CONEAT SUBMODULES AND CONEAT-FLAT MODULES

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Abstract. A submodule $N$ of a right $R$-module $M$ is called coneat if for every simple right $R$-module $S$, any homomorphism $N \to S$ can be extended to a homomorphism $M \to S$. $M$ is called coneat-flat if the kernel of any epimorphism $Y \to M \to 0$ is coneat in $Y$. It is proven that (1) coneat submodules of any right $R$-module are coclosed if and only if $R$ is right $K$-ring; (2) every right $R$-module is coneat-flat if and only if $R$ is right $V$-ring; (3) coneat submodules of right injective modules are exactly the modules which have no maximal submodules if and only if $R$ is right small ring. If $R$ is commutative, then a module $M$ is coneat-flat if and only if $M^+$ is $m$-injective. Every maximal left ideal of $R$ is finitely generated if and only if every absolutely pure left $R$-module is $m$-injective. A commutative ring $R$ is perfect if and only if every coneat-flat module is projective. We also study the rings over which coneat-flat and flat modules coincide.

1. Introduction

A subgroup $A$ of an abelian group $B$ is said to be neat in $B$ if $pA = A \cap pB$ for every prime integer $p$. The notion of neat subgroup was generalized to modules by Renault (see, [12]). Namely, a submodule $N$ of a right $R$-module $M$ is called neat in $M$, if for every simple right $R$-module $S$, $\text{Hom}(S, M) \to \text{Hom}(S, M/N) \to 0$ is epic. Dually, in [8], a submodule $N$ of a right $R$-module $M$ is called coneat in $M$ if $\text{Hom}(M, S) \to \text{Hom}(N, S) \to 0$ is epic for every simple right $R$-module $S$. The notions of neat and coneat are coincide over the ring of integers. By [8, Theorem], the commutative domains over which neat and coneat submodules coincide are exactly the domains with finitely generated maximal ideals (i.e., N-domains). This result was extended to certain commutative rings in [5]. Recently, modules related to neat and coneat submodules are considered by several authors. In [5], a right $R$-module $M$ is called absolutely neat (resp. coneat) if $M$ is a neat (resp. coneat) submodule of any module containing it. According to [16], a right $R$-module $M$ is $m$-injective
if for any maximal right ideal $I$ of $R$, any homomorphism $I \to M$ can be extended to a homomorphism $R \to M$. By Theorem 3.4, a right $R$-module $M$ is absolutely neat if and only if $M$ is $m$-injective.

A ring $R$ is called right $C$-ring if $\text{Soc}(R/I) \neq 0$ for each proper essential right ideal $I$ of $R$. Left perfect rings, right semiartinian rings and almost perfect domains are right $C$-rings.

A dual notion of $m$-injective modules has been studied in [1] and [2]. A module $M$ is called neat-flat if the kernel of any epimorphism $F \to M \to 0$ is a neat submodule of $F$. Closed submodules of any right $R$-module are neat, and neat submodules of any right $R$-module are closed if and only if $R$ is a right $C$-ring (see, [9, Theorem 5]). In [21], a module $M$ is called weak-flat if the kernel of any epimorphism $F \to M \to 0$ is a closed submodule of $F$. Hence, summing up we get, $R$ is a right $C$-ring if and only if every neat-flat right $R$-module is weak-flat.

We call $M$ coneat-flat if the kernel of any epimorphism $Y \to M \to 0$ is coneat in $Y$. In this paper, several characterizations of coneat submodules and coneat-flat modules are given. Some known results are generalized, and relations between coneat-flat modules and flat, $m$-injective, absolutely pure and projective modules are studied.

In Section 2, it is shown that a submodule $N$ of a right $R$-module $M$ is coneat if and only if for every maximal submodule $K$ of $N$, $N/K$ is a direct summand of $M/K$. A ring $R$ is a right $V$-ring if and only if submodules of right $R$-modules are coneat. $R$ is right small if and only if its absolutely coneat right modules are precisely those modules $M$ such that $M = \text{Rad}(M)$.

In Section 3, we prove that, a module $M$ is coneat-flat if and only if $M \cong P/N$ where $P$ is a projective $R$-module and $N$ is a coneat submodule of $P$. An $R$-module $M$ is coneat-flat if and only if and only if $M^+ \text{ is } m$-injective, over commutative rings. $R$ is a right $V$-ring if and only if every right $R$-module is coneat-flat.

In Section 4, we prove that, if $R$ is a left $C$-ring, then a right $R$-module $M$ is flat if and only if for each simple left $R$-module $S$. If $R$ is a commutative $C$-ring, then coneat-flat modules are only the flat modules, and the converse holds when $R$ is noetherian. $R$ is a left $N$-ring (i.e., maximal left ideals are finitely generated) if and only if every absolutely pure module is $m$-injective. A ring $R$ is left artinian if and only if $m$-injective left $R$-modules are precisely those modules $M$ with $M^+$ is projective.

In Section 5, we consider the projectivity of coneat-flat modules. We show that, if $R$ is right perfect, then every coneat-flat $R$-module is projective, the converse hold if $R$ is commutative. Finitely presented coneat-flat modules are projective, over semiperfect rings and over commutative rings.

