ON THE \((n, d)^{th}\) \(f\)-IDEALS

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Abstract. For a field \(K\), a square-free monomial ideal \(I\) of \(K[x_1, \ldots, x_n]\) is called an \(f\)-ideal, if both its facet complex and Stanley-Reisner complex have the same \(f\)-vector. Furthermore, for an \(f\)-ideal \(I\), if all monomials in the minimal generating set \(G(I)\) have the same degree \(d\), then \(I\) is called an \((n, d)^{th}\) \(f\)-ideal. In this paper, we prove the existence of \((n, d)^{th}\) \(f\)-ideal for \(d \geq 2\) and \(n \geq d + 2\), and we also give some algorithms to construct \((n, d)^{th}\) \(f\)-ideals.

1. Introduction

Throughout the paper, for a set \(A\), we use \(A_d\) to denote the set of the subsets of \(A\) with cardinality \(d\). For a field \(K\), let \(S = K[x_1, \ldots, x_n]\), and let \(I\) be a monomial ideal of \(S\). Denote by \(sm(S)\) (\(sm(I)\), respectively) the set of square-free monomials in \(S\) (in \(I\), respectively). As we know, there is a natural bijection between \(sm(S)\) and \(2^n\), denoted by

\[
\sigma : x_{i_1}x_{i_2} \cdots x_{i_k} \mapsto \{i_1, i_2, \ldots, i_k\},
\]

where \([n] = \{1, 2, \ldots, n\}\) for a positive integer \(n\). For other concepts and notations, see references [3, 5, 7, 8, 10, 11].

Constructing free resolutions of a monomial ideal is one of the core problems in combinatorial commutative algebra. A main approach to the problem is by taking advantage of properties of a simplicial complex, so it is important to have a research on properties of the complex corresponding to an ideals, see, e.g., references [4, 6, 9, 12]. There is an important class of ideals called \(f\)-ideals, whose facet complex \(\delta_F(I)\) and Stanley-Reisner complex \(\delta_N(I)\) have the same \(f\)-vector, where \(\delta_F(I)\) is generated by the set \(\sigma(G(I))\), and \(\delta_N(I) = \{\sigma(g) \mid g \in sm(S) \setminus sm(I)\}\). Note that the \(f\)-vector of a complex \(\delta_N(I)\), which is not easy to compute in general, is essential in the computation of the Hilbert series of \(S/I\). Since the correspondence of the complex \(\delta_F(I)\) and an ideal \(I\) is direct and clear, it is more easier to calculate the \(f\)-vector of \(\delta_F(I)\). So, it is convenient

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to calculate the Hilbert series and study other corresponding properties of \(S/I\) while \(I\) is an \(f\)-ideal.

The formal definition of an \(f\)-ideal appeared first in [1], and it is then studied in [2]. In [7], a monomial ideal \(I\) of \(K[x_1, \ldots, x_n]\) is called an \((n,d)^{th}\) ideal if the monomials in the minimal generating set \(G(I)\) have the same degree \(d\), and the \((n,d)^{th}\) \(f\)-ideals are characterized. General \(f\)-ideals are also studied in [7]. In [7], a bijection is introduced between square-free monomial ideals of degree 2 and simple graphs, and it is shown that \(V(n,2) \neq \emptyset\) holds for each \(n \geq 4\), where \(V(n,d)\) is the set of \((n,d)^{th}\) \(f\)-ideals. The structure of \(V(n,2)\) is determined, and the characterization of the unmixed \(f\)-ideals is also studied in [7]. Recall that an ideal \(I\) is called unmixed, if \(\text{codim}(P) = \text{codim}(I)\) holds for every prime ideal \(P\) minimal over \(I\).

In this paper, we give another characterization of unmixed \(f\)-ideals in part two. In Section 3, we generalize the aforementioned result of [7] by showing that \(V(n,d) \neq \emptyset\) for general \(d \geq 2\) and \(n \geq d + 2\). In Section 4, we introduce some algorithms to construct \((n,d)^{th}\) \(f\)-ideals, and we show an upper bound of the \((n,d)^{th}\) perfect number in Section 5. In Section 6, we show some examples of nonhomogeneous \(f\)-ideals, the existence of which was still open in [7].

The following notations, definitions and propositions are needed in this paper.

Let \(A\) be a set of square-free monomials in \(K[x_1, \ldots, x_n]\). The sets \(\sqcup(A)\) and \(\cap(A)\) are defined respectively by
\[
\sqcup(A) = \{gx_i \mid g \in A, x_i \nmid g, 1 \leq i \leq n\}
\]
and
\[
\cap(A) = \{h \mid 1 \neq h, h = g/x_i \text{ for some } g \in A \text{ and some } x_i \text{ with } x_i \mid g\}.
\]

**Definition 1.1** ([7, Definition 2.1]). Let \(S = K[x_1, \ldots, x_n]\), and let \(A \subseteq \text{sm}(S)_d\), where \(1 < d < n\). \(A\) is called an \((n,d)^{th}\) upper perfect set, if \(\sqcup(A) = \text{sm}(S)_{d+1}\) holds. Dually, \(A\) is called an \((n,d)^{th}\) lower perfect set, if \(\cap(A) = \text{sm}(S)_{d-1}\) holds. If \(A\) is both \((n,d)^{th}\) upper perfect and \((n,d)^{th}\) lower perfect, then \(A\) is called an \((n,d)^{th}\) perfect set, or alternatively, a perfect subset of \(\text{sm}(S)_{d}\). For a given pair of numbers \((n,d)\), the smallest number among cardinalities of \((n,d)^{th}\) perfect sets is called the \((n,d)^{th}\) perfect number, and is denoted by \(N_{(n,d)}\).

