ON THE QUASITORIC BRAID INDEX OF A LINK

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Abstract. We define new link invariants which are called the quasitoric braid index and the cyclic length of a link and show that the quasitoric braid index of link with \( k \) components is the product of \( k \) and the cycle length of link. Also, we give bounds of Gordian distance between the \((p, q)\)-torus knot and the closure of a braid of two specific quasitoric braids which are called an alternating quasitoric braid and a blockwise alternating quasitoric braid. We give a method of modification which makes a quasitoric presentation from its braid presentation for a knot with braid index 3. By using a quasitoric presentation of \( 10_{139} \) and \( 10_{124} \), we can prove that \( u(10_{139}) = 4 \) and \( \text{gcd}(10_{124}, K(3, 13)) = 8. \)

1. Introduction

A link is a disjoint union of circles which is embedded in \( \mathbb{R}^3 \) and a knot is a link with one component. There are various ways to describe a link, one of which is to present it via a braid. Given a braid, there is a link (the closure of the braid) corresponding to the braid. Alexander theorem say that every link is ambient isotopic to the closure of a braid. Toric braid is a braid which can be drawn on the standard torus. In 2002, Manturov [8] introduced the notion of a quasitoric braid which is a generalization of toric braid, and proved that every link can be presented as the closure of a quasitoric braid. By the virtue of the Manturov’s theorem, one can define the quasitoric braid index of link \( L \) as the minimum number of strands of a quasitoric braid which presents the given link \( L \).

In this paper, we will give a formula to calculate the quasitoric braid index and find a relationship between the quasitoric braid index and other link invariants, like the signature of a link.

2. Braids and quasitoric braids

Firstly, we recall the braid index and its properties, see [9] for details. Consider the point \( A_i = (i, 0, 0) \) and \( B_i = (i, 0, 1) \), where \( i = 1, 2, \ldots, n \), in \( \mathbb{R}^3 \). A
$n$-braid is defined as a set of pairwise nonintersecting ascending strands joining the points $A_1, A_2, \ldots, A_n$ to $B_1, B_2, \ldots, B_n$ in any order. The set $B_n$ of all $n$-braids is a group, called the $n$-braid group, under the product obtained by putting two braids $\beta_1$ and $\beta_2$ end to end. The following presentation of $B_n$ is given by Artin[2]:

$$\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for} \quad |i - j| \geq 2 \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for} \quad i = 1, \ldots, n - 1 \rangle,$$

where $\sigma_i$ is the $n$-braid given in Figure 1.

Since the strands of an $n$-braid $\beta$ connect the points $A_1, A_2, \ldots, A_n$ to $B_1, B_2, \ldots, B_n$, injectively for an $n$-braid $\beta$, there is an element $\pi(\beta)$ in the permutation group $\Sigma_n$ which is uniquely determined by each strand. Indeed, if $k$th strand of $\beta$ connects $A_i$ to $B_{i_k} (k = 1, \ldots, n)$, then the permutation $\pi(\beta)$, corresponding to $\beta$ is $(1 \ 2 \ \cdots \ n)$. For example, the permutation of $\beta$ in Figure 2(a) is $(1 \ 2 \ 3 \ 4 \ 5 \ 6)$. 

A braid is said to be pure if the permutation corresponding to the braid is the identity permutation. The closure of a braid $\tilde{\beta}$ is the link obtained from $\beta$ by joining the points $A_1, A_2, \ldots, A_n$ to the points $B_1, B_2, \ldots, B_n$ by parallel curves as shown in Figure 2(b).

**Proposition 2.1** (Alexander). Every link is ambient isotopic to the closure of a braid.

The braid index $b(L)$ of a link $L$ is the minimum number of strands of a braid which presents the given link $L$ and it is an invariant of link.
**Proposition 2.2** (Markov). Closures of two braids $\beta_1$ and $\beta_2$ represent ambient isotopic links if and only if $\beta_1$ can be transformed to $\beta_2$ by applying a finite sequence of Markov’s moves, which are depicted in Figure 3.

**Figure 3.** Markov’s moves.

**Definition 2.3** ([8]). The **toric braid** of type $(p, q)$ is the braid of the form $(\sigma_{p-1} \cdots \sigma_1)^q$ where $p$ is the number of strands and $q$ is an integer, see Figure 4(a). A braid $\beta$ is said to be **quasitoric** of type $(p, q)$ if it is obtained from the standard diagram of the toric braid of type $(p, q)$ by switching some crossings. Then a quasitoric braid of type $(p, q)$ can be expressed as $\beta_1 \beta_2 \cdots \beta_q$ where for each $\beta_j = \sigma_{p-1}^{e_{j,1}} \cdots \sigma_1^{e_{j,p-1}}$ and each $e_{j,i}$ is either 1 or $-1$, see Figure 4(b). A quasitoric braid of type $(p, q)$ is called a $p$-**quasitoric braid** shortly.