Throughout, $R$ is a ring with an identity element and all modules are unital right $R$-modules, unless otherwise stated. For an $R$-module $M$, the character module $\text{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$ is denoted by $M^+$. We use the notation $E(M)$,
Soc(M), Rad(M), for the injective hull, socle, radical of M respectively. By $N \leq M$, we mean that N is a submodule of M.

2. Characterization and closure properties of coneat submodules

In this section, several characterizations and some properties of coneat submodules are given. Recall that a submodule $K$ of $M$ is called small in $M$ (denoted by $K \ll M$) if $M \neq K + T$ for every proper submodule $T$ of $M$. A submodule $L \leq M$ is called coclosed in $M$ if $L/N \ll M/N$ implies $L = N$ for every $N \leq L$.

Proposition 2.1. For a submodule $N \leq M$ the following are equivalent.

(1) $N$ is coneat in $M$.

(2) If $K \leq N$ with $N/K$ finitely generated and $N/K \ll M/K$, then $K = N$.

(3) For any maximal submodule $K$ of $N$, $N/K$ is a direct summand of $M/K$.

(4) If $K$ is a maximal submodule of $N$, then there exists a maximal submodule $L$ of $M$ such that $K = N \cap L$.

Proof. (1) $\Rightarrow$ (4) Let $K$ be a maximal submodule of $N$ and $\pi : N \to N/K$ be the canonical epimorphism. By the hypothesis, there exists a homomorphism $f : M \to N/K$ such that $f|_N = \pi$. Then $\text{Ker } f$ is a maximal submodule of $M$ and $N + \text{Ker } f = M$. So that $N \cap \text{Ker } f$ is a maximal submodule of $N$. Then $\pi(N \cap \text{Ker } f) = f(N \cap \text{Ker } f) = 0$. Therefore $K = N \cap \text{Ker } f$.

(3) $\Rightarrow$ (1) Let $S$ be a simple right $R$-module and $f : N \to S$ a nonzero homomorphism. Since $f$ is an epimorphism, without loss of generality we may assume that $S = N/K$ for some maximal submodule $K$ of $N$. So that $\text{Ker } f$ is a maximal submodule of $N$. Then, by (3), $M/\text{Ker } f = (N/\text{Ker } f) \oplus (L/\text{Ker } f)$ for some $L \leq M$. Let $\tilde{f} : N/\text{Ker } f \to N/K$ be the isomorphism induced by $f$. Consider the canonical epimorphisms $\pi : M \to M/\text{Ker } f$ and $\pi' : M/\text{Ker } f \to N/\text{Ker } f$. Then the homomorphism $g = \tilde{f}\pi'\pi$ is the extension of $f$.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (2) Suppose $N/K$ is finitely generated and $N/K \ll M/K$ for some proper submodule $K \leq N$. Then there is a maximal submodule $T$ of $N$ such that $K \leq T$ and $N/T \ll M/T$, because $N/T$ is the image of $N/K$ under the canonical epimorphism $f : M/K \to M/T$, a contradiction.

(3) $\Leftrightarrow$ (4) is straight forward. $\square$

Properties of coclosed modules in [4, 3.7] are adapted to coneat submodules as follows. The proof is omitted.

Proposition 2.2. Let $K \leq L \leq M$ be submodules. Then the following hold.

(1) If $L$ is coneat in $M$, then $L/K$ is coneat in $M/K$.

(2) If $K \leq \text{Rad}(L)$ and $L/K$ is coneat in $M/K$, then $L$ is coneat in $M$.
Proposition 2.3. Let $L/K$ and $X = M/K$ be submodules of $M$. If $K$ is coneat in $M$ and $L/K$ is coneat in $M/K$, then $L$ is coneat in $M$.

Proof. Suppose $X$ is a submodule of $L$ such that $L/X$ finitely generated and $L/X$ is small in $M/X$. Firstly we will prove that $K/K \cap X$ is small in $M/K \cap X$.

Assume the contrary. Then there is an $R$-module $W$ such that

\[ K \cap X \leq W \quad \text{and} \quad W + K = M. \]

Suppose $L/[K + (W \cap X)]$ is not small in $M/[K + (W \cap X)]$. Then there is an $R$-module $Z$ such that $K + (W \cap X) \leq Z$ and $Z + L = M$. Since $K \leq Z$, $Z = Z \cap W + K$ by (\ast), and so $M = Z \cap W + L$. By smallness of $L/X$ is small in $M/X$, $Z \cap W + X = M$. Now $W = Z \cap W + X \cap W$, $W \leq Z$. Finally, since $Z + W = M$, $Z = M$. Recall that $L/K$ is coneat in $M/K$ and $L/[K + (W \cap X)]$ is epimorphic image of the finitely generated module $L/X$. Hence, $L = K + W \cap X$ by Proposition 2.1(2). By modular law, $X = K \cap X + W \cap X$, and $X \leq W$. Then $K + X = L$. Since $L/X$ is small in $M/X$, $W = M$ by (\ast). By our assumption $K$ is coneat in $M$, hence $K = K \cap X$ and $K \leq X$. Since $L/X$ is an epimorphic image of $L/K$ and $L/K$ is coneat in $M/K$, $L = X$ by Proposition 2.1(2), again.

\[ \Box \]

Proposition 2.4 ([15, Lemma 6.1]). Let $A$ be a submodule of an $R$-module $B$ and $i_A : A \hookrightarrow B$ be the inclusion map. For a right ideal $I$ of $R$, $A \cap IB = IA$ if and only if $R/I \otimes_A 1_{R/I \otimes A} \rightarrow R/I \otimes B$ is injective.