**Proposition 1.2** ([7, Theorem 2.3]). Let \(S = K[x_1, \ldots, x_n]\), and let \(I\) be an \((n,d)^{th}\) square-free monomial ideal of \(S\) with the minimal generating set \(G(I)\). Then \(I\) is an \(f\)-ideal if and only if \(G(I)\) is \((n,d)^{th}\) perfect and \(|G(I)| = \binom{n}{d}\) holds true.

**Proposition 1.3** ([7, Proposition 3.3]). \(V(n,2) \neq \emptyset\) if and only if \(n = 4k\) or \(n = 4k + 1\) for some positive integer \(k\).
Proposition 1.4 ([7, Proposition 5.3]). Let $S = K[x_1, \ldots, x_n]$. If $I$ is an $(n, d)$th $f$-ideal, then $I$ is unmixed if and only if $\text{sm}(S)_d \setminus G(I)$ is lower perfect in $\text{sm}(S)_d$.

In [7], a method for finding an $(n, 2)$th perfect set with the smallest cardinality is provided in the following: First, decompose the set $[n]$ into a disjoint union of two subsets $B$ and $\overline{B}$ uniformly, i.e., such that $|B| - |\overline{B}| \leq 1$ holds true. Second, for each such subset $B$, set $A = \{x_i x_j \mid \text{either } \{i, j\} \subseteq B, \text{ or } \{i, j\} \subseteq \overline{B}\}$. Then, $A$ is an $(n, 2)$th perfect set whose cardinality is equal to the $(n, 2)$th perfect number $N(n, 2)$, where

$$N(n, 2) = \begin{cases} k^2 - k, & \text{if } n = 2k; \\ k^2, & \text{if } n = 2k + 1. \end{cases}$$

Note that for any such subset $A$, a set $D$ with $A \subseteq D \subseteq \text{sm}(S)_2$ is also an $(n, 2)$th perfect set.

2. $(n, d)$th unmixed $f$-ideals

For a positive integer $d$ greater than 2, an $(n, d)$th $f$-ideal may be not unmixed, see Example 5.1 of [7] for a counterexample. So, it is interesting to characterize the unmixed $f$-ideals. In this section, we show a characterization of unmixed $f$-ideals by the corresponding simplicial complex, by taking advantage of the bijection $\sigma$ between square-free monomial ideals and simplicial complexes.

A simplicial complex $\Delta$ on $[n]$ is called a $d$-flag complex if every minimal nonface of $\Delta$ consists of $d$ elements of $[n]$. Note that a flag complex (see, e.g., [8, page 155]) is a 2-flag complex, as is just defined. For a simplicial complex $\Delta$ on $[n]$, the Alexander dual of $\Delta$, denoted by $\Delta^\vee$, is defined by $\Delta^\vee = \{[n] \setminus F \mid F \notin \Delta\}$, see [8] for details.

Proposition 2.1. Let $S = K[x_1, \ldots, x_n]$, and let $I$ be an $(n, d)$th square-free monomial ideal of $S$. Then $I$ is an $(n, d)$th unmixed $f$-ideal if and only if the following conditions hold:

1. $|G(I)| = \binom{n}{d}/2$.
2. $\dim \delta_F(I)^\vee = n - d - 1$.
3. $\langle \sigma(u) \mid u \in \text{sm}(S)_d \setminus G(I) \rangle$ is a d-flag complex.

Proof. We claim that the following two results hold true: First, the condition (2) holds if and only if $G(I)$ is lower perfect. Second, the condition (3) holds if and only if $G(I)$ is upper perfect and $\text{sm}(S)_d \setminus G(I)$ is lower perfect. If the above two results hold true, then it is easy to see that the conclusion holds by Propositions 1.2 and 1.4.

For the first claim, if $G(I)$ is lower perfect, then for each minimal nonface $F$ of $\delta_F(I)$, $|F| \geq d$ holds. By the definition of the Alexander dual, $H$ is a face
of $\delta x(I)^{\vee}$ if and only if $[n] \setminus H$ is a nonface of $\delta x(I)$. So, for each facet $L$ of $\delta x(I)^{\vee}$, $|L| \leq n - d$ holds true. Since $|G(I)| \neq \binom{n}{d}$, there exists some nonface of $\delta x(I)$ with cardinality $d$, or equivalently, there exists some facet of $\delta x(I)^{\vee}$ with cardinality $n - d$. Thus $\text{dim}(\delta x(I)^{\vee}) = n - d - 1$ holds.