**Figure 4.** Toric braid and quasitoric braid.

Note that the closure of the toric braid of type $(p, q)$ is the torus link $L(p, q)$. The following proposition says that every link is of the form of the closure of a quasitoric braid.

**Proposition 2.4** ([8]). **Any link can be presented by the closure of a quasitoric braid.**

**Definition 2.5.** The **quasitoric braid index** of a link $L$ is the minimum number of strands of a quasitoric braid which presents the given link $L$, and is denoted by $b_q(L)$. 
It is clear that \( b_q(L) \) is well-defined for any link \( L \) and is a link invariant. For example, the quasitoric braid index of the trefoil knot is 2, because the trefoil knot is the closure of \( \sigma_1^2 \) and nontrivial. It is clear that \( b(L) \leq b_q(L) \) for a link \( L \).

Let \( L = K_1 \cup K_2 \cup \cdots \cup K_k \) be a \( k \)-component link and \( \beta \) a braid presentation of \( L \). Then the closure of a braid \( \hat{\beta} \) is of the form \( \hat{\beta} = \hat{\beta}_1 \cup \hat{\beta}_2 \cup \cdots \cup \hat{\beta}_k \), where \( \hat{\beta}_i = K_i \) for \( 1 \leq i \leq k \). It is well-known that a permutation is presented by the product of cycles. Notice that the permutation of a braid \( \pi(\beta) \) of \( \beta \) is a product of \( k \)-cycles if and only if \( \hat{\beta} \) is a link with \( k \) components.

**Definition 2.6.** For a braid \( \beta \), the **cycle length** \( l_c(\beta) \) is the maximal length of cycles in the permutation of \( \beta \). The cycle length \( l_c(L) \) of a link \( L \) is the minimum of the cycle length \( l_c(\beta) \) among all braids \( \beta \) whose closures present the link \( L \). Indeed, \( l_c(L) = \min \{ l_c(\beta) \mid \hat{\beta} = L \} \).

For example, the permutation of \( \beta \) in Figure 2(a) is \((1243)\) and hence \( l_c(\beta) = 4 \). If \( K \) is equivalent to the trivial knot, clearly \( l_c(K) = 1 \). Notice that \( l_c(L) \) is a link invariant and that if \( K \) is a knot then \( l_c(K) \) equals the braid index \( b(K) \).

The following lemma shows some relations between the braid index of \( L \) and the cycle length of \( L \) with respect to the number of components of \( L \).

**Lemma 2.7.** Let \( L \) be a link with \( k \)-components.

1. \( b(L) = k \) if and only if \( l_c(L) = 1 \).
2. If \( b(L) = k + 1 \), then \( l_c(L) = 2 \).
3. If \( b(L) = k + n \) for a positive integer \( n \), then \( 2 \leq l_c(L) \leq n + 1 \).

*Proof.* (1) Suppose that \( b(L) = k \) and \( \beta \) is a braid presentation of a link \( L \) with \( b(\hat{\beta}) = k \). Since \( L \) is a link with \( k \)-components and the number of strands of \( \beta \) is \( k \), \( \beta \) is a pure braid. So, \( l_c(\beta) = 1 \). Since \( 1 \leq l_c(L) \leq l_c(\beta) = 1 \), we have \( l_c(L) = 1 \). Conversely, suppose that \( l_c(L) = 1 \). Let \( \beta \) be a braid presentation of \( L \) with \( l_c(\beta) = 1 \). Since \( L \) is a \( k \)-components link, \( \pi(\beta) = c_1 c_2 \cdots c_k \) for some cycles \( c_i \) (\( i = 1, 2, \ldots, k \)). Since \( l_c(\beta) = 1 \), the length of \( c_i \) is 1 for each \( i = 1, 2, \ldots, k \). Hence \( b(L) \leq k \cdot l_c(\beta) = k \) by the maximility of the cycle length. Since \( b(L) \geq k \), \( b(L) = k \).

(2) Since \( k + 1 = b(L) \leq k \cdot l_c(L) \) and \( l_c(L) \) is a positive integer, we have \( 2 \leq l_c(L) \). Let \( \beta \) be a braid presentation of link \( L \) with \( b(\hat{\beta}) = k + 1 \). Since \( L \) has \( k \) components, the braid permutation of \( \beta \) is shown as a product \( c_1 c_2 \cdots c_k \) of \( k \) cycles. Let \( m_i \) be the cycle length of \( c_i \). Since \( m_1 + m_2 + \cdots + m_k = k + 1 \) and \( m_i \) is a positive integer, the maximum of \( m_i \) is 2. Hence \( l_c(L) \leq l_c(\beta) = 2 \). Therefore, \( l_c(L) = 2 \).

