SYMMETRY OF COMPONENTS FOR RADIAL SOLUTIONS OF $\gamma$-LAPLACIAN SYSTEMS

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Abstract. In this paper, we give several sufficient conditions ensuring that any positive radial solution $(u, v)$ of the following $\gamma$-Laplacian systems in the whole space $\mathbb{R}^n$ has the components symmetry property $u \equiv v$

\[
\begin{aligned}
-\text{div}(|\nabla u|^{\gamma-2}\nabla u) &= f(u, v) \quad \text{in } \mathbb{R}^n, \\
-\text{div}(|\nabla v|^{\gamma-2}\nabla v) &= g(u, v) \quad \text{in } \mathbb{R}^n.
\end{aligned}
\]

Here $n > \gamma, \gamma > 1$.

Thus, the systems will be reduced to a single $\gamma$-Laplacian equation:

\[-\text{div}(|\nabla u|^{\gamma-2}\nabla u) = f(u) \quad \text{in } \mathbb{R}^n.
\]

Our proofs are based on suitable comparison principle arguments, combined with properties of radial solutions.

1. Introduction

In 2008, Li and Ma [10] studied the stationary Schrödinger system

(1.1) \[
\begin{aligned}
-\Delta u &= u^pv^q \quad \text{in } \mathbb{R}^n, \\
-\Delta v &= u^qv^p \quad \text{in } \mathbb{R}^n,
\end{aligned}
\]

and obtained a components symmetry result:

Proposition 1.1. Assume $n > \gamma, 1 \leq p, q \leq \frac{n+2}{n-2}$ and $p + q = \frac{n+2}{n-2}$. Then any $(L^{2\gamma}(\mathbb{R}^n))^2$-positive solution pair $(u, v)$ to (1.1) is radial symmetric, and hence $u \equiv v = a(b^2 + |x-x_0|^2)^{(2-n)/2}$ with $a, b > 0$ and $x_0 \in \mathbb{R}^n$.

The proof was achieved by the classification result in [4] and the method of moving planes based on the conformal invariant property. Afterwards, Lei and Li ([8]) studied the asymptotic radial symmetry and decay estimates of positive integrable solutions of

(1.2) \[
\begin{aligned}
-\text{div}(|\nabla u|^{\gamma-2}\nabla u) &= u^pv^q \quad \text{in } \mathbb{R}^n, \\
-\text{div}(|\nabla v|^{\gamma-2}\nabla v) &= v^pu^q \quad \text{in } \mathbb{R}^n,
\end{aligned}
\]

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where \( n > \gamma, \gamma > 1 \), and \( p, q > 0 \) and \( p + q = \frac{n\gamma}{n-\gamma} - 1 \).

In 2012, Quittner and Souplet studied the more general Laplacian systems (cf. [12])

\[
\begin{aligned}
-\Delta u &= f(u, v) \quad \text{in } \mathbb{R}^n, \\
-\Delta v &= g(u, v) \quad \text{in } \mathbb{R}^n,
\end{aligned}
\]

under some ‘monotonicity’ assumption

\[
(X - Y)[f(X, Y) - g(X, Y)] \leq 0, \quad X, Y \geq 0,
\]

and also obtained further interesting components symmetry results.

Such class of systems appears in the modeling of Bose-Einstein condensates which is described by the static Schrödinger equations [11]. The physical and mathematic background can be see in [1] and [3] and other related references.

In this paper, we expect to generalize those components symmetry property in [12] to the \( \gamma \)-Laplacian systems

\[
\begin{aligned}
-\text{div}(|\nabla u|^{\gamma-2}\nabla u) &= f(u, v) \quad \text{in } \mathbb{R}^n, \\
-\text{div}(|\nabla v|^{\gamma-2}\nabla v) &= g(u, v) \quad \text{in } \mathbb{R}^n.
\end{aligned}
\]

Here \( n > \gamma, \gamma > 1 \), and \( f, g \) satisfy (1.4) and other suitable growth assumptions on \( f, g \). In what follows, we assume that \( f, g : [0, \infty) \to \mathbb{R} \) are continuous.

For the \( \gamma \)-Laplacian equations with \( \gamma \neq 2 \), it seems difficult to handle the general classical solutions in view of its nonlinearity and degeneration. As a try for it, we only consider the radial classical solutions in this paper.