Conversely, assume $\text{dim}(\delta x(I)^{\vee}) = n - d - 1$. By a similar argument, one can see that the smallest cardinality of nonfaces of $\delta x(I)$ is $d$, hence $G(I)$ is lower perfect.

For the second claim, if $sm(S)_d \setminus G(I)$ is lower perfect, then for the complex $\Delta = \langle \sigma(u) \mid u \in sm(S)_d \setminus G(I) \rangle$, the cardinality of a nonface is not less than $d$. Since $G(I)$ is upper perfect, for each nonface $F$ of $\Delta$, there exists $v \in G(I)$ such that $\sigma(v) \subseteq F$. Note that $\sigma(v)$ is a nonface of $\Delta$, so all the minimal nonfaces of $\Delta$ have cardinality $d$. Hence $\Delta$ is a $d$-flag complex.

Conversely, assume that $\Delta = \langle \sigma(u) \mid u \in sm(S)_d \setminus G(I) \rangle$ is a $d$-flag complex. In a similar way, one can see that $G(I)$ is upper perfect and $sm(S)_d \setminus G(I)$ is lower perfect.

\section{Existence of $(n, d)$th $f$-ideals}

Let $x_{[n]} = x_1 x_2 \cdots x_n$. For a subset $M$ of $sm(S)_d$, denote $M' = \{x_{[n]}/u \mid u \in M\}$. The following lemma is essential in the proof of our main result in this section.

\textbf{Lemma 3.1.} $M$ is a lower (an upper, respectively) perfect subset of $sm(S)_d$ if and only if $M'$ is an upper (a lower, respectively) perfect subset of $sm(S)_{n-d}$.

\textbf{Proof.} For the necessary part, if $M$ is a lower perfect subset of $sm(S)_d$, then it follows from definition that $M'$ is a subset of $sm(S)_{n-d}$. In order to check that $M'$ is upper perfect, we will show for each monomial $u \in sm(S)_{n-d+1}$ that $u \in \sqcup(M')$ holds. This is equivalent to showing that there exists some $v \in M'$, such that $v \mid u$ holds. In fact, since $M$ is lower perfect, for the monomial $u' = x_{[n]}/u \in sm(S)_{d-1}$, there exists some $w \in M$ such that $u' \mid w$ holds. Let $v = x_{[n]}/w$. It is easy to see that $v | u$. Note that $v \in M'$, this shows that $M'$ is upper perfect. In a similar way, one can prove that $M'$ is lower perfect when $M$ is upper perfect. The sufficient part follows from the easy observation that $M'' = M$. \qed

\textbf{Corollary 3.2.} If $I$ is an $(n, d)$th square-free monomial ideal of $S$, then $I$ is an $f$-ideal if and only if $|G(I)| = \binom{n}{d}/2$ and $G(I)'$ is a perfect subset of $sm(S)_{n-d}$.

Denote $sm(S\{k\})_d = \{u \in sm(S)_d \mid x_k \mid u\}$, and $sm(S\{k\})_d = \{u \in sm(S)_d \mid x_k\}$.

Denote $sm(S(X))_d = \{u \in sm(S)_d \mid x_k \mid u \text{ for every } k \in X\}$, and let $sm(S(X))_d = \{u \in sm(S)_d \mid x_k \mid u \text{ for every } k \in X\}$.

\textbf{Definition 3.3.} For a subset $M$ of $sm(S\{k\})_d$, if $sm(S\{k\})_{d+1} \subseteq \sqcup(M)$ holds, then $M$ is called upper perfect without $k$. Dually, a subset $M$ of $sm(S\{k\})_d$ is
called lower perfect without $k$, if $\text{sm}(S\{k\})_{d-1} \subseteq \cap(M)$ holds. A subset $M$ of $\text{sm}(S\{k\})_d$ is called upper perfect containing $k$, if $\text{sm}(S\{k\})_{d+1} \subseteq \cup(M)$ holds; a subset $M$ of $\text{sm}(S\{k\})_d$ is called lower perfect containing $k$, if $\text{sm}(S\{k\})_{d-1} \subseteq \cap(M)$ holds. If $M$ is not only upper but also lower perfect without $k$, then $M$ is called perfect without $k$. Similarly, if $M$ is both upper and lower perfect containing $k$, then $M$ is called perfect containing $k$.

For a subset $X$ of $[n]$, we can define the upper perfect (lower perfect, perfect, respectively) set without $X$ (containing $X$) similarly. For a subset $A$ of $\text{sm}(S)_d$, let $A\{X\} = A \cap \text{sm}(S\{X\})_d$, and let $A\{X\} = A \cap \text{sm}(S\{X\})_d$.

**Proposition 3.4.** Let $A$ be a subset of $\text{sm}(S)_d$, and let $X = \{i_1, \ldots, i_j\}$ be a subset of $[n]$. Then the following statements hold:

1. $A\{X\} = A\{i_1\}\{i_2\} \cdots \{i_j\}$, and $A\{X\} = A\{i_1\}\{i_2\} \cdots \{i_j\}$;
2. If $A$ is upper perfect, then $A\{X\}$ is upper perfect without $X$;
3. If $A$ is lower perfect, then $A\{X\}$ is lower perfect containing $X$;
4. If $A$ is upper (lower, respectively) perfect without $X$, then $A'$ is lower (upper, respectively) perfect containing $X$. Furthermore, the converse also holds true.