An exact sequence $0 \rightarrow A \xrightarrow{i} B \rightarrow C$ is said to be coneat exact if $f(A)$ is a coneat submodule of $B$. A monomorphism $f : A \rightarrow B$ is said to be a coneat monomorphism, if the short exact sequence $0 \rightarrow A \xrightarrow{i} B \rightarrow B/f(A) \rightarrow 0$ is coneat exact. Neat-exact sequences are defined in the same manner.

Theorem 2.5. Let $R$ be a commutative ring and $f : N \rightarrow M$ be a monomorphism. The following are equivalent.

1. $f(N)$ is a coneat submodule of $M$.
2. $S \otimes_R N \xrightarrow{i_{S \otimes_R f}} S \otimes_R M$ is a monomorphism for each simple $R$-module $S$.
3. $mf(N) = f(N) \cap mM$ for each maximal ideal $m$ of $R$.

Proof. (1) $\Leftrightarrow$ (2) By [8, Proposition 3.1].
(2) $\Leftrightarrow$ (3) Follows by Proposition 2.4. $\Box$
Remark 2.6. If $N$ is a pure submodule of $M$, then $NI = N \cap MI$ for every left ideal of $R$ (see, [10, Corollary 4.92]). Therefore, over commutative rings, every pure submodule is coneat by Theorem 2.5(3). This fact will be used in the sequel.

Corollary 2.7. Let $R$ be a commutative ring. The following are equivalent.

1. $0 \to A \to B \to C \to 0$ is coneat exact.
2. $0 \to C^+ \to B^+ \to A^+ \to 0$ is neat exact.

Proof. By Theorem 2.5(2) and the adjoint isomorphism $(M \otimes N)^+ \cong \text{Hom}(M, N^+)$. □

Let $M$ be an $R$-module with $\text{Rad} M = M$. It is easy to see that $\text{Hom}(M, S) = 0$ for each simple module. Hence,

Corollary 2.8. Let $M$ be a right $R$-module with $\text{Rad}(M) = M$. Then $M$ is absolutely coneat.

A ring $R$ is said to be right small if $R \ll E(RR)$. A ring $R$ is small if and only if $E = \text{Rad}(E)$ for every injective $R$-module $E$ (see, [11, Proposition 3.3]).

Proposition 2.9. The following statements are equivalent for a ring $R$.

1. $R$ is a right small ring.
2. Absolutely coneat right $R$-modules are precisely those modules $N$ such that $\text{Rad}(N) = N$.

Proof. (1) $\Rightarrow$ (2) Let $E$ be the injective hull of $N$. Then $\text{Rad}(E) = E$ as $R$ is a small ring. Suppose $N$ is coneat in $E$. So that $\text{Rad}(N) = N \cap \text{Rad}(E) = N$ by Proposition 2.2(3). The rest of (2) by Corollary 2.8.

(2) $\Rightarrow$ (1) Every injective right $R$-module $E$ is absolutely coneat. Then (2) implies $\text{Rad}(E) = E$, and so $R$ is a small ring by [11, Proposition 3.3]. □

Let $R$ be a ring and $M$ be a nonzero $R$-module. $M$ is called coatomic if every proper submodule $N$ of $M$ is contained in a maximal submodule of $M$, i.e., $\text{Rad}(M/N) \neq 0$.

Proposition 2.10. Let $M$ be a module and $N$ be a coatomic submodule of $M$. Then $N$ is coneat in $M$ if and only if it is coclosed in $M$.

Proof. Suppose $N$ is coneat and $N/X \ll M/X$ for some proper submodule $X \leq N$. Since $N$ is coatomic, $X$ is contained in a maximal submodule, say $K$, of $N$. Then $N/K \ll M/K$, and this contradicts with the fact that $N$ is coneat. Hence $N$ is coclosed. The converse implication is obvious. □

In [19], a ring $R$ is called right $K$-ring if every non-zero small right $R$-module is coatomic. Dedekind domains and right max rings (i.e., every nonzero right $R$-module has a maximal submodule) are right $K$-rings.
Theorem 2.11. \( R \) is a right \( K \)-ring if and only if coneat submodules of any right \( R \)-module are coclosed.

Proof. For the necessity, let \( M \) be a non-zero small module and suppose \( M/K \) has no maximal submodules, i.e., \( \text{Rad}(M/K) = M/K \) for some proper submodule \( K \) of \( M \). Then \( M/K \) is small and coneat submodule in \( E(M/K) \). Hence \( M/K \) is coclosed in \( E(M/K) \) by (1). This gives a contradiction, since coclosed submodules are not small. Consequently, \( K \) is contained in a maximal submodule of \( M \), and so \( M \) is coatomic.

