ON THE CONVERGENCE OF SERIES OF MARTINGALE DIFFERENCES WITH MULTIDIMENSIONAL INDICES

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Abstract. Let \( \{X_n; n \geq 1\} \) be a field of martingale differences taking values in a \( p \)-uniformly smooth Banach space. The paper provides conditions under which the series \( \sum_{i \geq n} X_i \) converges almost surely and the tail series \( \{T_n = \sum_{i \geq n} X_i; n \geq 1\} \) satisfies \( \sup_{k \geq n} \|T_k\| = \mathcal{O}(b_n) \) and \( \sup_{k \geq n} \|T_k\| B_n \xrightarrow{P} 0 \) for given fields of positive numbers \( \{b_n\} \) and \( \{B_n\} \). This result generalizes results of A. Rosalsky, J. Rosenblatt [7], [8] and S. H. Sung, A. I. Volodin [11].

1. Introduction

Let \( \{X_n; n \geq 1\} \) be a sequence of random variables taking values in a real separable Banach space \( \mathbb{E} \) with norm \( \|\cdot\| \). If the series \( \sum_{i=1}^{\infty} X_i \) converges a.s., then the tail series
\[
T_n = \sum_{i=n}^{\infty} X_i, \quad n \geq 1
\]
is well-defined and \( \sup_{k \geq n} \|T_k\| = \mathcal{O}(b_n) \) as \( n \to \infty \).

A. Rosalsky, J. Rosenblatt [7], [8] and S. H. Sung, A. I. Volodin [11] investigated the rate in which \( \sup_{k \geq n} \|T_k\| = \mathcal{O}(b_n) \) and \( \sup_{k \geq n} \|T_k\| B_n \xrightarrow{P} 0 \), where \( \{b_n\} \) and \( \{B_n\} \) are given sequences of positive numbers.

The aim of this paper is to extend these results to the case where \( \{X_n, F_n; n \in \mathbb{N}^d\} \) is a field of \( \mathbb{E} \)-valued martingale differences. Under the assumption that \( \mathbb{E} \) is a \( p \)-uniformly smooth Banach space for some \( 1 \leq p \leq 2 \), we will provide sufficient conditions ensuring that \( \sum_{n \geq 1} X_n \) converges, \( \sup_{k \geq n} \|T_k\| = \mathcal{O}(b_n) \), \( \sup_{k \geq n} \|T_k\| B_n \xrightarrow{P} 0 \) and \( \sum_{n \geq 1} 1/n^p \mathbb{P}(\sup_{k \geq n} \|T_k\| > \varepsilon a_n) < \infty \) for every \( \varepsilon > 0 \), where \( T_n = S - \sum_{i \leq n} X_i \), \( \{b_n\}, \{B_n\}, \{a_n\} \) are given fields of positive numbers.

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2. Preliminaries and some useful lemmas

Throughout this paper, the symbol $C$ will denote a generic constant $(0 < C < \infty)$ which is not necessarily the same one in each appearance.

Let $E$ be a real separable Banach space. For a $E$-valued random variable $X$ and sub $\sigma$-algebra $G$ of $F$, the conditional expectation $E(X|G)$ is defined analogously to that in the real-random variable case and enjoys similar properties (see [10]).

$E, \| \cdot \|$ is said to be $p$-uniformly smooth $(1 \leq p \leq 2)$ if there exists a finite positive constant $C$ such that for all $E$-valued martingales $\{S_n; 1 \leq n \leq m\}$

\[
E\|S_m\|^p \leq C \sum_{n=1}^{m} E\|S_n - S_{n-1}\|^p.
\]

Clearly, every separable Banach space is of 1-uniformly smooth. If a real separable Banach space is of $p$-uniformly smooth for some $1 < p \leq 2$, then it is of $r$-uniformly smooth for all $r \in [1, p)$. A Hilbert space is 2-uniformly smooth and the space $L_p$ is $\min\{p, 2\}$-uniformly smooth (see [5], [6]).

Let $d$ be a positive integer, the set of all nonnegative integer $d$-dimensional lattice points will be denoted by $\mathbb{N}_0^d$ and the set of all positive integer $d$-dimensional lattice points will be denoted by $\mathbb{N}_0^d$. For $m = (m_1, \ldots, m_d) \in \mathbb{N}_0^d$, $n = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$, $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, denote $m + n = (m_1 + n_1, \ldots, m_d + n_d)$, $m - n = (m_1 - n_1, \ldots, m_d - n_d)$, $|n| = n_1 \cdot n_2 \cdots n_d$, $\|n\| = \min\{n_1, n_2, \ldots, n_d\}$, $1 = (1, \ldots, 1) \in \mathbb{N}_0^d$, $\bigvee_{i=1}^{d}(m_i < n_i)$ means that there is at least one of $m_1 < n_1, m_2 < n_2, \ldots, m_d < n_d$ holds. We write $m \preceq n$ (or $n \succeq m$) if $m_i \leq n_i$, $1 \leq i \leq d$; $m \prec n$ if $m \preceq n$ and $m \neq n$; $m \ll n$ (or $n \gg m$) if $\bigvee_{i=1}^{d}(m_i < n_i)$.

Let $(\Omega, F, P)$ be a probability space, $E$ be a real separable Banach space, $B(E)$ be the $\sigma$-algebra of all Borel sets in $E$. Let $\{X_n, 1 \preceq m \preceq n \preceq M\}$ be a (d-dimensional) field of $E$-valued random variables and $\{F_n, m \preceq n \preceq M\}$ be a (d-dimensional) field of nondecreasing sub-$\sigma$-algebras of $F$ with respect to the partial order $\preceq$ on $\mathbb{N}_0^d$ such that $X_n$ is $F_n$-measurable for all $m \preceq n \preceq M$, then $\{X_n, F_n, m \preceq n \preceq M\}$ is said to be an adapted field.