**Proof.** (1) and (2) are easy to see by the corresponding definitions.

In order to prove (3), it is sufficient to show that $A\{k\}$ is a lower perfect set containing $k$ for each $k \in [n]$. In fact, since $A$ is lower perfect, for each monomial $u \in \text{sm}(S\{k\})_{d-1}$, there exists a monomial $v$ in $A$ such that $u | v$. Note that $x_k | u$ holds, so $x_k | v$ also holds, which implies that $v \in \text{sm}(S\{k\})_d$ holds. Hence $A\{k\}$ is a lower perfect set containing $k$.

For (4), we only show that $A'$ is lower perfect containing $k$ when $A$ is upper perfect without $k$, and the remaining implications are similar to prove. In fact, for each monomial $u \in \text{sm}(S\{k\})_{n-d-1} \subseteq \text{sm}(S)_{n-d-1}$, $u' = x_{[n]}u \in \text{sm}(S)_{d+1}$. Note that $x_k | u$ implies $x_k \not| u'$ holds true, hence $u' \in \text{sm}(S\{k\})_{d+1}$ also hold. Since $A$ is upper perfect without $k$, there exists a monomial $v \in A$ such that $v \not| u'$ holds, hence $u | v'$ holds, where $v' = x_{[n]}v \in A'$. This completes the proof.

**Remark 3.5.** For a perfect subset $A$ of $\text{sm}(S)_d$, $A\{X\}$ needs not to be a lower perfect set without $X$, and $A\{X\}$ needs not to be an upper perfect set containing $X$, see the following for counterexamples:

**Example 3.6.** Let $S = K[x_1, \ldots, x_n]$, let

$$A = \{x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_3x_4x_5, x_1x_2x_6, x_2x_3x_6, x_2x_4x_6, x_4x_5x_6\},$$

and let $B = A \\backslash \\{x_1x_2x_6\}$. It is easy to see

- $A\{6\} = B\{6\} = \{x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_3x_4x_5\}$,
- $A\{6\} = \{x_1x_2x_6, x_1x_3x_6, x_2x_3x_6, x_4x_5x_6\}$, and
- $B\{6\} = \{x_1x_3x_6, x_2x_3x_6, x_4x_5x_6\}$. 

Also, it is direct to check that both \(A\) and \(B\) are perfect sets, and that both \(A\{6\}\) and \(B\{6\}\) are perfect sets without 6. Note that \(A\{6\}\) is a perfect set containing 6, but \(B\{6\}\) is not upper perfect.

By Proposition 3.4, we have the following example by mapping \(A\), \(B\) to \(A'\), \(B'\) respectively.

**Example 3.7.** Let \(S = K[x_1, \ldots, x_d]\), and let
\[
A' = \{x_1x_2x_3, x_1x_4x_5, x_2x_4x_5, x_3x_4x_5, x_1x_2x_6, x_3x_4x_6, x_4x_5x_6\},
\]
and \(B' = A' \setminus \{x_3x_4x_5\}\). It is easy to see that \(A'\{6\} = \{x_1x_2x_3, x_1x_4x_5, x_2x_4x_5, x_3x_4x_5\}\), \(B'\{6\} = \{x_1x_2x_3, x_1x_4x_5, x_2x_4x_5\}\), and \(A'\{6\} = B'\{6\} = \{x_1x_2x_6, x_3x_4x_6, x_3x_5x_6, x_4x_5x_6\}\). It is direct to check that both \(A'\) and \(B'\) are perfect sets, and that both \(A'\{6\}\) and \(A'\{6\}\) are perfect sets containing 6. Note that \(A'\{6\}\) is a perfect set without 6, but \(B'\{6\}\) is not lower perfect.

In order to obtain the main result of this section, we need a further fact.

**Lemma 3.8.** Let \(S = K[x_1, \ldots, x_n]\), and let \(A\) be a subset of \(sm(S)\). If \(A\{k\}\) is a perfect subset of \(sm(S\{k\})\) without \(k\), and \(A\{k\}\) is a perfect subset of \(sm(S\{k\})\) containing \(k\) for some \(k \in [n]\), then \(A\) is a perfect subset of \(sm(S)\).

**Proof.** In order to show \(A\) is an upper perfect subset of \(sm(S)\), it suffice to show that \(sm(S)_{d+1} \subseteq \sqcup(A)\). Note that \(sm(S)_{d+1} = sm(S\{k\})_{d+1} + sm(S\{k\})_{d+1}\), it suffice to show \(sm(S\{k\})_{d+1} \subseteq \sqcup(A)\) and \(sm(S\{k\})_{d+1} \subseteq \sqcup(A)\). Since \(A\{k\}\) is a perfect subset of \(sm(S\{k\})\) without \(k\), we have
\[
sm(S\{k\})_{d+1} \subseteq \sqcup(A\{k\}) \subseteq \sqcup(A).
\]
Similarly, \(sm(S\{k\})_{d+1} \subseteq \sqcup(A\{k\}) \subseteq \sqcup(A)\). This shows \(A\) is upper perfect. By a similar way, one can check that \(A\) is lower perfect. \(\Box\)

**Theorem 3.9.** For any integer \(d \geq 2\) and any integer \(n \geq d + 2\), there exists an \((n, d)\)th perfect set with cardinality less than or equal to \((\binom{n}{d})/2\).

