Let \( I \) and \( J \) be subfamilies of \( \mathcal{S} \) such that \( \Lambda[I] \subseteq J \). Then \( \Lambda^+_H[I^0] \subseteq J^0_H \) where \( I \triangleright I^0_H \) and \( J \triangleright J^0_H \).

**Proof.** Let \( f = h + \bar{g} \in I^0_H \). Since \( \Lambda^+_H[f] = \Lambda[h] + \Lambda[\bar{g}] \) and \( J \triangleright J^0_H \), it suffices to show that \( \Lambda[h] + \epsilon \Lambda[g] \in J \) for each \( |\epsilon| = 1 \). But \( \Lambda[h] + \epsilon \Lambda[g] = \Lambda[h + \epsilon g] \in \Lambda[I] \subseteq J \) since \( I \triangleright I^0_H \). \[ \square \]

Note that \( \mathcal{R}^0_H \not\subseteq \mathcal{S}^0_H \) and \( \mathcal{U}^0_H \not\subseteq \mathcal{K}^0_H \). Since \( \Lambda[\mathcal{R}] \subseteq \mathcal{W} \subseteq \mathcal{S}^* \) and \( \Lambda[\mathcal{U}] \subseteq \mathcal{V} \subseteq \mathcal{K} \). Theorem 4.2 gives the following two corollaries.
Corollary 4.3. \( \Lambda_H^+[R_H^0] \subset SS_H^0 \) and \( \Lambda_H^+[U_H^0] \subset SK_H^0 \).

Corollary 4.4. The classes \( R_H^0, U_H^0, W_H^0, V_H^0 \) are preserved under \( \Lambda_H^+ \).

The Alexander operator \( \Lambda \) provides a one-to-one correspondence between the classes \( S^{∗} \) and \( K \): \( f \in S^{∗} \) if and only if \( \Lambda[f] \in K \). A similar result holds for the positive harmonic Alexander operator which provides a one-to-one correspondence between the classes \( SS_H^{∗0} \) and \( SK_H^{0} \): \( f \in SS_H^{∗0} \) if and only if \( \Lambda_H^+[f] \in SK_H^{0} \). In particular, the classes \( SS_H^{∗0} \) and \( SK_H^{0} \) are preserved under \( \Lambda_H^+ \). In fact, the class \( SC_H^{0} \) is also preserved under \( \Lambda_H^+ \) since \( \Lambda[C] \subset C \), a result proved by Merkes and Wright [21].

Gao [11] proved that if \( f \in R \), then \( \text{Re}(\Lambda[f](z)/z) > (\pi^2/6) - 1 \approx 0.6449 \) \( (z \in D) \) and the function \( f \) given by (11) shows that the constant \( (\pi^2/6) - 1 \) cannot be replaced by any larger one. He also showed that if \( f \in A \) and \( \text{Re}f'(z) > (6 - \pi^2)/(24 - \pi^2) \approx -0.2738 \), then \( \Lambda[f] \in S^{∗} \). These results are generalized in context of positive harmonic Alexander operator.

Theorem 4.5. Let \( f = h + \overline{g} \in H \).

(i) If \( f \in R_H^0 \), then
\[
\text{Re} \left( \frac{\Lambda[b](z)}{z} \right) > \frac{|\Lambda[g](z)|}{|z|} + \frac{\pi^2}{6} - 1 \quad \text{for all } z \in D.
\]

(ii) If \( \text{Re} h'(z) > |g'(z)| + (6 - \pi^2)/(24 - \pi^2) \) \( \text{for all } z \in D \), then \( \Lambda_H^+[f] \in SS_H^{∗0} \).
Proof. Since $\mathcal{R} \supset \mathcal{R}_H^0$, it follows that $h + \epsilon g \in \mathcal{R}$ for each $|\epsilon| = 1$. Consequently
\[
\text{Re} \left( \frac{\Lambda[h](z)}{z} + \epsilon \frac{\Lambda[g](z)}{z} \right) = \text{Re} \frac{\Lambda[h + \epsilon g](z)}{z} > \frac{\pi^2}{6} - 1
\]
for each $z \in \mathbb{D}$ and $|\epsilon| = 1$. With appropriate choice of $\epsilon = \epsilon(z)$, we obtain (i).