(3) Since \( k + n = b(L) \leq k \cdot l_c(L) \) and \( l_c(L) \) is a positive integer, we have \( 2 \leq l_c(L) \). Let \( \beta \) be a braid presentation of link \( L \) with \( b(\hat{\beta}) = k + n \). Since \( L \) has \( k \) components, the braid permutation of \( \beta \) is presented by a product \( c_1 c_2 \cdots c_k \) of \( k \) cycles. Let \( m_i \) be the cycle length of \( c_i \). Since \( m_1 + m_2 + \cdots + m_k = k + n \)
and $m_i$ is a positive integer, the maximum of $m_i$ is $n + 1$. Hence $l_c(L) \leq l_c(\beta) = n + 1$. Therefore, $2 \leq l_c(L) \leq n + 1$.

Notice that the converse of Lemma 2.7(2) does not hold. Because, if $L$ is the closure of the braid $\beta = \sigma_1\sigma_2^{-1}\sigma_3\sigma_1\sigma_2^{-1}\sigma_3$, then $L$ is a 2-components link and $l_c(L) = 2$, but $b(L) = 4$.

If $L$ is a $k$-components link, then $L$ is a disjoint union of knots $K_i$ ($i = 1, 2, \ldots, k$). Since the cycle length $l_c(K_i)$ of a knot $K_i$ equals the braid index $b(K_i)$, we have

$$l_c(L) \geq \max\{b(K_i) \mid b(K_i) \text{ is the braid index of } K_i\}.$$ 

But, the equality does not holds, in general. For example, consider $L$ in Figure 5(a) which is a 2-components link. Since the diagram $D$ of $L$ in Figure 5(a) is a reduced and alternating diagram, $c(L) = 5$. Note that $b(D) = 3$. For, if $b(L) = 2$, then $L$ is presented as shown in Figure 5(b). Since $L$ has two components, the crossing number of $L$ is even which is a contradiction to the fact that the crossing number of $L$ is 5. Hence the braid index of $L$ is 3. Since each component of $L$ is trivial, the braid index of each component of $L$ is 1, but the cycle length of $L$ is 2, by Lemma 2.7(2).

![Figure 5](image_url)

**Figure 5.**

**Theorem 2.8.** Let $L(p, q)$ be the torus link of type $(p, q)$. Suppose that $1 \leq p \leq q$ and $k$ is the greatest common divisor of $p$ and $q$. Then

$$l_c(L(p, q)) = \frac{p}{k}.$$ 

**Proof.** Since $k$ is the greatest common divisor of $p$ and $q$, each component of $\hat{\beta}$ is the torus knot of type $(\frac{p}{k}, \frac{q}{k})$. Hence the braid index of each component of $\hat{\beta}$ is $\frac{p}{k}$. This means that the cycle length of each component of $\hat{\beta}$ is $\frac{p}{k}$. Thus
for any closure of a braid which represent $L(p,q)$, the maximum cycle length of the braid is greater than or equal to $\frac{p}{k}$. Indeed,

$$\frac{p}{k} \leq \ell_c(L(p,q)).$$

Notice that the braid presentation of $L(p,q)$ is of the form

$$\beta = (\sigma_{p-1}\sigma_{p-2}\cdots\sigma_1)^q.$$

Since the length of the permutation of each component of $\hat{\beta}$ is $\frac{p}{k}$, $\ell_c(\hat{\beta}) = \frac{p}{k}$. Thus we have

$$\ell_c(L(p,q)) \leq \ell_c(\beta) = \frac{p}{k}.$$

Therefore, we obtain

$$\ell_c(L(p,q)) = \frac{p}{k}. \quad \square$$

**Example 2.9.** $L(4,6)$ consists of two components, each of which is a $(2,3)$-torus knot. By Theorem 2.8, we have $\ell_c(L(4,6)) = \frac{4}{2}$.

![Figure 6. Example of $L(4,6)$.](image)

To investigate the relationship between the two invariants, the quasitoric braid index of a link and the cycle length of a link, we are going to acquaint the following proposition which is given by Manturov:

**Proposition 2.10 ([8]).** A $n$-strand braid whose permutation is a power of the cyclic permutation $(123\cdots n)$ is quasitoric.

By using Proposition 2.10, we can obtain the following.

**Theorem 2.11.** Let $L$ be a $k$-component link. Let $\ell_c(L)$ and $b_q(L)$ denote the cycle length and the quasitoric braid index of $L$, respectively. Then