**Proof.** We prove the result by induction on \(d\).

If \(d = 2\), the conclusion holds true for any integer \(n \geq 4\) by Proposition 1.3. In the following, assume \(d > 2\).

Assume that the conclusion holds true for any integer less than \(d\). For \(d\), we claim that the conclusion holds true for any integer \(n \geq d + 2\). We will show the result by induction on \(n\).

If \(n = d + 2\), then \((\binom{n}{d}) = (\binom{d}{2})\). Note that for any integer \(n \geq 4\), there exists an \((n, 2)\)th perfect set \(M\), such that \(|M| \leq (\binom{n}{2})/2\). By Lemma 3.1, \(M'\) is an \((n, d)\)th perfect set. It is clear that \(|M'| = |M| \leq (\binom{n}{2})/2 = (\binom{d}{2})/2\).

Now assume that the conclusion holds true for any integer less than \(n\). Then by Lemma 3.8, it will suffice to show that there is a perfect subset \(A\)
of $\text{sm}(S\{\hat{n}\})_d$ without $n$ and a perfect subset $B$ of $\text{sm}(S\{n\})_d$ containing $n$, such that $|A| \leq |\text{sm}(S\{\hat{n}\})_d|/2 = \binom{n-1}{d-1}/2$ and $|B| \leq |\text{sm}(S\{n\})_d|/2 = \binom{n-1}{d-1}/2$ hold.

Let $L = K[x_1, \ldots, x_{n-1}]$. Then clearly, $\text{sm}(S\{\hat{n}\})_d = \text{sm}(L)_d$ holds. By induction on $n$, there exists an $(n-1, d)^{th}$ perfect subset $A$ of $\text{sm}(L)_d$, such that $|A| \leq \binom{n-1}{d-1}/2$. It is easy to see that $A$ is a perfect subset of $\text{sm}(S\{\hat{n}\})_d$ without $n$. By induction on $d$, there exists an $(n-1, d-1)^{th}$ perfect subset $B_1$ of $\text{sm}(L)_{d-1}$, such that $|B_1| \leq \binom{n-1}{d-1}/2$ holds. Let $B = \{ux_n \mid u \in B_1\}$. It is easy to see that $B$ is a perfect subset of $\text{sm}(S\{n\})_d$ containing $n$, and $|B| = |B_1| \leq \binom{n-1}{d-1}/2$.

Let $D = A \cup B$. Note that $A = D\{\hat{n}\}$ and $B = D\{n\}$, by Lemma 3.8, $D$ is a perfect subset of $\text{sm}(S)_d$, and $|D| = |A| + |B| \leq \binom{n-1}{d-1}/2 + \binom{n-1}{d-1}/2 = \binom{n}{d}/2$. This completes the proof. $\square$

By Proposition 1.2 and Theorem 3.9, the following corollary is clear.

**Corollary 3.10.** For any integer $d \geq 2$ and any integer $n \geq d + 2$, $V(n, d) \neq \emptyset$ if and only if $2 \mid \binom{n}{d}$.

### 4. Algorithms for constructing examples of $(n, d)^{th}$ f-ideals

In this section, we will show some algorithms to construct $(n, d)^{th}$ f-ideals when $2 \mid \binom{n}{d}$. We discuss the following cases:

**Case 1:** $d = 2$. An $(n, 2)^{th}$ f-ideal is easy to construct by [7]. For reader’s convenience, we repeat it as the following: Decompose the set $[n]$ into a disjoint union of two subsets $B$ and $\overline{B}$ uniformly, namely, $|B| - |\overline{B}| \leq 1$. Then set $A = \{x_i x_j \mid i, j \in B, \text{ or } i, j \in \overline{B}\}$ to obtain an $(n, 2)^{th}$ perfect set. Note that $|A| = N_{(n, 2)} \leq \binom{n}{2}/2$, choose a subset $D$ of $\text{sm}(S)_{d} \setminus A$ randomly, such that $|D| = \binom{n}{2}/2 - N_{(n, 2)}$ holds. It is easy to see that $A \cup D$ is still a perfect set, and $|A \cup D| = \binom{n}{2}/2$. By Proposition 1.2, the ideal generated by $A \cup D$ is an $(n, 2)^{th}$ f-ideal. Note that each $(n, 2)^{th}$ f-ideal can be obtained in this way except $C_5$ by [7].

**Case 2:** $d > 2$ and $n = d + 2$.

**Algorithm 4.1.** In order to build an f-ideal $I \in V(d + 2, d)$, we obey the following steps:

Step 1: Calculate $\binom{d+2}{d}/2$. Note that $\binom{d+2}{d}/2 = \binom{d+2}{d}/2$.