For the sufficiency, suppose the contrary that, there is a module \( M \) and a submodule \( N \) of \( M \) which is coneat but not coclosed. Then there is a proper submodule \( K \) of \( N \) such that \( N/K \ll M/K \). By Proposition 2.2(1), \( N/K \) is a coneat submodule of \( M/K \). Then \( N/K \) is coatomic by the hypothesis, and so \( N/K \) is coclosed by Proposition 2.10, a contradiction. □

3. Coneat-flat modules

It is well known that, a right \( R \)-module \( M \) is flat if and only if any short exact sequence of the form \( 0 \to K \xrightarrow{f} N \to M \to 0 \) is pure exact, i.e., \( f(K) \) is a pure submodule of \( N \). It is natural to ask for which right \( R \)-modules \( P \) any short exact sequence ending with \( P \) is coneat exact? In this section several characterizations of such modules are given.

A right \( R \)-module \( M \) is called coneat-flat if the kernel of any epimorphism \( Y \to M \to 0 \) is a coneat submodule of \( Y \). Clearly, projective modules are coneat-flat but the converse need not be true in general (see, Theorem 5.1).

Theorem 3.1. The following are equivalent for an \( R \)-module \( M \):

1. \( M \) is coneat-flat.
2. \( \text{Ext}^1_R(M, S) = 0 \) for each simple \( R \)-module \( S \).
3. There is a coneat exact sequence \( 0 \to K \to L \to M \to 0 \) with \( L \) projective.
4. There is a coneat exact sequence \( 0 \to K \to L \to M \to 0 \) with \( L \) coneat-flat.

Proof. (1) \( \Rightarrow \) (2) Let \( E : 0 \to S \xrightarrow{\alpha} L \to M \to 0 \) be a short exact sequence with \( S \) simple right \( R \)-module. Since \( M \) is coneat-flat, \( S \) is coneat in \( L \), and there is a homomorphism \( \beta : L \to S \) such that the following diagram is commutative.

\[
\begin{array}{ccc}
E: & 0 & \longrightarrow & S & \longrightarrow & L & \longrightarrow & P & \longrightarrow & 0 \\
 & & \downarrow{1_S} & \downarrow{\gamma} & \downarrow{\beta} & & & & \\
 & S & \longrightarrow & L & \longrightarrow & M & \longrightarrow & 0 & \\
\end{array}
\]

Then \( 1_S = \beta \alpha \), and so the sequence \( E \) splits. Hence \( \text{Ext}^1_R(M, S) = 0 \).

(2) \( \Rightarrow \) (3) Assuming (2). There is a short exact sequence \( E : 0 \to C \to F \to M \to 0 \) with \( F \) free \( R \)-module. Applying \( \text{Hom}_R(-, S) \), we obtain the exact
sequence $0 \to \text{Hom}_R(M, S) \to \text{Hom}_R(F, S) \to \text{Hom}_R(C, S) \to \text{Ext}^1_R(M, S) = 0$.
That is, $\text{Hom}_R(\mathcal{E}, S)$ is exact for every simple $R$-module $S$, and so $\mathcal{E}$ is coneat exact.

$(3) \Rightarrow (4)$ is obvious.

$(4) \Rightarrow (1)$ Let $s : B \to M$ be any epimorphism. Consider the following commutative diagram.

\[ \begin{array}{cccccccc}
0 & 0 & \downarrow & \downarrow & K & K & 0
\end{array} \]

\[ \begin{array}{cccccccc}
0 & \text{Ker} s & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & L & 0
\end{array} \]

\[ \begin{array}{cccccccc}
0 & \text{Ker} s & \xrightarrow{s} & B & \xrightarrow{\beta} & M & 0
\end{array} \]

$\beta \alpha = st$ is coneat epimorphism, i.e., $\text{Ker}(st)$ is a coneat submodule of $X$, by Proposition 2.3. Then $s$ is coneat epimorphism by Proposition 2.2(1). This completes the proof.

By Theorem 3.1, we get the following.

**Corollary 3.2.** The class of coneat-flat modules is closed under extensions, direct sums, direct summands and coneat quotients. In particular, coneat-flat modules are closed under pure quotients over commutative rings.

**Proof.** Coneat-flat modules are closed under extensions, direct sums, direct summands and coneat quotients by Theorem 3.1, and under pure quotients by Remark 2.6 and Theorem 3.1.

**Proposition 3.3.** Let $R$ be a commutative ring and $M$ be an $R$-module. Then $M$ is coneat-flat if and only if $\text{Tor}_R(M, S) = 0$ for each simple $R$-module $S$.

**Proof.** Let $0 \to K \xrightarrow{i} F \to M \to 0$ be a short exact sequence with $F$ projective. Applying $- \otimes S$, we get

$0 = \text{Tor}(F, S) \to \text{Tor}(M, S) \to K \otimes S \xrightarrow{i \otimes 1_S} F \otimes S \to M \otimes S \to 0$.

Then $i \otimes 1_S$ is a monomorphism if and only if $\text{Tor}(M, S) = 0$. Now the proof is clear by Theorem 2.5 and Theorem 3.1.

**Proposition 3.4.** The following are equivalent for a right $R$-module $M$. 
(1) $M$ is $m$-injective.
(2) $M$ is a neat submodule of an $m$-injective module.
(3) $M$ is a neat submodule of every module containing it.
(4) $\text{Ext}^1_R(S, M) = 0$ for every simple right $R$-module $S$.