We say that a couple of nonnegative functions \((u, v)\) is semitrivial if one component is equal to 0 and the other is not (with the convention \(0^0 = 1\)).

**Theorem 1.2.** Let \( n > \gamma, \gamma > 1 \), and \( 0 \leq p, t \leq \frac{(n-1)\gamma}{n-\gamma} - 1 \). Assume that \( f, g \) satisfy (1.4), and for each \( \eta > 0 \), there exists \( c = c(\eta) > 0 \) such that

\[
f(u, v) \geq cu^p \quad \text{for all } v \geq \eta, u \geq 0,
\]

and

\[
g(u, v) \geq cv^t \quad \text{for all } u \geq \eta, u \geq 0.
\]

Then any nonnegative radial solution \((u, v)\) of (1.5) is either semitrivial or satisfies \(u \equiv v\).

The following corollary is a special case of Theorem 1.2 concerning the system

\[
\begin{aligned}
-\text{div}(|\nabla u|^{\gamma-2}\nabla u) &= u^p v^q \quad \text{in } \mathbb{R}^n, \\
-\text{div}(|\nabla v|^{\gamma-2}\nabla v) &= v^t u^s \quad \text{in } \mathbb{R}^n.
\end{aligned}
\]

Here \( n > \gamma, \gamma > 1 \), and \( p, q, t, s \geq 0 \).

**Corollary 1.3.** Let \( n > \gamma, \gamma > 1 \),

\[
q - t = s - p \geq 0,
\]
and

\[ 0 \leq p, t \leq \frac{(n-1)\gamma}{n-\gamma} - 1. \]  

Then any nonnegative radial solution \((u, v)\) of (1.8) is either semitrivial or satisfies \(u \equiv v\).

The following corollary is also a special case of Corollary 1.3 concerning the system (1.2).

**Corollary 1.4.** Assume \(n > \gamma, \gamma > 1\), and \(p + q = \frac{n\gamma}{n-\gamma} - 1\). Then any positive radial solution \((u, v)\) of (1.2) satisfies \(u \equiv v\).

**Remark 1.1.** (i) Comparing with the works of [5] and [10], the positive solutions of (1.2) with \(\gamma \neq 2\) may have not the radial symmetry property even if the exponent \(p + q\) satisfies the critical condition for Sobolev embedding. In fact, the \(\gamma\)-Laplacian equations have not the conformal invariant property except for some energy minimal solutions (ground states) (cf. [2]). Therefore, the classification result is hardly obtained.

(ii) When \(u \equiv v\), (1.2) is reduced to a single equation. According to [8] and [9], the integrable solutions of this single equation decay with the fast rate when \(|x| \to \infty\).

According to the conclusion pointed out in [12], the condition \(q - t = s - p\) in (1.9) is necessary. In addition, the following theorem implies that the conditions (1.9) and (1.10) are not purely technical.

**Theorem 1.5.** Let \(n > \gamma, \gamma > 1\) and \(p, q, t, s \geq 0\).

(i) Assume \(q - t = s - p \geq 0\). Then any nonnegative solution \((u, v)\) of (1.8) satisfies \(u \geq v\) or \(v \geq u\). Furthermore, if \(p + q \leq \frac{(n-1)\gamma}{n-\gamma} - 1\), the nonnegative radial solution is semitrivial.

(ii) Let \(q = t \geq \frac{n\gamma}{n-\gamma} - 1\) and \(p = s \geq 0\). Then there exists a positive solution \((u, v)\) of (1.8), such that \(u > v\) in \(\mathbb{R}^n\). More precisely, we have \(\lim_{|x| \to \infty} v(x) = 0\) and \(u \equiv v + 1\). Moreover, if \(q = s\), then the couple \((v, u)\) is also a solution.

(iii) Let \(p = t \geq \frac{n\gamma}{n-\gamma} - 1 - \frac{\gamma^2}{2(n-\gamma)}\) and \(q = s = p - (\gamma - 1)\). Then there exists a positive function \(w\) such that the couple \((u, v) = (cw, w/c)\) solves (1.8) for any \(c > 0\).