Step 2: As in the case 1, find a perfect subset $B$ of $\text{sm}(S)_2$ such that $|B| \leq \binom{d+2}{2}/2$, where $S = K[x_1, \ldots, x_{d+2}]$.

Step 3: Let $A = B'$. Then $A$ is a perfect subset of $\text{sm}(S)_d$ by Lemma 3.1, and $|A| = |B| \leq \binom{d+2}{2}/2 = \binom{d+2}{d}/2$.

Step 4: Choose a subset $D$ of $\text{sm}(S)_d \setminus A$ randomly, such that $|D| = \binom{d+2}{d}/2 - |A|$ holds. It is easy to see that $M = A \cup D$ is still a perfect set, and $|A \cup D| = \binom{d+2}{d}/2$. 

Step 5: Let $I$ be the ideal generated by $A \cup D$. By Proposition 1.2 again, $I$ is a $(d + 2, d)^{th}$ $f$-ideal.

Note that in this way, we construct almost all $(d + 2, d)^{th}$ $f$-ideals.

**Example 4.2.** Show an $f$-ideal $I \in V(8, 6)$.

Note that $8 = 6 + 2$, we obey the Algorithm 4.1.

Note that \(\binom{8}{3}/2 = 14\). Find a perfect subset $B$ of $sm(S)_2$ such that $|B| \leq \binom{8}{3}/2 = 14$, where $S = K[x_1, \ldots, x_8]$. It is easy to see that

$B = \{x_1, x_2, x_3, x_1x_4, x_2x_5, x_3x_6, x_7x_8, x_6x_7, x_6x_8, x_7x_8\}$

is a perfect subset of $sm(S)_2$, with $|B| = 12$. Let

$A = B' = \{x_1x_2x_3x_6x_7x_8, x_2x_3x_5x_6x_7x_8, x_2x_3x_5x_6x_7x_8, x_1x_2x_3x_6x_7x_8, x_1x_2x_3x_4x_5x_6x_7x_8, x_1x_2x_3x_4x_5x_6x_7x_8, x_1x_2x_3x_4x_5x_6x_7x_8\}$

$A$ is a perfect subset of $sm(S)_6$. Choose $D = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, then the ideal $I$ generated by $A \cup D$ is an $(8, 6)^{th}$ $f$-ideal.

**Case 3:** $d > 2$ and $n > d + 2$. Let $S^{[k]} = K[x_1, \ldots, x_k]$, and let $S = S^{[n]} = K[x_1, \ldots, x_n]$.

**Algorithm 4.3.** For an integer $n > d + 2$, we construct an $(n, d)^{th}$ $f$-ideal by using the following steps:

Step 1: Let $t = n$, $l = d$ and $E = \emptyset$. Set $B = \{B(t, l, E)\}$.

Step 2: Assign $C = B$, and denote $i = |C|$.

Step 3: Choose each $B(t, l, E) \in C$ one by one, deal with each one obeying the following rules:

- If $l = 2$ or $t = l + 2$, don’t change anything.
- If $l \neq 2$ and $t > l + 2$, then cancel $B(t, l, E)$ from $B$, and add $B(t - 1, l, E)$ and $B(t - 1, l - 1, E \cup \{t\})$ into $B$.
- After $i$ times, i.e., when $B(t, l, E)$ goes through all the element of $C$, make a judgement:
  - If $l = 2$ or $t = l + 2$ for each $B(t, l, E) \in B$, then go to Step 4, else return to Step 2.

Step 4: Choose $B(t, l, E) \in B$ one by one, deal with each one obeying the following rules:

- If $l = 2$, assign $B(t, l, E)$ a perfect subset of $sm(S^{[t]}_t)$ as Case 1.
- If $l \neq 2$ and $t = l + 2$, assign $B(t, l, E)$ a perfect subset of $sm(S^{[t]}_t)$ as Case 2.

Step 5: For each $B(t, l, E) \in B$, denote $B^*(t, l, E) = \{ux_E \mid u \in B(t, l, E)\}$, where $x_E = \prod_{j \in E} x_j$. Denote $B^* = \cup_{B(t, l, E) \in B} B^*(t, l, E)$. It is direct to check that $B^*$ is a perfect subset of $sm(S)_d$, and $|B^*| \leq \binom{n}{d}/2$. Choose a subset $D$ of $sm(S)_d \setminus B^*$ randomly, such that $|D| = \binom{n}{d}/2 - |B^*|$ holds.

Step 6: Let $I$ be the ideal generated by $B^* \cup D$. By Proposition 1.2 again, $I$ is an $(n, d)^{th}$ $f$-ideal.
Example 4.4. Show a $(6, 3)^{th}$ $f$-ideal.