Let $\{X_n, F_n, m \preceq n \preceq M\}$ be an adapted field. For $m \in \mathbb{N}_0^d$ ($m - 1 \preceq n \preceq M - 1$), we adopt the convention that $F_n = \{\emptyset, \Omega\}$ if there exists a positive integer $i$ ($1 \leq i \leq d$) such that $n_i = m_i - 1$ and set

$\mathcal{F}_n^{(i)} = \sigma\{F_k : k = (k_1, \ldots, k_d), m_j \leq k_j \leq M_j (j \neq i), \text{ and } k_i = n_i\}$

for all $1 \leq i \leq d$, and $\mathcal{F}_n^* = \sigma\{\mathcal{F}_n^{(i)} : 1 \leq i \leq d\}$.

The adapted field $\{X_n, F_n, m \preceq n \preceq M\}$ is said to be a field of martingale differences if $E(X_n|\mathcal{F}_{n-1}^{(i)}) = 0$ for all $m \preceq n \preceq M$, $1 \leq i \leq d$ (see [3]).

The adapted field $\{X_n, F_n, m \preceq n \preceq M\}$ is said to be strong adapted if $E(X_n|\mathcal{F}_{n-1}^{(i)})$ is $\mathcal{F}_n^{(i)}$-measurable for all $m \preceq n \preceq M$, $1 \leq i \leq d$. 
Remark 2.1. Let \( \{X_n, \mathcal{F}_n : m \leq n \leq M\} \) be a field of martingale differences. Then it is strong adapted. Conversely, let \( \{X_n, \mathcal{F}_n : m \leq n \leq M\} \) be strong adapted, when \( d = 1 \) then \( \{X_n - E(X_n|\mathcal{F}_{n-1}) : \mathcal{F}_n : m \leq n \leq M\} \) is a sequence of martingale differences, but when \( d > 1 \) then \( \{X_n - E(X_n|\mathcal{F}_{n-1}) : \mathcal{F}_n : m \leq n \leq M\} \) is not necessarily a field of martingale differences, because \( X_n - E(X_n|\mathcal{F}_{n-1}) \) may not be \( \mathcal{F}_n \)-measurable.

The adapted field \( \{X_n, \mathcal{F}_n, m \leq n \leq M\} \) is said to be strong* adapted if \( \{X_n \mathcal{I}_A, \mathcal{F}_n, m \leq n \leq M\} \) is strong adapted for all \( A \in \sigma(X_n) \).

Clearly, when \( \{X_n, \mathcal{F}_n, m \leq n \leq M\} \) is a field of martingale differences, then it is not necessarily a strong* adapted field.

The adapted field \( \{X_n, \mathcal{F}_n, m \leq n \leq M\} \) is said to be a field of strong* martingale differences if it is a strong* adapted and a field of martingale differences.

Remark 2.2. Let \( \{X_n, \mathcal{F}_n : m \leq n \leq M\} \) be a sequence of martingale differences, by \( E(X_n I_A|\mathcal{F}_{n-1}) = E(X_n I_A|\mathcal{F}_{n-1}) \in \mathcal{F}_{n-1} = \mathcal{F}_{n-1}(1) \). Then \( \{X_n, \mathcal{F}_n : m \leq n \leq M\} \) is a strong* martingale differences.

Example 1. Let \( \{X_n, n \in \mathbb{N}^d\} \) be a field of independent random variables with mean 0. Put \( \mathcal{F}_n = \sigma(X_k, k \leq n) \), then \( E(X_n|\mathcal{F}_{n-1}) = 0 \), \( E(X_n I_A|\mathcal{F}_{n-1}) = 0 \) for all \( A \in \sigma(X_n) \) and \( n \in \mathbb{N}^d \), \( 1 \leq i \leq d \). Therefore, \( \{X_n, \mathcal{F}_n, n \in \mathbb{N}^d\} \) is a field of strong* martingale differences.

Example 2. Let \( \{X_n, \mathcal{G}_n : n \geq 1\} \) be a sequence of martingale differences and set

\[
X_n = X_n \quad \text{if} \quad n = (n, n, \ldots, n) \quad \text{and} \quad X_n = 0 \quad \text{if} \quad n \neq (n, n, \ldots, n);
\]

\[
\mathcal{G}_n = \mathcal{G}_n \quad \text{if} \quad n = (n, n, \ldots, n) \quad \text{and} \quad \mathcal{G}_n = \{\emptyset, \Omega\} \quad \text{if} \quad n \neq (n, n, \ldots, n).
\]

Let \( \mathcal{F}_n = \sigma(\mathcal{G}_k, k \leq n) \) for all \( n \geq 1 \). Then \( \{X_n, \mathcal{F}_n : n \geq 1\} \) is a field of martingale differences. Moreover, for all \( n \geq 1 \), then

\[
E(X_n I_A|\mathcal{F}_{n-1}) = E(X_n I_A|\mathcal{G}_{n-1}) \in \mathcal{G}_n = \mathcal{F}_n
\]

if \( n = (n, n, \ldots, n) \) and \( E(X_n I_A|\mathcal{F}_{n-1}) = 0 \) \( \mathcal{F}_n \) if otherwise, for all \( A \in \sigma(X_n) \), so \( \{X_n, \mathcal{F}_n : n \geq 1\} \) is a field of strong* martingale differences.