Proof. (1) $\iff$ (4) Let $I$ be a right ideal of $R$. Then applying $\text{Hom}(\cdot, M)$ to the short exact sequence $0 \to I \to R \to R/I \to 0$, we get $0 \to \text{Hom}(R/I, M) \to \text{Hom}(R, M) \xrightarrow{i^*} \text{Hom}(I, M) \to \text{Ext}^1_R(R/I, M) \to \text{Ext}^1_R(R, M) = 0$. Then $i^*$ is epic if and only if $\text{Ext}^1_R(R/I, M) = 0$.

(2) $\iff$ (3) By [5, Theorem 3.3].

(3) $\iff$ (4) By [5, Theorem 3.4. (i)$\iff$(ii)]. □

Proposition 3.5. Let $R$ be a commutative ring. An $R$-module $M$ is coneat-flat if and only if $M^+$ is $m$-injective.

Proof. Let $S$ be a simple $R$-module. We have the standard isomorphism

$$\text{Ext}^1_R(S, M^+) \cong \text{Tor}^R_1(M, S)^+.$$  

Now, the proof is immediate by Proposition 3.3 and Proposition 3.4. □

Corollary 3.6. Let $R$ be a commutative ring. The class of coneat-flat modules is closed under pure submodules.

Proof. Let $0 \to A \to B \to C \to 0$ be a pure exact sequence of $R$-modules with $B$ coneat-flat. Then the short exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$ splits. By Proposition 3.5 the module $B^+$ is $m$-injective, and so $A^+$ is $m$-injective. Then $A$ is coneat-flat by Proposition 3.5, again. □

Proposition 3.7. The following statements are equivalent for a ring $R$.

(1) $R$ is a right V-ring.
(2) for every right $R$-module $M$ every submodule of $M$ is coneat in $M$.
(3) every right $R$-module is coneat-flat.

Proof. (1) $\Rightarrow$ (2) is clear, since every simple right $R$-module is injective by (1).

(2) $\Rightarrow$ (3) Let $M$ be a right $R$-module. Consider an epimorphism $f : F \to M$ with $F$ free right $R$-module. Then $\text{Ker} f$ is a coneat submodule of $F$ by (2). Therefore $M$ is coneat-flat by Theorem 3.1.

(3) $\Rightarrow$ (1) Let $S$ be a simple $R$-module and $E$ be an injective module containing $S$. By the hypothesis $E/S$ is coneat-flat. Hence the sequence $0 \to S \to E \to E/S \to 0$ splits by Theorem 3.1, and so $S$ is injective. □

4. When coneat-flat modules are flat

In this section, we study the flatness of coneat-flat modules, and the character of coneat-flat modules. We begin with the following. A module right $R$-module $M$ is called cotorsion if $\text{Ext}^1_R(F, M) = 0$ for any flat $R$-module $F$. 
Example 4.1. (1) Let \( R \) be a valuation domain with a non finitely generated maximal ideal \( P \). Then \( \text{Rad}(P) = P^2 = P \), and so \( P \) is a coneat submodule of \( R \) by Corollary 2.8. Hence \( R/P \) is coneat-flat by Theorem 3.1. On the other hand, \( R/P \) is not a flat \( R \)-module, since \( R/P \) is a torsion \( R \)-module.

(2) Let \( R \) be a regular ring that is not a right \( V \)-ring. Then there exists a flat module which is not coneat-flat by Proposition 3.7.

In light of Example 4.1, it is natural to consider the rings over which coneat-flat and flat modules coincide. We begin with the following lemma.

Lemma 4.2. Let \( R \) be a ring and \( S \) be a simple \( R \)-module. If \( R \) is commutative or semilocal, then \( S \) is cotorsion.

Proof. First suppose \( R \) is commutative and let \( I = \text{Ann}_R(S) \). Then clearly \( S \) is an \( R/I \)-module. Since \( R/I \) is simple, \( S \) is cotorsion as an \( R/I \)-module. So that \( S \) is a cotorsion \( R \)-module by [18, Proposition 3.3.3]. If \( R \) is semilocal, then \( J(R).S = 0 \) and so \( S \) is an \( R/J(R) \)-module. As \( R \) is semilocal, \( R/J(R) \) is semisimple and so \( S \) is a cotorsion \( R/J(R) \)-module. Now, \( S \) is a cotorsion \( R \)-module by [18, Proposition 3.3.3], again. □

Corollary 4.3. Suppose \( R \) is commutative or a semilocal ring. Then every flat module is coneat-flat.

Proof. Let \( S \) be a simple \( R \)-module. Then \( S \) is a cotorsion module by Lemma 4.2. Therefore \( \text{Ext}^1_R(M, S) = 0 \), and so \( M \) is coneat-flat by Theorem 3.1. □

Remark 4.4. A commutative domain \( R \) is called almost perfect if \( R/I \) is a perfect ring for each nonzero ideal \( I \) of \( R \). It is clear that almost perfect domains are \( C \)-rings. In [14], the authors prove that, if \( R \) is an almost perfect domain, then an \( R \)-module \( M \) is injective if and only if \( \text{Ext}^1_R(S, M) = 0 \) (i.e., \( M \) is \( m \)-injective) for each simple module \( S \). Actually, one of the characterization of right \( C \)-rings is the following: \( R \) is a right \( C \)-ring if and only if every \( m \)-injective right \( R \)-module is injective (see, [16, Lemma 4]).