Let $S = K[x_1, \ldots, x_6]$. By the above algorithm, we will choose a perfect subset $B(5, 3, \emptyset)$ of $sm(S[5])_3$ and a perfect subset $B(5, 2, \{6\})$ of $sm(S[5])_2$. Set

\[
B(5, 3, \emptyset) = \{x_3x_4x_5, x_2x_4x_5, x_1x_4x_5, x_1x_2x_3\}
\]

and

\[
B(5, 2, \{6\}) = \{x_1x_2, x_1x_3, x_2x_3, x_4x_5\}.
\]

Correspondingly,

\[
B^*(5, 3, \emptyset) = B(5, 3, \emptyset)
\]

and

\[
B^*(5, 2, \{6\}) = \{x_1x_2x_6, x_1x_3x_6, x_2x_3x_6, x_4x_5x_6\}.
\]

Hence

\[
B^* = \{x_3x_4x_5, x_2x_4x_5, x_1x_4x_5, x_1x_2x_3, x_1x_2x_6, x_1x_3x_6, x_2x_3x_6, x_4x_5x_6\}
\]

is a perfect subset of $sm(S)_3$. Note that $\binom{6}{3}/2 = 10$, and $|B^*| = 8$. Set

\[
D = \{x_1x_2x_4, x_1x_2x_5\}.
\]

The ideal $I$ generated by $B^* \cup D$ is a $(6, 3)^{th}$ $f$-ideal.

Note that the $(6, 3)^{th}$ $f$-ideal given in the above example is not unmixed. In fact, consider the simplicial complex $\sigma(sm(S)_3 \setminus G(I))$, and note that $\{1, 2\}$ is a nonface of $\sigma(sm(S)_3 \setminus G(I))$, which implies that $\sigma(sm(S)_3 \setminus G(I))$ is not a 3-flag complex. So, $I$ is not unmixed by Proposition 2.1.

5. An upper bound of the perfect number $N_{(n,d)}$

For a positive integer $k$ and a pair of positive integers $i \leq j$, denote by $Q^k_{i,j}$ the set of square-free monomials of degree $k$ in the polynomial ring $K[x_i, x_{i+1}, \ldots, x_j]$. Note that $Q^k_{i,j} = \emptyset$ holds for $i > j$. For a pair of monomial subsets $A$ and $B$, denote by $A \bullet B = \{uv \mid u \in A, v \in B\}$. If $B = \emptyset$, then assume $A \bullet B = A$. The following theorem gives an upper bound of the $(n, d)^{th}$ perfect number for $n > d + 2$.

Theorem 5.1. Given a integer $d > 2$, and a integer $n \geq d + 2$. The following statements about the perfect number $N_{(n,d)}$ hold:

1. If $n = d + 2$, then

\[
N_{(n,d)} = N_{(n,2)} = \begin{cases} 
  k^2 - k, & \text{if } n = 2k; \\
  k^2, & \text{if } n = 2k + 1.
\end{cases}
\]

2. If $n > d + 2$, then

\[
N_{(n,d)} \leq \sum_{i=5}^{n-d+2} N_{(i,2)} \binom{n-i-1}{d-3} + \sum_{j=3}^{d} N_{(j+2,2)} \binom{n-j-3}{d-j},
\]

where $\binom{n}{0} = 1$. 
Proof. By Lemma 3.1 and the equation (1.1) in Section 1, (1) is clear.

In order to prove (2), it will suffice to show that there exists a perfect set with cardinality \( t = \sum_{i=5}^{n-d+2} N(i,2) \binom{n-i-1}{d-3} + \sum_{j=3}^{d} N(j+2,2) \binom{n-j-3}{d-j} \).

Let \( P_{(i,2)} \) be an \((i,2)^{th}\) perfect set with cardinality \( N(i,2) \) for \( 5 \leq i \leq n-d+2 \), and let \( P_{(j+2,2)} \) be a \((j+2,2)^{th}\) perfect set with cardinality \( N(j+2,2) \) for \( 3 \leq j \leq d \). We claim that the set

\[
M = \left( \bigcup_{i=5}^{n-d+2} P_{(i,2)} \right) \cdot x_{i+1} \cdot Q_{(i+2,n)}^{d-3} \cup \left( \bigcup_{j=3}^{d} P_{(j+2,2)} \right) \cdot Q_{(j+4,n)}^{d-j}
\]

is an \((n,d)^{th}\) perfect set, with cardinality \( t \). It is easy to check that the cardinality of \( M \) is \( t \). It is only necessary to prove that \( M \) is perfect.

For each \( w \in \text{sm}(S)_{d+1} \), denote by \( n_{k}(w) \) the cardinality of the set \( \{ x_{i} \mid i \leq k \text{ and } x_{i} \mid w \} \). If \( n_{k}(w) \geq 4 \), then choose the smallest \( k \) such that \( n_{k+3}(w) = n_{k+2}(w) = k + 1 \). Clearly, \( 3 \leq k \leq d \). It is direct to check that \( w \) is divided by some monomial in \( P_{(k+2,k)} \cdot Q_{[k+4,n]}^{d-k} \). If \( n_{k}(w) \leq 3 \), then choose the smallest \( k \) such that \( n_{k}(w) = 3 \) and \( n_{k+1}(w) = 4 \). Clearly, \( 5 \leq k \leq n - d + 2 \). It is not hard to check that \( w \) is divided by some monomial in \( P_{(k+2,k)} \cdot x_{k+1} \cdot Q_{[k+4,n]}^{d-k} \).