$$b_q(L) = k \cdot \ell_c(L).$$
Proof. To show that $b_q(L) \leq k \cdot l_c(L)$, we will modify the Manturov’ proof in [8]. Let $L$ be a $k$-component link and $\beta(L)$ a $n$-braid presentation of $L$ with $l_c(\beta(L)) = m$. The permutation $\pi(L)$ of $\beta(L)$ can be written by the product of $k$ cycles. These cycles might contain different numbers of elements. Let us apply Markov moves for transforming these cycles. Since the first Markov move conjugates the braid, the number of elements in orbits does not change, but elements in orbits are permuted, because it conjugates the corresponding permutation of the braid. Since the second Markov move adds the element $n+1$ to the orbit which is containing the element $n$ and does not change other orbits, it increases the number of strands by one. Hence one can re-enumerate elements which are in the smallest cycles in such a way and then increase the number of elements in this cycle by one by using the second Markov move. By repeating this we can obtain the same number of elements for all cycles. So, there exists a braid $\alpha$ such that the permutation $\pi(\alpha)$ is shown by a power of $(12 \cdots km)$. Notice that, by the previous proposition, $\alpha$ is quasitoric. Moreover, $\alpha$ is Markov equivalent to $\beta(L)$. Hence we can obtain

$$b_q(L) = b_q(\hat{\beta}) \leq k \cdot m = k \cdot l_c(\beta(L)).$$

Since, for any braid presentation $\beta$ of link $L$, $b_q(L) \leq k \cdot l_c(\beta(L))$,

$$b_q(L) \leq k \cdot l_c(L).$$

Conversely, we claim that $b_q(L) \geq k \cdot l_c(L)$. Suppose that the quasitoric braid index $b_q(L)$ of the link $L$ is $p$. There exists $\beta$ in $B_p$ such that $\beta$ is a quasitoric braid of type $(p, q)$ with $\hat{\beta} = L$. Let $\alpha$ be the toric braid of type $(p, q)$. Since $\beta$ is obtained from $\alpha$ by switching some crossings and $L$ have $k$ components, the greatest common divisor of $p$ and $q$ is $k$ and the cycle length of $\alpha$ is $\frac{p}{k}$. Then, by the minimality of $l_c(L)$, we have

$$b_q(L) = k \cdot l_c(\alpha) = k \cdot l_c(\beta) \geq k \cdot l_c(\hat{\beta}) = k \cdot l_c(L).$$

Therefore, we have

$$b_q(L) = k \cdot l_c(L). \quad \square$$

The following corollary is a direct consequence from Theorem 2.11.

**Corollary 2.12.** For a knot $K$, the braid index and the quasitoric braid index of $K$ are the same. Indeed,

$$b(K) = b_q(K).$$

From Theorem 2.8 and Theorem 2.11, we get the following.

**Corollary 2.13.** Let $L(p, q)$ be the torus link of type $(p, q)$ and $1 \leq p \leq q$. Then

$$b_q(L(p, q)) = p.$$
3. Alternating quasitoric braids and blockwise alternating quasitoric braids

An $n$-braid $\beta$ is said to be homogeneous if $\beta$ is of the form $\sigma_{i_1}^{\alpha_1}\sigma_{i_2}^{\alpha_2}\cdots\sigma_{i_k}^{\alpha_k}$, with the condition that the exponent of the same generators are identical, where $1 \leq i_1, i_2, \ldots, i_k \leq n - 1$ and $\alpha_i = \pm 1$. In particular, a homogeneous braid is said to be alternating if the sign of the exponent of $\sigma_i$ is the opposite of the sign of the exponent of $\sigma_{i+1}$ for $i = 1, 2, \ldots, n - 2$. For example, the 4-braid $\beta = \sigma_2\sigma_3^{-2}\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_1^{-2}$ is homogeneous and alternating.

**Definition 3.1.** Let $\beta$ be a quasitoric braid of type $(p,q)$ of the form

$$\beta = (\sigma_p^{\varepsilon_{p-1}}\sigma_{p-2}^{\varepsilon_{p-2}}\cdots\sigma_1^{\varepsilon_1})^q,$$

where $p$ and $q$ are positive integers $p \geq 2$, and $\varepsilon_i = \pm 1$. $\beta$ is called an alternating quasitoric braid of type $(p, q)$, if the sign of the exponent of $\sigma_i$ is the opposite of the sign of the exponent of $\sigma_{i+1}$ for $i = 1, 2, \ldots, p - 2$.

![Figure 7. Alternating quasitoric braids of type (5,4).](image)

There are only two cases of an alternating quasitoric braid of type $(p, q)$. If $\beta$ is one of two cases, then the other $\beta^*$ is the mirror image of $\beta$, as seen in Figure 7.

Now we will acquaint a relationship between the crossing number $c(L)$ and the braid index $b(L)$ of links. For a non-splittable link $L$ with a crossing number $c(L)$, Ohyama [10] proved the inequality:

$$c(L) \geq 2(b(L) - 1).$$

**Remark 3.2.** As a consequence of Corollary 2.12, we have

$$c(K) \geq 2(b_q(K) - 1),$$

where $c(K)$ is the crossing number of $K$ and $b_q(K)$ is the quasitoric braid index of $K$.