Example 3. Let \( \{Y_n, n \in \mathbb{N}^d\} \) be a field of independent random variables with mean 0. Put \( \mathcal{F}_n = \sigma(Y_k, k \leq n) \) and \( X_n = \prod_{k \leq n} Y_k \), so \( \{X_n, n \in \mathbb{N}^d\} \) is not a field of independent random variables. If \( E X_n < \infty \) for all \( n \geq 1 \), then \( E(X_n|\mathcal{F}_n) = 0 \), \( E(X_n I_A|\mathcal{F}_{n-1}) = E(X_n I_A|\mathcal{F}_{n-1}) \in \mathcal{F}_n \) for all \( A \in \sigma(X_n) \), \( n \geq 1 \), \( 1 \leq i \leq d \). Therefore, \( \{X_n, \mathcal{F}_n, n \in \mathbb{N}^d\} \) is a field of strong* martingale differences.

In the sequel the following lemmas are useful.

Lemma 2.3. Let \( \mathbb{E} \) be a real separable \( p \)-uniformly smooth Banach space for some \( 1 \leq p \leq 2 \). Then there exits a positive constant \( C \) such that for all strong
adapted fields of $\mathbb{E}$-valued random variables $\{X_n, F_n : 1 \leq n \leq m\}$, we have

\[ E \max_{1 \leq k \leq m} \left\| \sum_{1 \leq j \leq k} (X_1 - E(X_j | F_{k-1}^n)) \right\|^p \leq C \sum_{1 \leq k \leq m} E \|X_k\|^p. \]

**Proof.** We will show that (2.2) holds by induction. Set

\[ S_k = \sum_{1 \leq j \leq k} (X_1 - E(X_j | F_{k-1}^n)). \]

Firstly, for $d = 1$, note that $\{\max_{1 \leq k \leq m} \|S_k\|, F_k : 1 \leq k \leq m\}$ is a nonnegative sub-martingale. Applying Doob’s inequality and by (2.1), we have (2.2). We assume that (2.2) holds for $d - 1$; we will show that it holds for $d$.

Denote

\[ k = (k_1, \ldots, k_{d-1}, k_d) = (k', k_d); \quad m = (m_1, \ldots, m_{d-1}, m_d) = (m', m_d); \]

with $k', m' \in \mathbb{N}^{d-1}$; set $Y_{k_d} = \max_{1 \leq k_d \leq m_d} \|S(k'; k_d)\|$ for each $1 \leq k_d \leq m_d$, we have

\[
E(S(k'; k_d) | F^d_{k'; k_d}) = E(S(k'; k_{d-1}) | F^d_{k'; k_{d-1}}) + \sum_{1 \leq m' \leq m} \left( E(X_{k'; k_d}) - E(X_{k'; k_d} | F^d_{k'; k_{d-1}} | F^d_{k'; k_{d-1}}) \right)
\]

and by $\{S(k'; k_d); F^d_{k'; k_d} : 1 \leq k_d \leq m_d\}$ being a strong adapted random field, it means that for each $1 \leq k' \leq m'$ then $\{S(k'; k_d); F^d_{k'; k_d} : 1 \leq k_d \leq m_d\}$ is a martingale, and we have that $\{Y_{k_d}; F^d_{k'; k_d} : 1 \leq k_d \leq m_d\}$ is a nonnegative sub-martingale sequence. Applying Doob’s inequality, we obtain

\[
E\left( \max_{1 \leq (k', k_d) \leq m} \|S(k'; k_d)\|^p \right) = E\left( \max_{1 \leq k_d \leq m_d} Y_{k_d}^p \right) \leq C \cdot EY_{m_d}^p
\]

Set

\[ X_{k'}^{d-1} = \sum_{1 \leq k_d \leq m_d} X_{k'; k_d} | F_{k'}^{d-1} = \sigma(F^{d-1}_{k'; k_d} : 1 \leq k_d \leq m_d). \]

Note that $F_{k'}^{d-1}, F_{k'}^n \in \mathbb{F}^d_{k'} - \{X_{k'}^{d-1}, X_{k'}^n \} \subseteq \mathbb{F}^{d-1}_{k'}$ for all $1 \leq k_d \leq m_d, 1 \leq i \leq d - 1$, then $\{X_{k'}^{d-1}, F_{k'}^{d-1} : 1 \leq k' \leq m'\}$ is a strong adapted field. Therefore, by the induction assumption,

\[
E \max_{1 \leq (k', k_d) \leq m} \|S(k'; k_d)\|^p \leq C \sum_{1 \leq k' \leq m'} E \|X_{k'}^{d-1}\|^p + \sum_{1 \leq k_d \leq m_d} E \left( X_{k'}^{d-1} - E(X_{k'; k_d} | F^d_{k'; k_{d-1}}) \right)^p \leq C \sum_{1 \leq k' \leq m'} E \|X_{k'}\|^p, \quad \text{by (1.1).} \]

\[ \square \]
Remark 2.4. If \( \{ X_n; n \geq 1 \} \) is a \( \mathbb{E} \)-valued martingale difference field, from Lemma 2.3, we obtain Lemma 1.1 in [3] (for \( p = q \)). Moreover, by Remark 2.1, Lemma 2.3 is stronger than Lemma 1.1 in [3] (for \( p = q \)).