For the proof of (ii), it is easy to see that $(h + \epsilon g)' > (6 - \pi^2)/(24 - \pi^2)$ in $\mathbb{D}$ for each $|\epsilon| = 1$. Hence $\Lambda[h + \epsilon g] \in \mathcal{S}^*$, or equivalently $\Lambda_H[f] \in \mathcal{S}^*_H$. □

![Figure 2. Graph of the function $\Lambda_H^+[L]$.](image)

As discussed earlier, we have the inclusion $\Lambda_H^+[\mathcal{S}^*_H] \subset \mathcal{S}K^0_H$. However, the inclusion $\Lambda_H^+[\mathcal{S}^*_H] \subset \mathcal{K}^0_H$ is not valid. To see this, note that the harmonic Koebe function $K$ given by (4) belongs to $\mathcal{S}^*_H$ and
\[
\Lambda_H^+[K](z) = \frac{1}{6} \left[ \frac{z(5 - 3z)}{(1 - z)^2} - \log(1 - z) \right] + \frac{1}{6} \left[ \frac{z(3z - 1)}{(1 - z)^2} - \log(1 - z) \right] = \frac{2}{3} \frac{z}{(1 - z)^2} + \frac{1}{3} \text{Im} \left( \frac{z - 3z^2}{1 - z} \right) - \frac{1}{3} \log |1 - z|, \quad z \in \mathbb{D}.
\]
The graph of the function $\Lambda_H^+[K]$ (see Figure 1) shows that the image domain is not even starlike. In particular, $\Lambda_H^+[\mathcal{S}^*_H] \not\subset \mathcal{S}^*_H$. Similarly, it can be shown that $\Lambda_H^+[\mathcal{K}^0_H] \not\subset \mathcal{K}^0_H$ by considering the harmonic half-plane mapping $L$ given by (5). Note that
\[
\Lambda_H^+[L](z) = \frac{1}{2} \left[ -\log(1 - z) + \frac{z}{1 - z} \right] + \frac{1}{2} \left[ -\log(1 - z) - \frac{z}{1 - z} \right] = -\log |1 - z| + i \text{Im} \left( \frac{z}{1 - z} \right).
\]
Clearly Figure 2 depicts that the image domain $\Lambda^+_H[L](D)$ is not convex.

Although the members of $\Lambda^+_H[K^0_H]$ need not map $D$ onto a convex domain, its members are necessarily univalent and close-to-convex in $D$ as seen by the following theorem.

**Theorem 4.6.** $\Lambda^+_H[K^0_H] \subset SC^0_H$.

**Proof.** Let $f = h + \bar{\epsilon}g \in K^0_H$. Then $h + \epsilon g \in \mathcal{C}$ for each $|\epsilon| = 1$ by [7, Theorem 5.7, p. 15]. Consequently $\Lambda[h] + \epsilon \Lambda[g] = \Lambda[h + \epsilon g] \in \mathcal{C}$ for each $|\epsilon| = 1$, as $\Lambda[\mathcal{C}] \subset \mathcal{C}$. Since $\mathcal{C} \triangleright SC^0_H$, we have $\Lambda^+_H[f] \in SC^0_H$. \hfill $\Box$

By Theorem 4.6, $\Lambda^+_H[L]$ is univalent and maps $D$ onto a close-to-convex domain. Using the technique of shear construction [7, Theorem 5.3, p. 14] and convolution of harmonic mappings, the authors [28] have further investigated certain properties of positive harmonic Alexander operator.

The failure of the implication $\Lambda^+_H[S^0_H] \subset K^0_H$ motivates to introduce the following definition.

**Definition 4.7.** Define another integral operator $\Lambda^-_H : \mathcal{H} \to \mathcal{H}$ by

$$\Lambda^-_H[f] = \Lambda[h] - \Lambda[g], \quad f = h + \bar{\epsilon}g \in \mathcal{H},$$

where $\Lambda$ is given by (2). We call $\Lambda^-_H$ the negative harmonic Alexander operator.

By [10, Lemma, p. 108], it follows that $\Lambda^-_H[S^0_H] \subset K^0_H$. In particular, the classes $S^0_H$ and $K^0_H$ are preserved under the operator $\Lambda^-_H$. Therefore the mappings

$$\Lambda^-_H[K](z) = \frac{2}{3(1-z)^2} + \frac{1}{3} \frac{z-3z^2}{(1-z)^2} - \frac{1}{3} i \arg(1-z)$$

and

$$\Lambda^-_H[L](z) = \text{Re} \left( \frac{z}{1-z} \right) - i \arg(1-z)$$

belong to $K^0_H$, where $K$ and $L$ are given by (4) and (5), respectively (see Figure 3).