An inequality can be given between a crossing number and a quasitoric braid index of the closure of an alternating quasitoric braid as the following.
Theorem 3.3. Let $\beta$ be an alternating quasitoric braid of type $(p, q)$ where $q \geq p \geq 2$. Then we have

$$c(\hat{\beta}) \geq q(b_q(\hat{\beta}) - 1).$$

Proof. Let $\beta$ be an alternating quasitoric braid of type $(p, q)$ and $q \geq p \geq 2$. Let $D$ be an alternating and reduced diagram of $\hat{\beta}$ with $q(p - 1)$ crossings. Let $\alpha$ be the toric braid of type $(p, q)$ and $k$ the greatest common divisor of $p$ and $q$. Since the cycle length of $\alpha$ is $p$ and $\beta$ is obtained from $\alpha$ by switching some crossings, we have

$$p = k \cdot l_c(\alpha) = k \cdot l_c(\hat{\beta}) = b_q(\hat{\beta}).$$

Hence

$$c(\hat{\beta}) = q(p - 1) \geq q \cdot (b_q(\hat{\beta}) - 1).$$

The following proposition gives us an invariant of the closure of a braid which come from an alternating braid.

Proposition 3.4 ([9]). Let $\beta = \sigma_1^{e_1} \sigma_2^{e_2} \cdots \sigma_k^{e_k}$ be an alternating braid. If every generator that appears in $\beta$ does so at least twice, then

$$\exp(\beta) = \sum_{i=1}^{k} e_i$$

is an invariant of the closure of $\beta$.

Theorem 3.5. Let $\beta$ be an alternating quasitoric braid of type $(p, q)$ where $p$ and $q$ are relatively prime and $p, q \geq 2$.

(1) If $p$ is even, then $\hat{\beta}$ is not amphicheiral.

(2) If $p$ is odd, then $\hat{\beta}$ is amphicheiral.

Proof. It is clear that $\hat{\beta}$ is a knot since $p$ and $q$ are relatively prime.

(1) Since $p$ is even, $\beta$ is presented with odd generators, $\sigma_1, \sigma_2, \ldots, \sigma_{p-1}$, say $\beta = (\sigma_1^{-1} \sigma_2 \cdots \sigma_{p-1}^{-1})^q$ so that $\beta^* = (\sigma_{p-1}^{-1} \sigma_{p-2}^{-1} \cdots \sigma_1^{-1})^q$. Since $\beta$ and $\beta^*$ are both alternating, and since every generator which is appeared in $\beta$ and $\beta^*$ does so at least twice, $\exp$ is an invariant of their closures. Suppose that $\hat{\beta}$ is equivalent to its mirror image $\hat{\beta}^*$,

$$-q = \exp(\beta) = \exp(\beta^*) = q,$$

which is a contradiction. Therefore, $\hat{\beta}$ is not amphicheiral.

(2) Since $p$ is odd, $\beta$ can be presented with even generators, $\sigma_1, \ldots, \sigma_{p-1}$, say $\beta = (\sigma_1^{-1} \sigma_2 \cdots \sigma_{p-1})^q$ so that $\beta^* = (\sigma_{p-1}^{-1} \sigma_{p-2}^{-1} \cdots \sigma_1^{-1})^q$, see Figure 8 for example. Note that $\hat{\beta}$ and $\hat{\beta}^*$ are the mirror image of each other. (c) is obtained from $\hat{\beta}^*$ by rotating 180 degree around the center $P$ of $\hat{\beta}^*$. Note that (c) is equivalent to $\hat{\beta}$. It means that $\hat{\beta}$ is equivalent to $\hat{\beta}^*$. Since $p$ and $q$ are relatively prime, it is clear that $\hat{\beta}$ and $\hat{\beta}^*$ are knots. Therefore $\hat{\beta}$ is an amphicheiral knot. \square
Gordian distance is a metric characterizing the difference between two links $L$ and $L'$. The pattern of local transformation which is shown in Figure 9(a) is called the $\times$-move and Figure 9(b) is called the zero-linking twist.

For oriented links, it is clear that the $\times$-move is the zero-linking twist. It is well-known that for oriented links, $L$ and $L'$, with the same number of components, $L$ can be changed to $L'$ by a finite number of zero-linking twists, see [5] in detail.

**Definition 3.6 ([5]).** For oriented links $L$ and $L'$ with the same number of components,

1. The $\times$-distance $d^\times(L, L')$ is defined by the least number of the $\times$-move which are needed to move $L$ to $L'$.
2. The $\tau$-distance $d^\tau(L, L')$ is defined by the least number of the zero-linking twists which are needed to move $L$ to $L'$.

For oriented links $L$ and $L'$ with different numbers of components, the $\times$-distance and $\tau$-distance of $L$ and $L'$ are defined by

$$d^\tau(L, L') = d^\times(L, L') = \infty.$$  

In [5], Kawauchi introduced an inequality between the $\times$-distance and $\tau$-distance of links with the same number of components.
Proposition 3.7 ([5]). For oriented links \( L \) and \( L' \),
\[
d^\times(L, L') \geq d^\sigma(L, L') \geq \frac{|\sigma(L) - \sigma(L')|}{2},
\]
where \( \sigma \) is the signature of a link.