### 2. Proof of Theorem 1.2

The properties of our main study of the radial solution \(U(x) = u(r)\) for

\[ -\Delta_s U := -\text{div}(|\nabla U|^{\gamma-2}\nabla U) \geq 0, \]

in our arguments is contained in the following lemma.

**Lemma 2.1.** Let \(U \geq 0\) belong to \(C^2(\mathbb{R}^n)\). If \(U(x) = u(r)\) is a radial solution of (2.1), then

(i) \(u'(r) \leq 0\) for \(r > 0\);
(ii) $u(r) \geq l := \lim_{R \to \infty} u(R)$ for $r \geq 0$.

Proof. Clearly, if $U$ solves (2.1), then $u$ is a radial solution of

$$-(r^{n-1}|u'|^{\gamma-2}u')' \geq 0, \quad r \geq 0.$$  

By integrating on both sides of (2.2) from 0 to $R$ with $R > 0$, we obtain

$$|u'(R)|^{\gamma-2}u(R) \leq 0,$$

and hence (i) is verified.

In addition, $u$ is nonincreasing and nonnegative. Thus, $l$ is well-defined and (ii) is proved. $\square$

The following lemma plays the key role in this paper. The idea of the proof comes from [12] which appears in Souplet’s earlier paper (cf. Lemma 2.7 of [14]).

**Lemma 2.2.** Assume that $f, g$ satisfy

$$f(X, Y) \geq g(X, Y), \quad 0 \leq X \leq Y.$$  

If $(u, v)$ is a nonnegative radial solution of (1.5) such that

$$\lim_{R \to \infty} \inf v(R) = 0,$$

then $v \leq u$ in $\mathbb{R}^n$.

Proof. Let $w = v - u$. By (2.3), we have

$$\Delta \gamma v - \Delta \gamma u = f - g \geq 0 \quad \text{in} \quad \{w \geq 0\}.$$  

We prepare a standard smooth replacement of the positive part function. Let $H \in C^2(\mathbb{R})$ be a function with the following properties

$$0 \leq H(t) \leq t_+ = \max(t, 0) \quad \text{for} \quad t \in \mathbb{R}, \quad H'(t), H''(t) > 0 \quad \text{for} \quad t > 0.$$  

We then set

$$h(R) := H(w(R)) \quad \text{for} \quad R > 0.$$  

Using (2.6), we have

$$0 \leq h(R) \leq w_+(R) \leq v(R), \quad R > 0$$

Consequently, in view of (2.4), we have

$$\lim_{r \to \infty} \inf h(r) = 0.$$  

It follows that there exists a sequence $R_i \to \infty$ such that $h'(R_i) < 0$. According to Lemma 2.1(i) the integral mean value theorem, we get

$$|\partial_r u|^{\gamma-1} - |\partial_r v|^{\gamma-1} = (\gamma - 1) \int_0^1 [t|\partial_r u| + (1-t)|\partial_r v|]^{\gamma-2} dt \partial_r w.$$
In view of $h'(R_i) < 0$, there holds
\[
\int_{\partial B_{R_i}} (|\partial_r u|^{-\gamma} - |\partial_r v|^{-\gamma})H'(w)d\theta
\]
(2.8)
\[= (\gamma - 1)|S^{n-1}|h'(R_i)R_i^{n-1}\int_0^1 [t|\partial_r u| + (1-t)|\partial_r v|]^{-\gamma-2}dt \leq 0.\]

On the other hand, when $\gamma > 2$, we have
\[
|\nabla w|^\gamma \leq c(|\nabla v|^{-2}\nabla v - |\nabla u|^{-2}\nabla u)\nabla (v - u);
\]
and when $1 < \gamma \leq 2$, we have
\[
(|\nabla v| + |\nabla u|)^{-2}\nabla |\nabla w|^2 \leq c(|\nabla v|^{-2}\nabla v - |\nabla u|^{-2}\nabla u)\nabla (v - u).
\]

