SECOND-ORDER SYMMETRIC DUALITY IN
MULTIOBJECTIVE PROGRAMMING OVER CONES

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ABSTRACT. In this paper, some omissions in Mishra and Lai [13], have been pointed out and their corrective measures have been discussed briefly.

1. Introduction

A pair of primal and dual problems in mathematical programming is called symmetric if the dual of the dual is the primal problem. Dorn [6] introduced the concept of symmetric duality in quadratic programming. His results were extended to nonlinear convex programming problems by Dantzig et al. [4] and later by Bazaraa and Goode [3] over arbitrary cones.

Mangasarian [11] introduced the concept of second-order duality for nonlinear problems. Since then, many authors [1, 2, 7, 9, 15, 16] have worked on second-order symmetric duality. Mishra and Lai [13] studied Mond-Weir type second-order multiobjective symmetric duality for the following pair of problems:

\[(P) \quad K\text{-minimize} \quad f(x, y) - \frac{1}{2} p^T \nabla_{yy} f(x, y)p
\]
\[\text{subject to} \quad -\nabla_y (\lambda^T f)(x, y) - \nabla_{yy} (\lambda^T f)(x, y) \in C_2^*,
\]
\[y^T [\nabla_y (\lambda^T f)(x, y) + \nabla_{yy} (\lambda^T f)(x, y)] \geq 0,
\]
\[\lambda \in K^*, \quad x \in C_1.
\]

\[(D) \quad K\text{-maximize} \quad f(u, v) - \frac{1}{2} q^T \nabla_{xx} f(u, v)q
\]
\[\text{subject to} \quad \nabla_x (\lambda^T f)(u, v) + \nabla_{xx} (\lambda^T f)(u, v) \in C_1^*,
\]
\[u^T [\nabla_x (\lambda^T f)(u, v) + \nabla_{xx} (\lambda^T f)(u, v)] \leq 0,
\]
\[\lambda \in K^*, \quad v \in C_2,
\]

where \(f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^k\) is a twice differentiable function of \(x\) and \(y\), \(C_1\) and \(C_2\) are closed convex cones with nonempty interiors in \(\mathbb{R}^n\) and \(\mathbb{R}^m\), respectively.

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$K$ is a closed convex pointed cone in $\mathbb{R}^k$ such that $\text{int } K \neq \emptyset$ and $K^*$ is its positive polar cone.

The first term in the objectives of (P) and (D) is a $k$-vector, while the second term is not a $k$-vector. Therefore the models and so the results in [13] seem to be erroneous. Some of the other observations are as follows:

(i) For a vector function $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^k$, it is not clear that what do the authors mean by $\nabla_{xx} f(x, y)$ and $\nabla_{yy} f(x, y)$.

(ii) It is well known that the weak duality theorem gives a relation between the objective functions of the primal and dual problems. It is not so in [13] as the second-order terms in the two objective functions are missing from the conclusion of the weak duality theorem.

(iii) In the strong duality theorem, the assumption that $\nabla_{yyy}(\lambda^T f)(\bar{x}, \bar{y})$ is negative definite is meaningless since $\nabla_{yyy}(\lambda^T f)(\bar{x}, \bar{y})$ is not a matrix.

(iv) The authors simply state that the proof of their strong duality theorem follows on the lines of [5], while the proof in [5] is full of errors (see [7]).

(v) The definitions of $K$-strongly $K$-second-order pseudoinvex functions seem to be inappropriate due to the absence of a second-order derivative term (see [1, 7, 15]).

2. Notations and preliminaries

Let $C_1$ and $C_2$ be closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, with nonempty interiors. Let $\nabla_x f_i$ ($\nabla_y f_i$) denote $n \times 1$ ($m \times 1$) gradient vector with respect to first (second) vector variable and let $\nabla_{xy} f_i$ denote the $n \times m$ matrix. All vectors shall be considered as column vectors.