By Theorem 3.5, we can give bounds of \( \times \)-distance and \( \tau \)-distance between the \((p, q)\)-torus knot \( K(p, q) \) and the closure of an alternating quasitoric braid.

Theorem 3.8. Let \( \beta \) be an alternating quasitoric braid of type \((p, q)\) where \( p \) and \( q \) are relatively prime integers and \( q > p \geq 2 \). If \( p \) is odd, then
\[
\frac{(p - 1) \cdot q}{2} \geq d^\times(K(p, q), \hat{\beta}) \geq d^\tau(K(p, q), \hat{\beta}) \geq \frac{|\sigma(K(p, q))|}{2},
\]
where \( \sigma(K(p, q)) \) is the signature of the \((p, q)\)-torus knot.

Proof. Let \( \beta \) be an alternating quasitoric braid of type \((p, q)\), then
\[
\beta = (\sigma_{p-1} \sigma_{p-2} \cdots \sigma_1)^q \quad \text{or} \quad \beta = (\sigma_{p-1} \sigma_{p-2} \cdots \sigma_1^{-1})^q.
\]
Since \( p \) and \( q \) are relatively prime, it is clear that \( \hat{\beta} \) is a knot. Since \( K(p, q) \) can be presented as the closure of the braid \((\sigma_{p-1} \sigma_{p-2} \cdots \sigma_1)^q\), \( K(p, q) \) is obtained from \( \hat{\beta} \) by changing \( \frac{(p - 1) \cdot q}{2} \) crossings. This means that
\[
\frac{(p - 1) \cdot q}{2} \geq d^\times(K(p, q), \hat{\beta}).
\]
Moreover, we already show that if \( p \) is odd, then \( \hat{\beta} \) is amphicheiral, in Theorem 3.5. So, the signature of \( \hat{\beta} \) is 0. Therefore,
\[
d^\tau(K(p, q), \hat{\beta}) \geq \frac{|\sigma(K(p, q))|}{2}. \quad \square
\]

Example 3.9. Figure 10(a) is a knot diagram of \( 8_{18} \) and Figure 10(b) is a diagram of the \((3, 4)\)-torus knot. By using the recurrence relation which is given by Gordon-Litherland-Murasugi [4], one can obtain that \( \sigma(K(3, 4)) = -6 \). Hence
\[
4 = \frac{(3 - 1) \cdot 4}{2} \geq d^\times(8_{18}, 8_{19}) \geq d^\tau(8_{18}, 8_{19}) \geq \frac{|-6|}{2} = 3.
\]

If \( p \) is even, then \( \hat{\beta} \) and the mirror image \( \widehat{\beta^*} \) of \( \hat{\beta} \) are not equivalent (Theorem 3.5), and hence the signature of \( \hat{\beta} \) and \( \beta^* \) are not equal to 0, in general.

Remark 3.10. For relatively prime positive integers \( p \) and \( q \) with \( q > p \geq 2 \), put \( \beta = (\sigma_{p-1} \sigma_{p-2} \cdots \sigma_1)^q \) and \( \beta^* = (\sigma_{p-1} \sigma_{p-2} \cdots \sigma_1^{-1})^q \). If \( p \) is even, then we have the following:
\[
\frac{p \cdot q}{2} \geq d^\times(L(p, q), \hat{\beta}) \quad \text{and} \quad \frac{(p - 2) \cdot q}{2} \geq d^\times(L(p, q), \widehat{\beta^*}).
\]

Now we will consider a class of knots whose signature is zero.
Definition 3.11. For positive integers $p$ and $q$, let $X = \sigma_{p-1}\sigma_{p-2}\cdots\sigma_2\sigma_1$ and $-X = \sigma_{p-1}^{-1}\sigma_{p-2}^{-1}\cdots\sigma_2^{-1}\sigma_1^{-1}$. A braid $\beta$ is called a blockwise alternating quasitoric braid of type $(p, q)$, if it is of the form

$$\beta = \prod_{i=0}^{q-1}(-1)^iX.$$
equivalent to $\hat{\beta}^\ast$. Since $p$ and $q$ are relatively prime, it is clear that $\hat{\beta}$ and $\hat{\beta}^\ast$ are knots. Therefore $\hat{\beta}$ is an amphicheiral knot. □

Figure 12.

As a consequence of the above theorem, we have the following.