Lemma 2.5. Let \( \{ X_n; n \geq 1 \} \) be a field of \( \mathbb{E} \)-valued random variables. Then

\[
(2.3) \quad P \left( \sup_{k \geq m} \|X_k\| > \epsilon \right) = \lim_{n \to \infty} P \left( \max_{m \leq k \leq n} \|X_k\| > \epsilon \right),
\]

\[
(2.4) \quad P \left( \lim_{n \to \infty} \inf \|X_n\| > \epsilon \right) \leq \lim_{n \to \infty} P (\|X_n\| > \epsilon).
\]

Proof. 1. Remark that for \( d = 1 \), by the continuity from below theorem, we have (2.3). Assume that (2.3) holds for \( d = D-1 \geq 1 \), we with to show that for \( d = D \). Let \( m = (m_1, m_2, \ldots, m_d) = (m_1, m_1), \) \( k = (k_1, k_2, \ldots, k_d) = (k_1, k_1), \) \( n = (n_1, n_2, \ldots, n_d) = (n_1, n_1), \) by the continuity from below theorem, we have

\[
P \left( \sup_{k \geq m} \|X_k\| > \epsilon \right) = P \left( \lim_{n_1 \to \infty} \sup_{k_1 \geq m_1} \max_{m_1 \leq k_1 \leq n_1} \|X_k\| > \epsilon \right) = \lim_{n_1 \to \infty} P \left( \sup_{k_1 \geq m_1} \max_{m_1 \leq k_1 \leq n_1} \|X_k\| > \epsilon \right).
\]

By the induction assumption,

\[
P \left( \sup_{k \geq m} \|X_k\| > \epsilon \right) = \lim_{n_1 \to \infty} \frac{P \left( \sup_{k \geq m} \|X_k\| > \epsilon \right)}{\|n_1\| \to \infty} = \lim_{n_1 \to \infty} P \left( \max_{m \leq k \leq n} \|X_k\| > \epsilon \right).
\]

2. By Theorem 8.1.3 of Chow and Teicher [1] and the same argument as in the proof of (2.3), we have (2.4). □

Lemma 2.6. Let \( \{ X_n; n \geq 1 \} \) be a field of \( \mathbb{E} \)-valued random variables. Then,

\( X_n \) converges a.s. as \( \|n\| \to \infty \) if only if for all \( \epsilon > 0, \)

\[
(2.5) \quad \lim_{\|n\| \to \infty} P \left( \sup_{k \geq 0} \|X_{n+k} - X_n\| > \epsilon \right) = 0.
\]

Proof. Necessity. Suppose that \( X_n \to X \) a.s. as \( \|n\| \to \infty \). Then (2.5) holds, by the following inequality

\[
\sup_{k \geq 0} \|X_{n+k} - X_n\| \leq \sup_{m \geq n} \|X_m - X\| + \|X_n - X\|.
\]

Sufficiency. Suppose (2.5) holds, let \( n' = (n, n, \ldots, n) \), \( k' = (k, k, \ldots, k) \), we have \( n \to \infty \) if and only if \( \|n'\| \to \infty \). Set \( Y_n = X_{n'} \) for all \( n \geq 1 \). Then for an arbitrary \( \epsilon > 0, \)

\[
\lim_{n \to \infty} P \left( \sup_{k \geq 0} \|Y_{n+k} - Y_n\| > \epsilon \right) = \lim_{\|n'\| \to \infty} P \left( \sup_{k' \geq 0} \|X_{n'+k'} - X_{n'}\| > \epsilon \right) = 0
\]
which implies that $Y_n$ converges a.s. to a certain random variable $X$ as $n \to \infty$, i.e., $X_n'$ converges a.s. to $X$ as $\|n\| \to \infty$. Now we prove $X_n \to X$ a.s. as $\|n\| \to \infty$.

For an arbitrary $\varepsilon > 0$,

$$P \left( \sup_{n \geq n'} \|X_n - X\| > \varepsilon \right) \leq P \left( \sup_{n \geq n'} \|X_n - X_n'\| > \varepsilon/2 \right) + P (\|X_n' - X\| > \varepsilon/2) \to 0 \text{ as } n \to \infty,$$

so $X_n \to X$ a.s. as $\|n\| \to \infty$.  

\[\Box\]

3. Main results

Let $\{X_n, n \geq 1\}$ be a field of random variables in Banach space $E$. Put $S_n = \sum_{k \leq n} X_k$ for all $n \geq 1$. The series $\sum_{n \geq 1} X_n$ is said to converge a.s. if the field of $E$-valued random variables $\{S_n, n \geq 1\}$ converges a.s.. In this case, put

$$S = \lim_{\|n\| \to \infty} S_n$$

and

$$T_n = S - S_n = \sum_{k > n} X_k$$

(set $S_0 = 0$). We have

$$\sup_{k \geq n} \|T_k\| \overset{P}{\to} 0 \quad \text{as } \|n\| \to \infty.$$

The following theorems provide sufficient conditions a.s. for the convergence of $\sum_{n \geq 1} X_n$ as well as the rate of convergence to 0 of $\sup_{k \geq n} \|T_k\|$.

\begin{theorem}
Let $E$ be a $p$-uniformly smooth Banach space for some $1 \leq p \leq 2$, $\{X_n, \mathcal{F}_n : n \in \mathbb{N}\}$ be a field of $E$-valued martingale differences. Let $\{b_n\}, \{B_n\}$ be fields of positive constants, such that $b_n = o(1)$, $B_n = o(1)$ as $\|n\| \to \infty$.

(1) If

$$\sum_{k > n} E\|X_k\|^p = O(b_n^p),$$

then $\sum_{n \geq 1} X_n$ converges a.s. and

$$\sup_{k \geq n} \|T_k\| = O_P(b_n).$$

(2) If

$$\sum_{k > n} E\|X_k\|^p = o(B_n^p)$$

then $\sum_{n \geq 1} X_n$ converges a.s. and

$$\sup_{k \geq n} \|T_k\| = O_P(b_n^p).$$

\end{theorem}
as \( \|n\| \to \infty \), then \( \sum_{n \geq 1} X_n \) converges a.s. and

\[
(3.4) \quad \frac{\sup_{k \geq n} \|T_k\|}{b_n} \to 0
\]
as \( \|n\| \to \infty \).