**Definition 2.1** ([14]). The positive polar cone $C^*$ of a cone $C$ is defined by

$$C^* = \{ z : x^T z \geq 0 \text{ for all } x \in C \}.$$  

We consider the following multiobjective programming problem:

\begin{align*}
(P1) \quad & \text{minimize } f(x) \quad \text{subject to } \quad x \in X^o = \{ x \in S : -g(x) \in Q \},
\end{align*}

where $S \subseteq \mathbb{R}^n$, $f : S \rightarrow \mathbb{R}^k$, $g : S \rightarrow \mathbb{R}^m$, $K$ and $Q$ are closed convex pointed cones with nonempty interiors in $\mathbb{R}^k$ and $\mathbb{R}^m$, respectively.

**Definition 2.2** ([10]). A point $\bar{x} \in X^o$ is an efficient solution of (P1) if there exists no $x \in X^o$ such that $f(\bar{x}) - f(x) \in K \setminus \{0\}$.

For the definitions of $K$-η-bonvex, $K$-η-pseudobonvex and second-order $F$-pseudoconvex functions, refer to [8].

3. Mond-Weir type second-order symmetric duality

We consider the following pair of Mond-Weir type second-order multiobjective symmetric dual programming problems which also aims to correct the dual
Then we will use Theorem 3.1 required in the proof of strong and converse duality theorems. However, these restrictions are pair considered in [13]:

**Primal (MP)**

\[ K-\text{minimize} \quad \{ f_1(x, y) - \frac{1}{2} p_1^T \nabla_{yy} f_1(x, y)p_1, \ldots, f_k(x, y) - \frac{1}{2} p_k^T \nabla_{yy} f_k(x, y)p_k \} \]

subject to \( \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(x, y) + \nabla_{yy} f_i(x, y)p_i) \in C^*_2 \),

\( y^T \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(x, y) + \nabla_{yy} f_i(x, y)p_i) \geq 0, \)

\( \lambda \in \text{int}K^*, x \in C_1, \)

**Dual (MD)**

\[ K-\text{maximize} \quad \{ f_1(u, v) - \frac{1}{2} q_1^T \nabla_{xx} f_1(u, v)q_1, \ldots, f_k(u, v) - \frac{1}{2} q_k^T \nabla_{xx} f_k(u, v)q_k \} \]

subject to \( \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v)q_i) \in C^*_1 \),

\( u^T \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v)q_i) \leq 0, \)

\( \lambda \in \text{int}K^*, v \in C_2, \)

where \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)^T \in \mathbb{R}^k \), \( C_1 \times C_2 \subset S_1 \times S_2 \) and for \( i = 1, 2, \ldots, k \),

(i) \( f_i : S_1 \times S_2 \to \mathbb{R} \) is a thrice differentiable function of \( x \) and \( y \),

(ii) \( p_i \) and \( q_i \) are vectors in \( \mathbb{R}^m \) and \( \mathbb{R}^n \), respectively.

We will use \( p = (p_1, p_2, \ldots, p_k) \) and \( q = (q_1, q_2, \ldots, q_k) \).

**Duality theorems**

We do not need to restrict \( x \in C_1 \) and \( v \in C_2 \) in the programs (MP) and (MD) respectively for proving Theorems 3.1 and 3.2. However, these restrictions are required in the proof of strong and converse duality theorems.

**Theorem 3.1** (Weak duality). Let \( (x, y, \lambda, p) \) be feasible for (MP) and \( (u, v, \lambda, q) \) be feasible for (MD). Let

(i) \( f(\cdot, v) \) be \( K_{\eta_1} \)-bonvex in the first variable at \( u \),

(ii) \( -f(x, \cdot) \) be \( K_{\eta_2} \)-bonvex in the second variable at \( y \), and

(iii) \( \eta_1(x, u) + u \in C_1 \) and \( \eta_2(v, y) + y \in C_2 \).