**Theorem 3.13.** Let $\beta$ be a blockwise alternating quasitoric braid of type $(p,q)$ where $p$ and $q$ are relatively prime and $q > p \geq 2$ and $\sigma(K(p,q))$ the signature of $(p,q)$-torus knot. If $q$ is even, then we have

$$\frac{(p - 1) \cdot q}{2} \geq d^\times(K(p,q), \hat{\beta}) \geq d^\tau(K(p,q), \hat{\beta}) \geq \frac{\sigma(K(p,q))}{2}.$$  

**Proof.** Since $q$ is even, we can put $q = 2k$ for a positive integer $k$. Let $\beta$ be a blockwise alternating quasitoric braid of type $(p,q)$, then

$$\hat{\beta} = ((\sigma_{p-1}\sigma_{p-2}\sigma_{p-3}\cdots\sigma_1)(\sigma_{p-1}^{-1}\sigma_{p-2}^{-1}\sigma_{p-3}^{-1}\cdots\sigma_1))^k.$$  

Since $p$ and $q$ are relatively prime, it is clear that $\hat{\beta}$ is a knot. Since $K(p,q)$ can be presented as the closure of the braid

$$((\sigma_{p-1}\sigma_{p-2}\sigma_{p-3}\cdots\sigma_1)(\sigma_{p-1}^{-1}\sigma_{p-2}^{-1}\sigma_{p-3}^{-1}\cdots\sigma_1))^k;$$

$K(p,q)$ is obtained from $\hat{\beta}$ by changing $(p - 1) \cdot k$ crossings. This means that $\frac{(p - 1) \cdot q}{2} \geq d^\times(K(p,q), \hat{\beta})$. Moreover, if $q$ is even, then $\hat{\beta}$ is amphicheiral by Theorem 3.5. So, the signature of $\hat{\beta}$ is 0. Therefore, $d^\tau(K(p,q), \hat{\beta}) \geq \frac{\sigma(K(p,q))}{2}$. □

4. Applications: Knots with braid index 3

Corollary 2.12 says that the braid index and the quasitoric braid index of a knot are the same. From now on, we will explain a method to find a quasitoric braid presentation of a knot with braid index 3.

Let $K$ be a knot with braid index 3. Without loss of generality, we may assume that $K$ has a braid presentation of the form

$$\beta = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_{2m}}^{\varepsilon_{2m}},$$
where each $i_j$ is either 1 or 2 and each $e_j$ is either 1 or $-1$, for a positive integer $m$. For, if $K$ is the closure of $\beta \in B_3$, then the permutation of $\beta$ is either (123) or (132) because $K$ is a knot. Since both either (123) and (132) are even permutations, the word length of $\beta$ must be even.

We claim that every word $w$ with word length 2 is a quasitoric braid word. If $w$ is either $\sigma_1\sigma_1^{-1}, \sigma_1\sigma_1^{-1}\sigma_1\sigma_1^{-1}, \sigma_2\sigma_2^{-1}, \sigma_2\sigma_2^{-1}\sigma_2, \sigma_2\sigma_2^{-1}\sigma_2\sigma_2^{-1}, \sigma_2\sigma_2^{-1}\sigma_2\sigma_2^{-1}\sigma_2\sigma_2^{-1}, \sigma_2\sigma_2^{-1}\sigma_2\sigma_2^{-1}\sigma_2\sigma_2^{-1}\sigma_2\sigma_2^{-1}$, clearly $w$ is quasitoric. If $w$ is either $\sigma_1\sigma_1, \sigma_1\sigma_1^{-1}, \sigma_1\sigma_1^{-1}\sigma_1\sigma_1^{-1}, \sigma_2\sigma_2^{-1}, \sigma_2\sigma_2^{-1}\sigma_2, \sigma_2\sigma_2^{-1}\sigma_2\sigma_2^{-1}$, then it can be presented as a product of quasitoric braid words as seen in the following table:

<table>
<thead>
<tr>
<th>braid word</th>
<th>quasitoric braid word</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_1\sigma_1$</td>
<td>$\sigma_2\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1$</td>
</tr>
<tr>
<td>$\sigma_1\sigma_2$</td>
<td>$\sigma_2\sigma_1\sigma_2\sigma_1^{-1}$</td>
</tr>
<tr>
<td>$\sigma_1\sigma_2^{-1}$</td>
<td>$\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1}$</td>
</tr>
<tr>
<td>$\sigma_1^{-1}\sigma_1$</td>
<td>$\sigma_2\sigma_1\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1}$</td>
</tr>
<tr>
<td>$\sigma_1^{-1}\sigma_2$</td>
<td>$\sigma_2\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1^{-1}$</td>
</tr>
<tr>
<td>$\sigma_1^{-1}\sigma_2^{-1}$</td>
<td>$\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}$</td>
</tr>
<tr>
<td>$\sigma_2\sigma_2$</td>
<td>$\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}$</td>
</tr>
<tr>
<td>$\sigma_2^{-1}\sigma_2$</td>
<td>$\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1}$</td>
</tr>
<tr>
<td>$\sigma_2^{-1}\sigma_2^{-1}$</td>
<td>$\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1}$</td>
</tr>
</tbody>
</table>