Proof. (1) Set \( S_n = \sum_{k \geq n} X_k \). For an arbitrary \( \varepsilon > 0 \), set \( n = (n_1, \ldots, n_i, \ldots, n_d) \), \( k = (k_1, \ldots, k_i, \ldots, k_d) = (k_i, k_i', \ldots, k_i') \), \( j = (j_1, \ldots, j_i, \ldots, j_d) = (j_i, j_i, \ldots, j_i) \) for all \( 1 \leq i \leq d \). We have that

\[
P \left( \sup_{k \geq 0} \| S_{n+k} - S_n \| > \varepsilon \right)
= P \left( \sup_{k \geq 0} \| \sum_{n \leq j \leq n+k} X_j \| > \varepsilon \right)
\leq \sum_{i=1}^d P \left( \sup_{k \geq 0} \left( \sum_{1 \leq j \leq n_i, j_i=n_i, 1 \leq j_i' \leq n_i'+k_i'} X_{(j_i, j_i', j_i')} \right) > \varepsilon/d \right).
\]

Applying the Markov inequality and Lemma 2.3, we obtain

\[
P \left( \sup_{k \geq 0} \| \sum_{1 \leq j \leq n_i, j_i=n_i, 1 \leq j_i' \leq n_i'+k_i'} X_{(j_i, j_i', j_i')} \| > \varepsilon/d \right)
\leq \frac{\varepsilon^p}{\varepsilon^p} E \left( \sum_{k \geq 0} \| \sum_{1 \leq j \leq n_i, j_i=n_i, 1 \leq j_i' \leq n_i'+k_i'} X_{(j_i, j_i', j_i')} \|^p \right)
\leq C \sum_{1 \leq j \leq n_i, j_i=n_i, 1 \leq j_i' \leq n_i'+k_i'} E \| X_{(j_i, j_i', j_i')} \|^p.
\]

Then, using (3.1) or (3.3), we have that

\[
P \left( \sup_{k \geq 0} \| S_{n+k} - S_n \| > \varepsilon \right) \leq C \sum_{i=1}^d \sum_{1 \leq j \leq n_i, j_i=n_i, 1 \leq j_i' \leq n_i'+k_i'} E \| X_{(j_i, j_i', j_i')} \|^p
= C \sum_{i \geq n} E \| X_j \|^p = o(1) \text{ as } \|n\| \to \infty,
\]
which implies that \( S_n \) converges a.s as \( \|n\| \to \infty \) (by Lemma 2.6). Then \( \sum_{n \geq 1} X_n \) converges a.s. Thus, the tail series \( \{ T_n = \sum_{k \geq n} X_k \} \) is a well-defined field of random variables.

Next, to prove that (3.1) implies (3.2), observe that for \( K > 0 \)

\[
sup_{n \geq 1} P \left( \frac{\sup_{k \geq n} \| T_k \|}{b_n} > K \right)
\]
\[
\sup_{n \geq 1} P \left( \sup_{k \geq n} \left\| \sum_{i \geq k} X_i \right\| > K b_n \right) \\
= \sup_{n \geq 1} \lim_{N \to \infty} P \left( \max_{n \leq k \leq N} \left\| \sum_{i \geq k} X_i \right\| > K b_n \right) \quad \text{(by Lemma 2.5)}
\]
\[
\leq \sup_{n \geq 1} \lim_{N \to \infty} P \left( \max_{n \leq k \leq N} \left\| \sum_{i \geq k} X_i \right\| > K b_n \right)
\]
\[
\leq \sup_{n \geq 1} \liminf_{M \to \infty} P \left( \max_{n \leq k \leq M} \left\| \sum_{i \geq k} X_i \right\| > K b_n \right) \quad \text{(by Lemma 2.5)}
\]
\[
\leq \sup_{n \geq 1} \liminf_{M \to \infty} P \left( \sum_{n \leq k \leq M} \left\| \sum_{i \geq k} X_i \right\| > K b_n \right)
\]
\[
\leq 2 \sup_{n \geq 1} \lim_{M \to \infty} P \left( \sum_{n \leq k \leq M} \left\| \sum_{i \geq k} X_i \right\| > K b_n \right)
\]
\[
\leq \sup_{n \geq 1} \frac{2^{p+1}}{K^p b_n^p} \lim_{M \to \infty} E \max_{n \leq k \leq M} \left\| \sum_{i \geq 1 \leq k} X_i \right\|^p \quad \text{(by the Markov inequality)}
\]
\[
\leq \sup_{n \geq 1} \frac{2^{p+1}}{K^p b_n^p} \lim_{M \to \infty} \sum_{n \leq k \leq M} E \|X_i\|^p \quad \text{(by Lemma 2.3)}
\]
\[
= \sup_{n \geq 1} \frac{C}{K^p b_n^p} \sum_{n \leq k \leq M} E \|X_i\|^p \leq \frac{C}{K^p} \quad \text{(by (3.1))} \to 0 \quad \text{as} \quad K \to \infty.
\]

(2) The proof that (3.3) implies (3.4) is the same as that in (1). \(\square\)

Remark 3.2. It should be noted that:

- In the case \(d = 1\), Theorem 3.1 reduces to Corollary 2 in [9].
- The proof of Theorem 3.1 closely follows the pattern of Theorem 1 in [8].
- The primary mode of the convergence given by (3.4) of Theorem 3.1 was introduced in [4] in the case \(d = 1\) for the tail series of a convergent series of random variables.