Then

\[ \{ f_1(x, y) - \frac{1}{2} p_1^T \nabla_{xx} f_1(x, y)p_1, \ldots, f_k(x, y) - \frac{1}{2} p_k^T \nabla_{xx} f_k(x, y)p_k \} \notin K \setminus \{0\}. \]

**Proof.** The proof follows on the lines of Theorem 3.2 [8]. □

The following weak duality theorem can also be proved.
Theorem 3.2 (Weak duality). Let \((x, y, \lambda, p)\) be feasible for (MP) and \((u, v, \lambda, q)\) be feasible for (MD). Let

\[(a_1) \quad \lambda^T f(\cdot, v) \text{ be } \eta_1\text{-pseudobovex in the first variable at } u,
(a_2) \quad -\lambda^T f(x, \cdot) \text{ be } \eta_2\text{-pseudobovex in the second variable at } y,
(a_3) \quad \eta_1(x, u) + u \in C_1 \text{ and } \eta_2(v, y) + y \in C_2,
\]
or

\[(b_1) \quad \lambda^T f(\cdot, v) \text{ be second-order } F\text{-pseudocovex at } u,
(b_2) \quad -\lambda^T f(x, \cdot) \text{ be second-order } F\text{-pseudocovex at } y,
(b_3) \quad F_{x,u}(\xi) + u^T \xi \geq 0 \text{ for } \xi \in C_1^\ast \text{ and } F_{v,y}(\eta) + y^T \eta \geq 0 \text{ for } \eta \in C_2^\ast.
\]

Then

\[
\{f_i(u, v) - \frac{1}{2}q_i^T \nabla_{xx} f_i(u, v) q_i, \ldots, f_k(u, v) - \frac{1}{2}q_k^T \nabla_{xx} f_k(u, v) q_k\}
\]

\[-\{f_i(x, y) - \frac{1}{2}p_i^T \nabla_{yy} f_i(x, y) p_i, \ldots, f_k(x, y) - \frac{1}{2}p_k^T \nabla_{yy} f_k(x, y) p_k\} \not\in K \setminus \{0\}.
\]

In the following theorems (MP)\(\lambda\) and (MD)\(\lambda\) respectively denote the problems (MP) and (MD) when \(\lambda\) is fixed to be \(\hat{\lambda}\).

Theorem 3.3 (Strong duality). Let \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})\) be an efficient solution for (MP). Suppose that

\[\text{(i)} \quad \nabla_{yy} f_i(\bar{x}, \bar{y}) \text{ is positive definite for } i = 1, 2, \ldots, k \text{ and } \sum_{i=1}^k \lambda_i \bar{p}_i^T \nabla_y f_i(\bar{x}, \bar{y}) \geq 0 \text{ or } \nabla_{yy} f_i(\bar{x}, \bar{y}) \text{ is negative definite for } i = 1, 2, \ldots, k \text{ and } \sum_{i=1}^k \lambda_i \bar{p}_i^T \nabla_y f_i(\bar{x}, \bar{y}) \leq 0,
\]

\[\text{(ii)} \quad \text{the set } \{\nabla_y f_i(\bar{x}, \bar{y}) + \nabla_{yy} f_i(\bar{x}, \bar{y}) \bar{p}_i, i = 1, 2, \ldots, k\} \text{ is linearly independent}, \text{ and}
\]

\[\text{(iii)} \quad R^*_+ \subseteq K.
\]

Then, \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)\) is feasible for (MD)\(\lambda\), and the objective function values of (MP) and (MD)\(\lambda\) are equal. Furthermore, if the hypotheses of Theorem 3.1 or Theorem 3.2 are satisfied for all feasible solutions of (MP)\(\lambda\) and (MD)\(\lambda\), then \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})\) is an efficient solution for (MD)\(\lambda\).

Proof. Since \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})\) is an efficient solution for (MP), by the Fritz-John necessary optimality conditions [14], there exist \(\alpha \in K^\ast, \beta \in C_2, \gamma \in \mathbb{R}_+, \) such that the following conditions are satisfied at \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})\) (for simplicity, we write \(\nabla_x f_i, \nabla_{xy} f_i\) instead of \(\nabla_x f_i(\bar{x}, \bar{y}), \nabla_{xy} f_i(\bar{x}, \bar{y})\) etc.):

\[
(x - \bar{x})^T \sum_{i=1}^k \alpha_i (\nabla_x f_i - \frac{1}{2}(\nabla_x (\nabla_{yy} f_i) \bar{p}_i))^T \bar{p}_i
\]