By using the above claim inductively, one can change any braid word of braid index 3 to a quasitoric braid word. For example, for $\beta = \sigma_1\sigma_1\sigma_2^{-1}\sigma_1\sigma_1^{-1}\sigma_1\sigma_1^{-1}\sigma_2^{-1}$, the corresponding quasitoric braid presentation is

\[(\sigma_2\sigma_1\sigma_2\sigma_1^{-1})(\sigma_2^{-1}\sigma_1)(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}) (\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}).\]

**Lemma 4.1.** Let $\beta$ be a quasitoric braid of type $(p, q)$ and $K(p, q)$ the torus knot of type $(p, q)$. Then

\[
(1) \quad u(K(p, q)) \leq u(\tilde{\beta}) + d^\infty(\tilde{O}, K(p, q)).
\]

**Proof.** By the triangle inequality, $d^\infty(\tilde{O}, K(p, q)) \leq d^\infty(\tilde{O}, \tilde{\beta}) + d^\infty(\tilde{\beta}, K(p, q))$, where $\tilde{O}$ is the trivial knot. \(\square\)

In [6] and [7], Kronheimer and Mrowka used Gauge theory to prove that the unknotting number of any algebraic knot is equal to the genus of the Milnor fiber. A consequence of the result gives the unknotting number of torus knots of type $(p, q)$:

\[
(2) \quad u(K(p, q)) = \frac{(p-1)(q-1)}{2}.
\]

By using the quasitoric braid presentation together with Kronheimer and Mrowka’s result, we can calculate Gordian distance between the knot $K(3, 5)$ and the $(3, 13)$-torus knot $K(3, 13)$. 

Theorem 4.2. The Gordian distance between the torus knot $K(3, 5)$ and the torus knot $K(3, 13)$ is

$$d^\times(K(3, 5), K(3, 13)) = 8.$$  

Proof. It is well-known that $K(3, 5)$ has a braid presentation $\beta = \sigma_1\sigma_1\sigma_1\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2$. By using our algorithm, we can get a quasitoric braid presentation of $K(3, 5):$

$$\sigma_2\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_2\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1^{-1}.$$  

Since $K(3, 5) = \beta$, we have, by Lemma 4.1,

$$u(K(3, 13)) - u(K(3, 5)) \leq d^\times(K(3, 5), K(3, 13)).$$  

Since $u(K(3, 13)) = 12$ and $u(10_{124}) = 4$ by the equality (2), we have

$$8 \leq d^\times(K(3, 5), K(3, 13)).$$  

On the other hand, since a braid presentation of $K(3, 13)$ is given by

$$\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1,$$

by comparing the braid words of $K(3, 5)$ and $K(3, 13)$, one can see that $K(3, 5)$ can be obtained from $K(3, 13)$ by 8 crossing changes. Indeed,

$$d^\times(K(3, 5), K(3, 13)) \leq 8.$$  

Therefore, we have the result. □

Our results can be used to determine the unknotting number of a knot. The calculation of the unknotting number of a knot is very difficult in general. It is known that

$$2u(K) \leq c(K) - 1,$$

where $c(K)$ denotes the crossing number of $K$. In [11], Taniyama showed that the equality $2u(K) = c(K) - 1$ holds only when $K$ is a $(2, p)$-torus knot.

As an application of our results, we can decide the unknotting number of the knot $10_{139}$.

Theorem 4.3. The unknotting number of the knot $10_{139}$ is 4.
Proof. Since $10_{139} \neq K(2,p)$, Taniyama’s result says that $2u(10_{139}) < c(10_{139}) - 1$. Indeed,

$$u(10_{139}) \leq 4.$$

Notice that the braid index of $10_{139}$ is 3. In fact, $10_{139}$ has a braid presentation:

$$\sigma_1 \sigma_1 \sigma_2 \sigma_1 \sigma_1 \sigma_2 \sigma_2.$$

By applying our algorithm, one can get a quasitoric braid presentation of $10_{139}$:

$$\sigma_2 \sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_2 \sigma_1 \sigma_2 \sigma_1.$$

By Lemma 4.1, we have $u(K(3,13)) - d^\infty(10_{139}, K(3,13)) \leq u(10_{139})$. Since $u(K(3,13)) = 12$,

$$12 - d^\infty(10_{139}, K(3,13)) \leq u(10_{139}).$$

On the other hand, since a braid presentation of $K(3,13)$ is given by

$$\sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1,$$

by comparing the braid words of $10_{139}$ and $K(3,13)$, one can see that $10_{139}$ can be obtained from $K(3,13)$ by 8 crossing changes. Indeed, $d^\infty(10_{139}, K(3,13)) \leq 8$. Therefore, we have the result.

\[\Box\]

Remark 4.4. The unknotting number of the knot $10_{139}$ was calculated by Gibson and Ishikawa [3] before.

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