Example 4. Let \(\{V_n, n \geq 1\}\) be a field independent, identically distributed, mean 0 random variables in a \(p\)-uniformly smooth Banach space \(E\) \((1 \leq p \leq 2)\) such that \(E\|V_1\|^p < \infty\), let \(\{a_n, n \geq 1\}\) be a field of nonzero constants such that \(\sum_{n \geq 1} |a_n|^p < \infty\), set \(X_n = a_n V_n\) for \(n \geq 1\). Then \(\{X_n, F_n; n \geq 1\}\) is
a field of martingale difference. By taking $b_n = \left( \sum_{i \gg n} |a_i|^p \right)^{1/p} \alpha_n^{-1}$, where \( \{a_n, n \geq 1\} \) is any field of positive numbers, we have that

$$\frac{\sum_{i \gg n} \|X_i\|^p}{b_n} = \alpha_n \|X_1\|^p \to 0.$$  

If \( \sup_{n \geq 1} \alpha_n < \infty \), then (3.1) holds, by Theorem 3.1, we have that

$$\sup_{k \geq n} \|T_k\| = o_P(b_n).$$  

If \( \alpha_n \to 0 \) as \( \|n\| \to \infty \), then (3.3) holds, by Theorem 3.1, we have that

$$\sup_{k \geq n} \|T_k\| \xrightarrow{P} 0 \quad \text{as} \quad \|n\| \to \infty.$$  

Next, we establish the rate of convergence of series of strong\(^*\) martingale difference fields, with the field of positive Borel function \( \{\phi_n, n \geq 1\} \) which have a property similar to that of the sequence of functions in [2] of Hong and Tsay, i.e.,

\[
C_n u^\lambda_n \leq \Phi_n(u) \leq D_n v^\mu_n \quad \text{for all} \quad u \geq v > 0,
\]

where \( C_n \geq 1, D_n \geq 1, \lambda_n \geq 1, 0 < \mu_n \leq p \).

Note that the array of functions \( \{\Phi_n, n \geq 1\} \) with \( \Phi_n(x) = x^p, p \geq 1 \) satisfies the condition (3.5).

**Theorem 3.3.** Let \( \mathbb{E} \) be a \( p \)-uniformly smooth Banach space for some \( 1 \leq p \leq 2 \), let \( \{X_n, \mathcal{F}_n; n \in \mathbb{N}\} \) be a field of \( \mathbb{E} \)-valued strong\(^*\) martingale differences. Let \( \{\Phi_n, n \geq 1\} \) be a field of positive Borel functions which satisfies the conditions (3.5) and \( \Phi_n(u) \leq \Phi_m(u) \) for all \( n \ll m \). Let \( \{b_n\}, \{B_n\} \) be fields of positive constants, such that \( \Phi_n(b_n) = o(1), \Phi_n(B_n) = o(1) \) as \( \|n\| \to \infty \).

(1) If

\[
\sum_{k \gg n} A_k E \Phi_k(\|X_k\|) = O(\Phi_n(b_n)),
\]

where \( A_n = \max\{A_n, D_n\} \), then \( \sum_{n \geq 1} X_n \) converges a.s. and the series \( \{T_n = \sum_{k \gg n} X_k\} \) satisfies the relation

\[
\sup_{k \geq n} \|T_k\| = O_P(b_n).
\]

(2) If

\[
\sum_{k \gg n} A_k E \Phi_k(\|X_k\|) = o(\Phi_n(B_n))
\]

as \( \|n\| \to \infty \) (where \( A_n = \max\{A_n, D_n\} \)), then \( \sum_{n \geq 1} X_n \) converges a.s. and the series \( \{T_n = \sum_{k \gg n} X_k\} \) obeys the limit law

\[
\sup_{k \gg n} \|T_k\| \xrightarrow{P} 0 \quad \text{as} \quad \|n\| \to \infty.
\]
Proof. (1) For each \( n \geq 1 \), set \( Y_n = X_n I(\|X_n\| \leq 1) \), \( Z_n = X_n I(\|X_n\| > 1) \), \( U_n = Y_n - E(Y_n|\mathcal{F}_n^n) \), \( V_n = Z_n - E(Z_n|\mathcal{F}_n^n) \), \( S_n^1 = \sum_{k \leq n} U_k \), and \( S_n^2 = \sum_{k \leq n} V_k \). Then \( X_n = U_n + V_n \) and \( S_n = S_n^1 + S_n^2 \). Moreover, since \( (X_n, \mathcal{F}_n, n \geq 1) \) is a field of strong” martingale differences, it is clear that \( \{U_n, \mathcal{F}_n, n \geq 1\} \) and \( \{V_n, \mathcal{F}_n, n \geq 1\} \) are strong adapted fields.

By the proof of Theorem 3.1, we have

\[
P \left( \sup_{k \geq 0} \|S_{n+k}^1 - S_n^1\| > \varepsilon \right) = C \sum_{i \geq n} E\|Y_i\|^p \leq C \sum_{i \geq n} E\|Y_i\|^{\alpha n}
\]

\[
\leq C \sum_{i \geq n} D_k \cdot E\Phi_i(\|Y_i\|) \leq C \sum_{i \geq n} A_i \frac{E\Phi_i(\|X_i\|)}{\Phi_i(1)}
\]

\[
\leq C \frac{1}{\Phi_i(1)} \sum_{i \geq n} A_i E\Phi_i(\|X_i\|) < o(1) \quad \text{as} \quad \|n\| \to \infty.
\]

Then \( S_n^1 \) converges a.s. as \( \|n\| \to \infty \) (by Lemma 2.6). Next, by the proof of Theorem 3.1, we have

\[
P \left( \sup_{k \geq 0} \|S_{n+k}^2 - S_n^2\| > \varepsilon \right) = C \sum_{i \geq n} E\|Z_i\| \leq C \sum_{i \geq n} E\|Z_i\|^{\alpha n}
\]

\[
\leq C \sum_{i \geq n} \frac{1}{C_i} \cdot E\Phi_i(\|Z_i\|) \leq C \sum_{i \geq n} A_i \frac{E\Phi_i(\|X_i\|)}{\Phi_i(1)}
\]

\[
\leq C \frac{1}{\Phi_i(1)} \sum_{i \geq n} A_i E\Phi_i(\|X_i\|) < o(1) \quad \text{as} \quad \|n\| \to \infty.
\]

Then \( S_n^2 \) converges a.s. as \( \|n\| \to \infty \) (by Lemma 2.6). By \( S_n = S_n^1 + S_n^2 \), which implies \( S_n \) converges a.s as \( \|n\| \to \infty \), then \( \sum_{n \geq 1} X_n \) converges a.s. Thus, the tail series \( \{T_n = \sum_{k \geq n} X_k; n \geq 1\} \) is a well-defined field of random variables.