Hence \( M \) is upper perfect.

For each \( w \in \text{sm}(S)_{d-1} \), if \( n_{k}(w) \geq 2 \), then choose the smallest \( k \) such that \( n_{k+3}(w) = n_{k+2}(w) = k - 1 \). Clearly, \( 3 \leq k \leq d \). It is direct to check that \( w \) divides some monomial in \( P_{(k+2,k)} \cdot Q_{[k+4,n]}^{d-k} \). If \( n_{k}(w) \leq 1 \), then choose the smallest \( k \) such that \( n_{k}(w) = 1 \) and \( n_{k+1}(w) = 2 \). Clearly, \( 5 \leq k \leq n - d + 2 \). It is not hard to check that \( w \) divides some monomial in \( P_{(k+2,k)} \cdot x_{k+1} \cdot Q_{[k+4,n]}^{d-k} \). Hence \( M \) is lower perfect.

\( \square \)

Figure 1 may help to interpret the above theorem intuitively. In this figure, there is a boundary consisting of the line \( l = 2 \) and the line \( t = l + 2 \). From the point \((d,n)\) to a point of the boundary, every directed chain \( C \) denotes a set of monomials \( M(C) \) by the following rules:

1. Every arrow of \( C \) is from \((l,t)\) to either \((l,t-1)\) or \((l-1,t-1)\).
2. If the arrow is from \((l,t)\) to \((l,t-1)\), then each monomial in \( M(C) \) is not divided by \( x_{t} \). Correspondingly, if it is from \((l,t)\) to \((l-1,t-1)\), then each monomial in \( M(C) \) is divided by \( x_{t} \).
3. Each point \((l,t)\) of the boundary is a \((l,t)^{th}\) perfect set.

Actually, the figure shows us a class of \((n,d)^{th}\) perfect sets. If we choose each point \((l,t)\) of the boundary to be a \((l,t)^{th}\) perfect set with cardinality \( N(l,t) \), then the cardinality of the \((n,d)^{th}\) perfect set corresponding to this figure is exactly

\[
\sum_{i=5}^{n-d+2} N(i,2) \binom{n-i-1}{d-3} + \sum_{j=3}^{d} N(j+2,2) \binom{n-j-3}{d-j}
\]

Example 5.2. Calculation of the \((6,3)^{th}\) perfect number.

Let \( A \) be a \((6,3)^{th}\) perfect set. By Proposition 3.4(3), \( A(6) \) is a lower perfect set containing 6. Note that for the monomials of \( \{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\} \), each monomial in \( A(6) \) is divided by at most two of them. So, \( |A(6)| \geq 3 \). By Proposition 3.4(2), \( A(6) \) is an upper perfect set without 6. As the discussion
above, $|A\{\hat{6}\}| \geq 3$. Hence $|A| \geq |A\{\hat{6}\}| + |A\{6\}| \geq 6$. Actually, it is direct to check that the following set

$$B = \{x_1x_2x_3, x_1x_2x_4, x_3x_4x_5, x_1x_3x_6, x_2x_3x_6, x_5x_4x_6\}$$

is a $(6,3)^{th}$ perfect set with cardinality 6. Thus $N_{(6,3)} = 6$. Note that the upper bound given by Proposition 5.1(2) is 8, and is not bad for the perfect number in the case.

6. Nonhomogeneous $f$-ideal

In [7], a characterization of $f$-ideals in general case is shown, but it is still not easy to show an example of nonhomogeneous $f$-ideal, i.e., the $f$-ideal $I$ with the property that monomials in $G(I)$ do not have a same degree. In fact, the interference from monomials of different degree makes the computation complicated. Anyway, we finally worked out the following example:

**Example 6.1.** Let $S = K[x_1, x_2, x_3, x_4, x_5]$, and let

$$I = (x_1x_2, x_3x_4, x_1x_3x_5, x_2x_4x_5).$$
It is direct to check that
\[ \delta_F(I) = \langle \{1, 2\}, \{3, 4\}, \{1, 3, 5\}, \{2, 4, 5\} \rangle \]
and
\[ \delta_N(I) = \langle \{1, 3\}, \{2, 4\}, \{1, 4, 5\}, \{2, 3, 5\} \rangle. \]
It is easy to see they have the same \( f \)-vector, and hence \( I \) is an \( f \)-ideal, which is clearly nonhomogeneous.

After this nontrivial example, clearly there are a lot of nonhomogeneous \( f \)-ideals. We will show another example to end this paper.

**Example 6.2.** Let \( S = K[x_1, x_2, x_3, x_4, x_5, x_6] \), and let
\[ I = \langle x_1x_2, x_2x_3, x_1x_3x_4x_5, x_1x_4x_6, x_1x_5x_6, x_2x_4x_6 \rangle. \]
Note that
\[ \delta_N(I) = \langle \{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 4\}, \{2, 5, 6\}, \{3, 4, 6\}, \{3, 5, 6\} \rangle. \]
It is direct to check that \( I \) is also a nonhomogeneous \( f \)-ideal.

**References**

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