Proposition 4.5. Let \( R \) be a left \( C \)-ring. A right \( R \)-module \( M \) is flat if and only if \( \text{Tor}^1_R(M, S) = 0 \) for each simple left \( R \)-modules \( S \).

Proof. Necessity is clear. For the sufficiency assume that \( \text{Tor}^1_R(M, S) = 0 \) for each simple left \( R \)-modules \( S \). Then \( 0 = \text{Tor}^1_R(M, S)^+ \cong \text{Ext}^1_R(S, M^+) \) implies \( M^+ \) is \( m \)-injective by Theorem 3.4. Therefore \( M^+ \) is injective, because \( R \) is a left \( C \)-ring. Hence \( M \) is flat by [7, Theorem 3.2.10]. □

Proposition 4.6. Let \( R \) be a commutative ring. Consider the following statements.

(1) \( R \) is a \( C \)-ring.

(2) Coneat-flat \( R \)-modules are flat.

Then (1) \( \Rightarrow \) (2). If \( R \) is a noetherian, then (2) \( \Rightarrow \) (1).
Proof. (1) $\Rightarrow$ (2) By Corollary 3.3 and Proposition 4.5.

(2) $\Rightarrow$ (1) Let $M$ be an $m$-injective $R$-module. Then $M^+$ is flat by the hypothesis and Theorem 4.10. As $R$ is noetherian, $M$ is injective by [3, Theorem 2]. Hence $R$ is a $C$-ring. □

Theorem 4.7. The following are equivalent for a commutative ring $R$.

1. Every cotorsion-flat module is flat.
2. Flat modules are precisely those modules $M$ satisfying

$$\text{Ext}^1_R(M, \prod_{i\in I} S_i) = 0,$$

where the $S_i$’s are all the non-isomorphic simple modules.

Proof. (1) $\Rightarrow$ (2) By Lemma 4.2, simple modules are cotorsion. Then $\prod_{i\in I} S_i$ is cotorsion, since cotorsion modules are closed under direct products. Hence, if $M$ is flat, then $\text{Ext}^1_R(M, \prod_{i\in I} S_i) = 0$. Conversely, suppose $\text{Ext}^1_R(M, S_i) = 0$ for each $i \in I$. So that $M$ is cotorsion-flat by Theorem 3.1. Hence $M$ is flat by (1).

(2) $\Rightarrow$ (1) Suppose $M$ is cotorsion-flat. Then $\text{Ext}^1_R(M, S_i) = 0$ for each simple $R$-module $S_i$. So that $\text{Ext}^1_R(M, \prod_{i\in I} S_i) = 0$ for any index set $I$ and simple $R$-modules $S_i$. Hence $M$ is flat by (2).

Proposition 4.8. Let $R$ be a commutative $N$-ring and $M$ be an arbitrary $R$-module. Then the following hold.

1. $M$ is $m$-injective if and only if $M^+$ is cotorsion-flat.
2. $M$ is $m$-injective if and only if $M^{++}$ is $m$-injective.
3. $M$ is cotorsion-flat if and only if $M^{++}$ is cotorsion-flat.
4. Any direct product of cotorsion-flat modules is cotorsion-flat.
5. Any direct product of copies of $R$ is cotorsion-flat.
6. The class of $m$-injective modules is closed under pure quotients.

Proof. (1) An $R$-module $M$ is $m$-injective if and only if $M^+$ is cotorsion-flat by [13, Theorem 9.51], since $R$ is an $N$-ring.

(2) $M$ is $m$-injective if and only if $M^+$ is cotorsion-flat by (1), and $M^+$ is cotorsion-flat if and only if $M^{++}$ is $m$-injective by Proposition 3.5.

(3) If $M$ is cotorsion-flat, then $M^+$ is $m$-injective by Proposition 3.5. So $M^{++}$ is $m$-injective by (2), and hence $M^{++}$ is cotorsion-flat. Conversely, if $M^{++}$ is cotorsion-flat, then $M$ is cotorsion-flat by Corollary 3.6, since $M$ is a pure submodule of $M^{++}$.

(4) Let $(M_i)_{i\in J}$ be a family of cotorsion-flat $R$-modules. Since the class of cotorsion-flat modules is closed under direct sums, $\bigoplus_{i\in J} M_i$ is flat. So $(\bigoplus_{i\in J} M_i)^{++} \cong (\prod_{i\in J} M_i^+)^+$ is cotorsion-flat by (3). Since $\oplus_{i\in J} M_i^+$ is a pure submodule of $\prod_{i\in J} M_i^+$, $(\oplus_{i\in J} M_i^+)^+$ is a direct summand of $(\prod_{i\in J} M_i^+)^+$, and so $(\oplus_{i\in J} M_i^+)^+ \cong \prod_{i\in J} M_i^{++}$ is cotorsion-flat. Since cotorsion-flat modules are closed
under pure submodules and $\prod_{i \in J} M_i$ is a pure submodule of $\prod_{i \in J} M_i^{++}$, the module $\prod_{i \in J} M_i$ is coneat-flat.