It is worth to remark that Theorems 4.2, 4.5 and 4.6 continue to hold for the negative harmonic Alexander operator $\Lambda^-_H$.

### 4.2. Harmonic Libera operator

Similar to Definition 4.1, we introduce the notion of harmonic Libera operator as follows.

**Definition 4.8.** Define an integral operator $\Theta_H : \mathcal{H} \to \mathcal{H}$ by

$$\Theta_H[f] = \Theta[h] + \Theta[g], \quad f = h + \bar{\epsilon}g \in \mathcal{H},$$

where $\Theta$ is the Libera operator defined by (3). We call $\Theta_H$ the harmonic Libera operator.
The linearity of the operator $\Theta_H$ and the inclusions $\Theta[S^*] \subset S^*$, $\Theta[K] \subset K$, $\Theta[C] \subset C$ (see [18]) show that Theorems 4.2 and 4.6 hold for harmonic Libera operator $\Theta_H$ as well. Thus we obtain the following theorem.

**Theorem 4.9.** Let $\mathcal{I}$ and $\mathcal{J}$ be subfamilies of $\mathcal{S}$ such that $\Theta[\mathcal{I}] \subset \mathcal{J}$. Then $\Theta_H[\mathcal{I}^0_H] \subset \mathcal{J}^0_H$ where $\mathcal{I}^0_H$ and $\mathcal{J}^0_H$ are harmonic analogues of $\mathcal{I}$ and $\mathcal{J}$, respectively. In particular, the classes $SS^*_H$, $SK^*_H$ and $SC^*_H$ are preserved under $\Theta_H$. Moreover, $\Theta_H[K^0_H] \subset SC^*_H$.

Mocanu [24] proved that $\Theta[R] \subset S^*$. Therefore, by Theorem 4.9, we have $\Theta_H[R^0_H] \subset SS^*_H$. Unlike positive harmonic Alexander operator (Corollary 4.3), the inclusion $\Theta_H[U^0_H] \subset SK^*_H$ is not valid in general. This can be seen by considering the function $f_0(z) = z + \bar{z}^2/2 \in U^0_H$. Note that $\Theta_H[f_0](z) = z + \bar{z}^2/3$ and the analytic function $z + \bar{z}^2/3 \not\in K$.

The classes $R^0_H$, $W^0_H$, $U^0_H$ and $V^0_H$ are also preserved under $\Theta_H$. This can be seen directly from Theorem 4.9 or by observing that we can write $\Theta_H[f] = f\tilde{*}\phi$ where $\tilde{*}$ is defined by (10), $\phi \in K$ is given by

\[\phi(z) = z + \sum_{n=2}^{\infty} \frac{2}{n+1} z^n = -2 - \frac{2}{z} \log(1-z), \quad z \in \mathbb{D}\]

and using the convolution results of these classes stated in Section 3. From the inclusion $\Theta_H[K^0_H] \subset SC^*_H$, it follows that the mapping

\[\Theta_H[L](z) = \frac{z}{1-z} + \frac{z - 2}{1-z} \frac{2}{z} \log(1-z)\]
\[ = 2 \text{Re} \left( \frac{z}{1-z} \right) - 2 \left( \frac{1}{1-z} + \frac{1}{z} \log(1-z) \right) \]

belongs to \( SC_0^\mathcal{H} \), where \( L \in \mathcal{K}_0^\mathcal{H} \) is given by (5) (see Figure 4). However, \( \Theta_H[L] \not\in \mathcal{K}_0^\mathcal{H} \), which shows that \( \Theta_H[\mathcal{K}_0^\mathcal{H}] \not\subset \mathcal{K}_0^\mathcal{H} \).

Recall that a function \( f \in \mathcal{H} \) is convex in the direction of the real (resp. imaginary) axis if the intersection of the image domain \( f(\mathbb{D}) \) with each horizontal (resp. vertical) line is connected. For further investigation of results regarding harmonic Libera operator, we need to prove the following theorem.