\[+ \sum_{i=1}^k \bar{\lambda}_i (\nabla_{yx} f_i + \nabla_x (\nabla_{yy} f_i) \beta - \gamma \bar{y})] \geq 0 \text{ for all } x \in C_1,
\]
\[(y - \bar{y})^T \left( \sum_{i=1}^{k} \alpha_i (\nabla_y f_i - \frac{1}{2} \nabla_y (\nabla_{yy} f_i)\bar{p}_i) \right) \]
\[+ \sum_{i=1}^{k} \lambda_i (\nabla_{yy} f_i + \nabla_y (\nabla_{yy} f_i)\bar{p}_i)(\beta - \gamma \bar{y}) \]
\[-\gamma \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_y f_i + \nabla_{yy} f_i\bar{p}_i) \geq 0 \text{ for all } y \in \mathbb{R}^m, \]
\[\left[ (\beta - \gamma \bar{y})^T (\nabla_y f_i + \nabla_{yy} f_i\bar{p}_i) \right](\lambda_i - \bar{\lambda}_i) \geq 0, \]
\[i = 1, 2, \ldots, k \text{ for all } \lambda \in \text{int} K^*, \]
\[\left[ (\beta - \gamma \bar{y})\bar{\lambda}_i - \alpha_i \bar{p}_i \right]^T \nabla_{yy} f_i = 0, \quad i = 1, 2, \ldots, k, \]
\[\beta^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_y f_i + \nabla_{yy} f_i\bar{p}_i) = 0, \]
\[\gamma \bar{y}^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_y f_i + \nabla_{yy} f_i\bar{p}_i) = 0, \]
\[(\alpha, \beta, \gamma) \neq 0. \]
\[\square \]

Following the proof of Theorem 3.4 [8], it can be proved that \((\bar{x}, \bar{y}, \bar{q})\) is an efficient solution of \((MD)_{\bar{\lambda}}\).

**Remark 3.1.** In multiobjective programming for weak duality theorems, one requires same \(\lambda\) for the primal and dual feasible solutions and so for strong duality theorems, \(\bar{\lambda}\) corresponding to the optimal (weak efficient, efficient or properly efficient) solution of the primal problem is required to be fixed in the dual problem. Therefore the above proof gives that \((\bar{x}, \bar{y}, \bar{q} = 0)\) is an efficient solution for \((MD)_{\bar{\lambda}}\) and not that \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{q} = 0)\) is an efficient solution for \((MD)\). In the literature [12, 13, 16] optimality for the dual problem (Wolfe or Mond-Weir type) has been claimed, which is not correct.

**Theorem 3.4** (Converse duality). Let \((\bar{u}, \bar{v}, \bar{\lambda}, \bar{q})\) be an efficient solution for \((MD)\). Suppose that

(i) \(\nabla_{xx} f_i(\bar{u}, \bar{v})\) is positive definite for all \(i = 1, 2, \ldots, k\) and
\[\sum_{i=1}^{k} \lambda_i \bar{q}_i^T \nabla_{xx} f_i(\bar{u}, \bar{v}) \geq 0 \text{ or} \]
\[\nabla_{xx} f_i(\bar{u}, \bar{v})\) is negative definite for all \(i = 1, 2, \ldots, k\) and
\[\sum_{i=1}^{k} \lambda_i \bar{q}_i^T \nabla_{xx} f_i(\bar{u}, \bar{v}) \leq 0, \]
(ii) the set \(\{\nabla_x f_i(\bar{u}, \bar{v}) + \nabla_{xx} f_i(\bar{u}, \bar{v})\bar{q}_i, i = 1, 2, \ldots, k\}\) is linearly independent, and
(iii) \(\mathbb{R}_+^k \subseteq K.\)
Then \((\bar{u}, \bar{v}, \bar{p} = 0)\) is feasible for \(\text{(MP)}_{\lambda}\), and the objective function values of \(\text{(MP)}_{\lambda}\) and \(\text{(MD)}\) are equal. Furthermore, if the hypotheses of Theorem 3.1 or Theorem 3.2 are satisfied for all feasible solutions of \(\text{(MP)}_{\lambda}\) and \(\text{(MD)}_{\lambda}\), then \((\bar{u}, \bar{v}, \bar{p})\) is an efficient solution for \(\text{(MP)}_{\lambda}\).

**Proof.** Follows on the lines of Theorem 3.3. \(\square\)

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