(5) By (4).

(6) Take any pure exact sequence $0 \to A \to B \to C \to 0$ with $B$ $m$-injective. Then we have a split exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. By (1), $B^+$ is coneat-flat, and so $C^+$ is coneat-flat. Then $C$ is $m$-injective by (1), again. $\square$

An $R$-module $M$ is called absolutely pure if it is pure in every module containing it as a submodule. It is well known that, a ring $R$ is left noetherian if and only if every absolutely pure left $R$-module is injective.

Proposition 4.9. $R$ is a left N-ring if and only if every absolutely pure left $R$-module is $m$-injective.

Proof. ($\Rightarrow$) Let $M$ be an absolutely pure left $R$-module. Since $R$ is a left N-ring, $\text{Ext}^1_R(S, M) = 0$ for each simple left $R$-module $S$. That is, $M$ is $m$-injective.

($\Leftarrow$) Let $S$ be a simple left $R$-module. Then $\text{Ext}^1_R(S, M) = 0$ for each absolutely pure left $R$-module $M$ by the assumption. Then $S$ is finitely presented by [6, Proposition]. $\square$

Theorem 4.10. Let $R$ be a ring. The following statements are equivalent.

1. (a) $M$ is a flat right $R$-module if and only if $\text{Tor}^1_R(M, S) = 0$ for each simple left $R$-module $S$,

   (b) $R$ is a left N-ring.

2. $M$ is an $m$-injective left $R$-module if and only if $M^+$ is flat.

3. $M$ is an $m$-injective left $R$-module if and only if $M$ is an absolutely pure left $R$-module.

Proof. (1) $\Rightarrow$ (2) Let $M$ be a left $R$-module and $S$ be a simple left $R$-module. Suppose $M$ is $m$-injective. Then $0 = \text{Ext}^1_R(S, M)^+ \cong \text{Tor}^1_R(M^+, S)$ by [13, Theorem 9.51], and so $M^+$ is flat by (1). Conversely suppose $M^+$ is flat. Then $M^{++}$ is injective by [13, Theorem 3.52], and so $M$ is absolutely pure, since $M$ is pure in $M^{++}$. Therefore $M$ is $m$-injective by Proposition 4.9.

(2) $\Rightarrow$ (3) Firstly, we shall prove that a right $R$-module $M$ is flat if and only if $M^{++}$ is flat. Then $R$ is left coherent by [3, Theorem 1]. Suppose $M$ is a flat right $R$-module. Then $M^+$ is ($m$-)injective, and so $M^{++}$ is flat by (2). Now, conversely suppose $M^{++}$ is a flat right $R$-module. Then $M$ is flat, since $M$ is pure submodule of $M^{++}$ and flat modules closed under pure submodules.

Let $M$ be a left $R$-module. Then $M^+$ is flat if and only if $M$ is absolutely pure by [3, Theorem 1], since $R$ is left coherent. Hence the rest of (3) follows by (2).

(3) $\Rightarrow$ (1) Suppose $\text{Tor}^1_R(M, S) = 0$ for each simple left $R$-module $S$. Then $\text{Ext}^1_R(S, M^+) = 0$, and so $M^+$ is $m$-injective. Then $M^+$ is absolutely pure by (3). Therefore $M^+$ is injective, since it is pure-injective. Thus $M$ is flat. This proves (a), and (b) follows by Proposition 4.9. $\square$
Proposition 4.11. Let $R$ be a commutative ring. Consider the following statements.

(1) $R$ is a $C$-ring.
(2) Concat-flat $R$-modules are flat.

Then (1) \( \Rightarrow \) (2). If $R$ is a noetherian, then (2) \( \Rightarrow \) (1).

Proof. (1) \( \Rightarrow \) (2) By Proposition 3.3 and Proposition 4.5.

(2) \( \Rightarrow \) (1) Let $M$ be an $m$-injective $R$-module. Then $M^+$ is flat by the hypothesis and Theorem 4.10. As $R$ is noetherian, $M$ is injective by [3, Theorem 2]. Hence $R$ is a $C$-ring.

It is easy to see that, a left $N$-ring and left semiartinian ring is left noetherian. The following is a slight generalization of this fact.

Corollary 4.12. If $R$ is a left $N$-ring and a left $C$-ring, then $R$ is left noetherian.

Proof. By Proposition 4.5 and Theorem 4.10, a left $R$-module $M$ is $m$-injective if and only if it is absolutely pure. So that every absolutely pure left module is injective. Hence $R$ is left noetherian.

Note that, Corollary 4.12, generalizes [5, Theorem 4.1 (ii)\( \Rightarrow \) (i)].

In [3, Theorem 4], the authors proves that, $R$ is left artinian if and only if a left module $M$ is injective exactly when $M^+$ is projective. We show that, this result still holds if we replace $m$-injective by injective.

Theorem 4.13. Let $R$ be a ring. The following are equivalent.

(1) $R$ is left artinian.
(2) A left $R$-module $M$ is $m$-injective if and only if $M^+$ is projective.

Proof. (1) \( \Rightarrow \) (2) $R$ is a left $C$-ring by (1), and so $m$-injective modules are injective. Now, (2) follows by [3, Theorem 4].