**Theorem 4.10.** Let \( f = h + \bar{g} \in \mathcal{H} \) with \( h(z) + g(z) = z/(1-z) \) and \( \psi \in \mathcal{K} \). If

\[ \text{Re}(1-z)^2 h'(z) > 1/2 \quad \text{for} \quad z \in \mathbb{D}, \]

then \( f \ast \psi \in SC_0^\mathcal{H} \) and is convex in the direction of the imaginary axis, where \( \ast \) is defined by (10).

**Proof.** To apply [28, Lemma 1.1] to the function \( f \ast \psi \), we need to show that \( f \ast \psi \) is sense-preserving and \( h \ast \psi + g \ast \psi \) is univalent and convex in the direction of imaginary axis. Since \( h \ast \psi + g \ast \psi = (h + g) \ast \psi = z/(1-z) \ast \psi = \psi \in \mathcal{K} \), it only remains to show that the dilatation \( w_{f \ast \psi} = (g \ast \psi)'/(h \ast \psi)' \) of \( f \ast \psi \) satisfies \( |w_{f \ast \psi}| < 1 \) or equivalently \( \text{Re}(1 - w_{f \ast \psi})/(1 + w_{f \ast \psi}) > 0 \) in \( \mathbb{D} \). Using the identity \( \psi = h \ast \psi + g \ast \psi \), it is easy to deduce that

\[ \text{Re} \left( \frac{1-w_{f \ast \psi}}{1+w_{f \ast \psi}} \right) = \text{Re} \left( \frac{(h \ast \psi)' - (g \ast \psi)'}{(h \ast \psi)' + (g \ast \psi)'} \right) = 2 \text{Re} \left( \frac{(h \ast \psi)'}{\psi'} \right) - 1. \]
Since we can write
\[
\frac{\psi}{\psi'} = \frac{\psi \cdot \frac{z}{(1-z)^2}}{\psi' \cdot \frac{z}{1-z}} \left[(1-z)^2 h'(z)\right],
\]
where \(\psi \in \mathcal{K}, z/(1-z)^2 \in \mathcal{S}^*\) and \(\text{Re}(1-z)^2 h'(z) > 1/2\), it follows that \(\text{Re}((h * \psi)' / \psi') > 1/2\) for all \(z \in \mathbb{D}\) by [32, Theorem 2.4, p. 54]. Hence (14) shows that the expression \(\text{Re}(1 - w_{f \ast \psi})/(1 + w_{f \ast \psi})\) is strictly positive in \(\mathbb{D}\). \(\square\)

Since \(\Theta_H[f] = f \tilde{\phi}\) where \(\phi \in \mathcal{K}\) is given by (13), Theorem 4.10 gives the following corollary.

**Corollary 4.11.** Let \(f = h + \bar{g} \in \mathcal{H}\) with \(h(z) + g(z) = z/(1-z)\) and \(\text{Re}(1-z)^2 h'(z) > 1/2\) for \(z \in \mathbb{D}\). Then \(\Theta_H[f] \in \mathcal{S}^0_H\) and is convex in the direction of the imaginary axis.

The harmonic half-plane mapping \(L = M + \bar{N}\) given by (5) satisfies \(M(z) + N(z) = z/(1-z)\) and \(\text{Re}(1-z)^2 M'(z) = \text{Re}(1/(1-z)) > 1/2\), so \(\Theta_H[L] \in \mathcal{S}^0_H\) and is convex in the direction of the imaginary axis (which is clearly evident from Figure 4) by Corollary 4.11. We give another example illustrating Corollary 4.11.