Next, to prove that (3.6) implies (3.7), for each \( n \geq 1 \), and \( n \geq i \), set \( Y'_i = X_i I(\|X_i\| \leq b_n) \), \( Z'_i = X_i I(\|X_i\| > b_n) \), \( U'_n = Y'_n - E(Y'_n|\mathcal{F}_n^n) \), \( V'_n = Z'_n - E(Z'_n|\mathcal{F}_n^n) \). Then \( X_n = U'_n + V'_n \). Moreover, by \( \{X_n, \mathcal{F}_n, n \geq 1\} \) being a strong” martingale difference field, then it is clear that \( \{U'_n, \mathcal{F}_n, n \geq 1\} \) and \( \{V'_n, \mathcal{F}_n, n \geq 1\} \) are strong adapted fields.

By the proof of Theorem 3.1, we observe that for \( K > 0 \),

\[
\sup_{n \geq 1} P \left( \frac{\sup_{k \geq n} \|T_k\|}{b_n} > K \right) \leq 2 \sup_{n \geq 1} \liminf_{M \to \infty} P \left( \frac{\max_{n \leq k \leq M} \|\sum_{n \leq i \leq k} X_i\|}{b_n} > \frac{K}{2} \right)
\]

\[
\leq 2 \sup_{n \geq 1} \liminf_{M \to \infty} \left( P \left( \frac{\max_{n \leq k \leq M} \|\sum_{n \leq i \leq k} U'_i\|}{b_n} > \frac{K}{4} \right) + P \left( \frac{\max_{n \leq k \leq M} \|\sum_{n \leq i \leq k} V'_i\|}{b_n} > \frac{K}{4} \right) \right)
\]
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\[ \leq 2 \sup_{n \geq 1} \lim_{M \to \infty} \left( \frac{4^p}{K^p b_n^p} E \left\{ \max_{n \leq k \leq M} \left\| \sum_{n \leq i \leq k} U_i' \right\|^p \right\} \right. \\
\quad + \frac{4}{K b_n} E \left\{ \max_{n \leq k \leq M} \left\| \sum_{n \leq i \leq k} V_i' \right\| \right\} \left) \right. \\
\leq 2 \sup_{n \geq 1} \lim_{\| M \| \to \infty} \left( \frac{4^p}{K^p b_n^p} \sum_{n \leq i \leq M} E \| U_i' \|^p \right. \\
\quad + \left. \frac{4}{K b_n} \sum_{n \leq i \leq M} E \| V_i' \| \right) \text{ (by Lemma 2.3)} \\
\leq C \sup_{n \geq 1} \left( \frac{1}{K^p} \sum_{n \leq i} \frac{E \| Y_i' \|^p}{b_n} + \frac{1}{K} \sum_{n \leq i} \frac{E \| Z_i \|}{b_n} \right) \\
\leq C \sup_{n \geq 1} \left( \frac{1}{K^p} \sum_{n \leq i} \frac{E \| Y_i' \|^p}{b_n} + \frac{1}{K} \sum_{n \leq i} \frac{E \| Z_i \|^\lambda_n}{b_n} \right) \\
\leq C \sup_{n \geq 1} \left( \frac{1}{K^p} \sum_{i \geq n} D_i \cdot E \frac{\Phi_i(\| Y_i' \|)}{\Phi_i(b_n)} + \frac{1}{K} \sum_{i \geq n} \frac{1}{C_i} \cdot E \frac{\Phi_i(\| Z_i \|)}{\Phi_i(b_n)} \right) \\
\leq C \sup_{n \geq 1} \frac{1}{\Phi_n(b_n)} \left( \frac{1}{K^p} + \frac{1}{K} \right) \sum_{i \geq n} A_i E \Phi_n(\| X_i \|) < o(1) \text{ as } K \to \infty.

(2) The proof that (3.8) implies (3.9) is the same as that in (1).

When \( d = 1 \), by Remark 2.1, we have the following corollary.

**Corollary 3.3.1.** Let \( E \) be a \( p \)-uniformly smooth Banach space for some \( 1 \leq p \leq 2 \), \( \{ X_n, F_n; n \in \mathbb{N} \} \) be a sequence of \( E \)-valued martingale differences. Let \( \{ \Phi_n; n \geq 1 \} \) be a sequence of positive Borel functions which satisfies the following two conditions

\[ C_n \frac{u^{\lambda_n}}{v^{\lambda_n}} \leq \frac{\Phi_n(u)}{\Phi_n(v)} \leq D_n \frac{u^{\mu_n}}{v^{\mu_n}} \text{ for all } u \geq v > 0, \]

where \( C_n \geq 1, D_n \geq 1, \lambda_n \geq 1, 0 < \mu_n \leq p, \)

\[ \Phi_n(u) \leq \Phi_m(u) \text{ for all } n > m. \]

Let \( \{ b_n \}, \{ B_n \} \) be sequences of positive constants, such that \( \Phi_n(b_n) = o(1), \)

\( \Phi_n(B_n) = o(1). \)

(1) If

\[ \sum_{k \geq n+1} A_k E \Phi_k(\| X_k \|) = O(\Phi_n(b_n)), \]
then \( \sum_{n \geq 1} X_n \) converges a.s. and
\[
\sup_{k \geq n+1} \|T_k\| = O_P(b_n),
\]
where \( T_n = \sum_{k \geq n+1} X_k \).