(2) \( \Rightarrow \) (1) Firstly, we show that a left $R$-module $M$ is $m$-injective if and only if $M$ is absolutely pure.

Let $M$ be an absolutely pure left $R$-module. Consider the pure exact sequence $0 \rightarrow M \rightarrow E(M) \rightarrow E(M)/M \rightarrow 0$. Then the short exact sequence $0 \rightarrow (E(M)/M)^+ \rightarrow E(M)^+ \rightarrow M^+ \rightarrow 0$ splits. Then $E(M)^+$ is projective, and hence $M^+$ is projective. By (2), $M$ is $m$-injective. Conversely, let $M$ be an $m$-injective left $R$-module. Since $M$ is pure in $M^+$ and $M^{++}$ is injective, $M$ is absolutely pure.

Then a left $R$-module $M$ is $m$-injective if and only if $M$ is absolutely pure if and only if $M^+$ is projective. By [3, Theorem 3], $R$ is right perfect, and so it is a left $C$-ring, i.e., $m$-injective left $R$-modules are injective. Hence $R$ is left artinian by [3, Theorem 4] and (2).
5. When coneat-flat modules are projective

In this section, we shall consider when coneat-flat modules are projective. We begin with the following result.

**Theorem 5.1.** Consider the following statements.

1. $R$ is a right perfect ring.
2. Every coneat-flat right $R$-module is projective.

Then (1) $\Rightarrow$ (2). If $R$ is either commutative or semilocal, then (2) $\Rightarrow$ (1).

**Proof.** (1) $\Rightarrow$ (2) Let $P$ be a coneat-flat module. Consider a short exact sequence $0 \to K \to F \to P \to 0$ with $F$ free module. Since $R$ is perfect, $F$ is supplemented by [17, 43.9]. So $K$ has a supplement in $F$, that is, $K + N = F$ and $A \cap N \ll N$ for some submodule $N$ of $F$. On the other hand, $K$ is coatomic, as $R$ is a perfect ring. Then $K$ is a coclosed submodule of $F$ by Proposition 2.10. So that $K \cap N \ll K$. Hence $K$ and $N$ are mutual supplements, and so $K \oplus N = F$ by [17, 41.15]. Therefore $N \cong F/K \cong P$ is projective.

(2) $\Rightarrow$ (1) Let $M$ be a flat module. By Corollary 4.3, $M$ is coneat-flat, and so $M$ is projective by (2). Hence $R$ is a perfect ring. $\square$

The following is an immediate consequence of Theorem 5.1.

**Corollary 5.2.** Let $R$ be a perfect ring. Then an $R$-module $P$ is projective if and only if $\text{Ext}^1_R(P, S) = 0$ for every simple $R$-module $S$.

An epimorphism $f : N \to M$ is said to be a small cover of $M$ if $\text{Ker} f \ll N$. Moreover, if $N$ is projective, then $f$ is called a projective cover.

**Proposition 5.3.** Let $R$ be a ring and $M$ be a right $R$-module with a projective cover $f : P \to M$. Set $K = \text{Ker} f$. Then $M$ is a coneat-flat module if and only if $\text{Rad}(K) = K$.

**Proof.** ($\Rightarrow$) Assume $\text{Rad}(K) \neq K$. Then $K$ has a maximal submodule, say $A$. By Proposition 2.1, there exists a maximal submodule $L$ of $P$ such that $A = K \cap L$. Then $K \leq \text{Rad} P$ implies $K = K \cap \text{Rad}(P) \leq K \cap L = A$. Contradiction. Hence (2) holds.

($\Leftarrow$) By Corollary 2.8 and Theorem 3.1. $\square$

**Corollary 5.4.** Let $R$ be a semiperfect ring. Then finitely presented coneat-flat modules are projective.

**Lemma 5.5.** Let $R$ be a commutative ring and $M$ be a coneat-flat $R$-module. Then, for all maximal ideals $m$ of $R$, $M_m$ is a coneat-flat $R_m$-module.

**Proof.** Since $M$ is a coneat-flat $R$-module, there is a short exact sequence $0 \to K \to F \to M \to 0$ where $K$ is coneat submodule of $F$ with $F$ is a projective $R$-module by Theorem 3.1. By exactness of localization, for all maximal ideals $m$ of $R$, the sequence $0 \to K_m \to F_m \to M_m \to 0$ is exact. Since $mK = K \cap mF$
for all maximal ideals $m$ of $R$, we have $m_nK_m = K_m \cap m_FM_m$. Therefore $M_n$ is a coneat-flat $R_m$-module by Theorem 2.5.

\[ \square \]

**Corollary 5.6.** Let $R$ be a commutative ring. Then a finitely presented $R$-module $M$ is coneat-flat if and only if it is projective.

**Proof.** Sufficiency is clear. For the necessity, suppose $M$ is coneat-flat. Let $m$ be a maximal ideal of $R$. Then $M_m$ is a coneat-flat $R_m$-module by Lemma 5.5. So that $M_m$ is projective (and so flat) over $R_m$ by Corollary 5.4. Then $M$ is flat by [10, page 160, Exercise 14]. Therefore $M$ is projective by [10, Theorem 4.30].

\[ \square \]

**References**

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