**Example 4.12.** Consider the harmonic function \(\Psi(z) = \psi_1 + \psi_2 \in \mathcal{H}\) where
\[
\psi_1(z) = \frac{1}{4} \log \left(\frac{1+z}{1-z}\right) + \frac{1}{2} \frac{z}{1-z}, \quad \text{and} \quad \psi_2(z) = -\frac{1}{4} \log \left(\frac{1+z}{1-z}\right) + \frac{1}{2} \frac{z}{1-z}.
\]
In fact, \(\Psi \in \mathcal{K}^0_H\) and is constructed by shearing the conformal mapping \(l(z) = z/(1-z)\) in the direction of imaginary axis with dilatation \(w_{\Psi}(z) = z\). Note that \(\text{Re}(1-z)^2 \psi_1'(z) = \text{Re}(1/(1+z)) > 1/2\) for \(z \in \mathbb{D}\). Hence, by Corollary 4.11, \(\Theta_H[\Psi] = \Theta[\psi_1] + \Theta[\psi_2] \in \mathcal{S}^0_H\) and is convex in the direction of the imaginary axis, where
\[
\Theta[\psi_1](z) = \frac{1}{2} \log \left(\frac{1+z}{1-z}\right) + \frac{1}{2z} \log(1-z^2) - 1 - \frac{1}{z} \log(1-z)
\]
and
\[
\Theta[\psi_2](z) = -\frac{1}{2} \log \left(\frac{1+z}{1-z}\right) - \frac{1}{2z} \log(1-z^2) - 1 - \frac{1}{z} \log(1-z).
\]
The images of radial segments and concentric circles inside \(\mathbb{D}\) under \(\Psi\) and \(\Theta_H[\Psi]\) are shown in Figure 5.

**Example 4.13.** If \(K = H + \bar{G} \in \mathcal{S}^0_H\) is the harmonic Koebe function given by (4), then
\[
\Theta_H[K] = \Theta[H] + \Theta[G] = \frac{2}{3} \text{Re} \left(\frac{3z - 1}{z(1-z)^2} + \frac{1}{z} - 1\right) - \frac{2}{z} \left(\frac{z}{1-z} + \log(1-z)\right).
\]
Figure 5. Images of the functions Ψ and Θ_H[Ψ].

Figure 6. Graph of the function Θ_H[K].

Figure 6 clearly depicts that the image domain Θ_H[K](𝔻) is not starlike. This shows that Θ_H[𝒮_H^0] ∉ 𝒮_H^0.

However, it can be shown that Θ_H[K] ∈ 𝒮_H^0 and is convex in the direction of the real axis. To see this, note that Θ_H[K] = K_φ where φ ∈ K is given by (13). Since \( H - G = z/(1 - z)^2 \) and \( zφ' \in 𝒮^* \) is convex in the direction of real axis (see Figure 7(A)), it follows that \( Hφ - Gφ = zφ' \) is univalent and convex in the direction of real axis. Moreover, the dilatation \( w_Kφ = (Gφ')/(Hφ') \) of \( Kφ \) satisfies

\[
\text{Re} \left( \frac{1 + w_Kφ}{1 - w_Kφ} \right) = 2 \text{Re} \left( \frac{(Hφ')'}{(zφ')'} \right) - 1 = 2 \text{Re} \left( \frac{φH + k}{φk + k} \right) - 1
\]

which is clearly positive in \( 𝔻 \) (the dashed line in Figure 7(B) represents the line \( \text{Re} z = 1/2 \)), where \( k(z) = z/(1 - z)^2 \) is the Koebe function. By [28, Lemma 1.1], Θ_H[K] = Kφ ∈ 𝒮_H^0 and is convex in the direction of the real axis.

We close this section with the following remark.
Figure 7. Mapping properties of the function $\phi$.

Remark 4.14. As discussed earlier, the classes $S^*_0 H$ and $K^0_H$ are not preserved under $\Theta_H$. Analogous to Definition 4.7, if we define another notion of harmonic Libera operator $\tilde{\Theta}_H : H \to H$ by $\tilde{\Theta}_H[f] = \Theta[h] - \Theta[g]$ where $f = h + \bar{g} \in H$ and $\Theta$ is the Libera operator defined by (3), then also $\tilde{\Theta}_H[S^*_0 H] \not\subset S^*_0 H$ and $\tilde{\Theta}_H[K^0_H] \not\subset K^0_H$. This can be observed by Figure 8 which depicts the graph of the function

$$\tilde{\Theta}_H[L] = \Theta[M] - \Theta[N] = 2i \text{Im} \left( \frac{z}{1 - z} \right) + 2 \left( \frac{1}{1 - z} + \frac{1}{z} \log(1 - z) \right)$$

where $L$ is given by (5). The image domain $\tilde{\Theta}_H[L](\mathbb{D})$ is not even starlike.

Figure 8. Graph of the function $\tilde{\Theta}_H[L]$. 
CONSTRUCTION OF SUBCLASSES OF UNIVALENT HARMONIC MAPPINGS

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