(2) If
\[
\sum_{k \geq n+1} A_k E\Phi_k(\|X_k\|) = o((\Phi_n(B_n))),
\]
then\( \sum_{n \geq 1} X_n \) converges a.s. and
\[
\frac{\sup_{k \geq n+1} \|T_k\|}{B_n} \rightarrow 0,
\]
where \( T_n = \sum_{k \geq n+1} X_k \).

Remark 3.4. Let \( \{X_n, F_n, n \geq 1\} \) be a sequence of real-valued independent random variables with \( EX_n = 0, n \geq 1 \). Let \( \{g_n(x), n \geq 1\} \) be a sequence of functions defined on \([0, \infty)\) such that
\[
0 \leq g_n(0) \leq g_n(x), 0 < g_n(x) \uparrow \infty \text{ as } n \uparrow \infty \text{ for each } x > 0
\]
and
\[
\frac{g_n(x)}{x} \uparrow, \frac{g_n(x)}{xp} \downarrow \text{ on } (0, \infty), \ n \geq 1, \ \text{ for some } 1 < p \leq 2.
\]
In Corollary 3.3.1, taking \( \Phi_n = g_n \) for all \( n \geq 1 \), with \( \lambda_n = 1, \mu_n = p, C_n = 1, D_n = 1, n \geq 1 \), we obtain Theorem 2 in [11].

Finally, we establish the rate of complete convergence of the tail series of martingale difference fields.

Theorem 3.5. Let \( E \) be a \( p \)-uniformly smooth Banach space for some \( 1 \leq p \leq 2 \), \( \{X_n, F_n; n \in \mathbb{N}\} \) be a field of \( E \)-valued martingale differences. Let \( \{a_n\} \) be a field of positive constants, such that either \( a_n \leq a_m \) for all \( n \leq m \) or \( a_n \geq a_m \) for all \( n < m \) and \( \sup_n a_{2^n}/a_{2^n+1} \leq M < \infty \). If
\[
(3.10) \quad \sum_{n \geq 1} \varphi(n)E\|X_n\|^p < \infty,
\]
where \( \varphi(n) = \sum_{2^k \leq n} \frac{1}{2^k} \), then for all \( \varepsilon > 0 \),
\[
(3.11) \quad \sum_{n \geq 1} \frac{1}{n} P(\sup_{k \geq n} \|T_k\| > \varepsilon a_n) < \infty.
\]

Proof. For all \( n \geq 2 \) then \( \varphi(n) \geq \frac{1}{2^2} > 0 \), so \( \sum_{k \geq n} E\|X_k\|^p = o(1) \) as \( \|n\| \rightarrow \infty \). By proof of Theorem 3.1, we have \( \sum_{n \geq 1} X_n \) converges a.s. Thus, the tail series \( \{T_n = \sum_{k \geq n} X_k; n \geq 1\} \) is a well-defined field of random variables.

Next, to prove that (3.10) implies (3.11), we note that
\[
\sum_{n \geq 1} \frac{1}{n} P(\sup_{k \geq n} \|T_k\| > \varepsilon a_n) = \sum_{n \geq 2^n} \sum_{n \geq 2^{n+1}} \frac{1}{n} P(\sup_{k \geq n} \|T_k\| > \varepsilon a_n)
\]
Applying Lemma 2.3, Lemma 2.5, the Markov inequality and the same argument as the proof of Theorem 3.1, we see

\[
\sum_{n \geq 1} 1_{|n|} P(\sup_{k \geq n} \|T_k\| > \varepsilon |n|^{\alpha}) = \sum_{n \geq 1} \frac{M_p}{\varepsilon^{p-1} \alpha^{2p}} \sum_{i \geq 2^n} E\|X_i\|^p \leq C \sum_{n \geq 1} \varphi(n)\|X_n\|^p < \infty.
\]

\[\Box\]

**Corollary 3.5.1.** Let \( E \) be a \( p \)-uniformly smooth Banach space for some \( 1 \leq p \leq 2 \), \( \{X_n, \mathcal{F}_n; n \in \mathbb{N}\} \) be a field of \( E \)-valued martingale differences. If

\[
\sum_{n \geq 1} E\|X_n\|^p < \infty,
\]

then for all \( \alpha > 0, \varepsilon > 0 \),

\[
\sum_{n \geq 1} \frac{1}{|n|} P(\sup_{k \geq n} \|T_k\| > \varepsilon |n|^{\alpha}) < \infty.
\]

**Proof.** Put \( a_n = |n|^{\alpha} \) then \( \varphi(n) \leq \sum_{n \geq 1} \frac{1}{|n|^{\alpha}} = \left(\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}\right)^d < \infty \) so (3.12) implies (3.10). By Theorem 3.5 we get (3.13). \( \Box \)

**Corollary 3.5.2.** Let \( E \) be a \( p \)-uniformly smooth Banach space for some \( 1 \leq p \leq 2 \), \( \{X_n, \mathcal{F}_n; n \in \mathbb{N}\} \) be a sequence of \( E \)-valued martingale differences. If

\[
\sum_{n=1}^{\infty} E\|X_n\|^p \log_2 n < \infty,
\]

then for all \( \alpha > 0, \varepsilon > 0 \),

\[
\sum_{n=1}^{\infty} \frac{1}{n} P(\sup_{k \geq n} \|T_k\| > \varepsilon) < \infty.
\]

**Proof.** Put \( a_n = 1 \) then for \( d = 1 \) we have \( \varphi(n) \leq \log_2 n \). Hence (3.14) implies (3.10). By Theorem 3.5 we obtain (3.15). \( \Box \)

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