Therefore,
(i) when $\gamma > 2$, by (2.8) and (2.9), we have
\[
0 \leq \int_{B_{R_i}} H''(w)|\nabla w|^\gamma dx
\]
\[\leq \int_{B_{R_i}} H''(w)(|\nabla v|^{-2}\nabla v - |\nabla u|^{-2}\nabla u)\nabla (v - u)dx
\]
\[= \int_{B_{R_i}} (|\nabla v|^{-2}\nabla v - |\nabla u|^{-2}\nabla u)\nabla H'(w)dx
\]
\[= \int_{\partial B_{R_i}} (|\partial_r u|^{-1} - |\partial_r v|^{-1})H'(w)ds - \int_{B_{R_i}} (\Delta_\gamma v - \Delta_\gamma u)H'(w)dx
\]
\[\leq - \int_{B_{R_i}} (f-g)H'(w)dx \leq 0.
\]

(ii) When $1 < \gamma \leq 2$, by the same argument of (i), from (2.10) we also deduce that
\[
0 \leq \int_{B_{R_i}} (|\nabla v| + |\nabla u|)^{-2}H''(w)|\nabla w|^2 dx \leq 0.
\]

Therefore, for $\gamma > 1$, we always have $\nabla w = 0$ on $\mathbb{R}^n$, which implies that $w$ is a constant. Going back to (2.7) and (2.4), we conclude that $w_+ = 0$, and hence $w \leq u$. \qed

**Proof of Theorem 1.2.** In view of Lemma 2.2, it suffices to show that either ($u, v$) is semitrivial or
\[
(2.11) \quad \lim_{R \to \infty} \inf u(R) = \lim_{R \to \infty} \inf v(R) = 0.
\]
Assume, for instance, that the first limit does not hold. Then there exists $C > 0$ such that $u \geq C$ in $\mathbb{R}^n$ by (ii) of Lemma 2.1. Thus, $-\Delta_\gamma v \geq \bar{c}v^r$ in $\mathbb{R}^n$ by assumption (1.7). According to the Liouville type results in [13], it is known that $v \equiv 0$ by virtue of $r \leq \frac{(n-1)\gamma}{n-\gamma} - 1$. The proof is complete. \qed
Remark 2.1. A couple \((u, 0)\) is a semitrivial solution of (1.8) if and only if \(r > 0\), and either \(q > 0\) and \(u\) is a \(p\)-harmonic function (i.e., it solves 
\[-\text{div}(\nabla u|^{\frac{2}{n\gamma - 2}} \nabla u) = 0,\]

or \(q = 0\), \(n > \gamma\), \(p \geq \frac{n\gamma}{n\gamma - 1}\), and \(u\) solves
\[-\text{div}(\nabla u|^{\frac{2}{n\gamma - 2}} \nabla u) = u^{p} \text{ in } \mathbb{R}^{n}\]
(the existence is showed in [13]). A symmetric statement of course still holds for semitrivial solutions of the form \((0, v)\).

3. Proof of Theorem 1.5

First we state a Pohozaev type result.

Lemma 3.1. If the boundary value problem
\[
\begin{cases}
-\Delta_{\gamma} v = f(v) = (1 + v)^p v^q, & x \in B_{R}, \\
v = 0, & x \in \partial B_{R},
\end{cases}
\]

(3.1)

has positive radial solutions, then
\[
\int_{B_{R}} v f(v) dx < \frac{n\gamma}{n - \gamma} \int_{B_{R}} F(v) dx.
\]

Here \(B_{R} = B_{R}(0)\) and \(F(v) = \int_{0}^{v} f(t) dt\).

Proof. Let \(v\) be the positive solution of the boundary value problem (3.1).

We multiply the equation in (3.3) by \((x \cdot \nabla v)\) and integrate over \(B_{R}\). Using integration by parts, we obtain
\[
- \int_{B_{R}} \Delta_{\gamma} v (x \cdot \nabla v) dx
\]
\[
= - \int_{\partial B_{R}} (|\nabla v|^{\gamma - 2} \nabla v \cdot \nu)(\nabla v \cdot x) ds + \int_{B_{R}} (|\nabla v|^{\gamma - 2} \nabla v) \nabla (\nabla v \cdot x) dx
\]
\[
= - R \int_{\partial B_{R}} (|\nabla v|^{\gamma - 2} \frac{\partial v}{\partial \nu})^2 ds + \int_{B_{R}} |\nabla v|^\gamma dx + \frac{1}{\gamma} \int_{B_{R}} x \cdot \nabla (|\nabla v|^{\gamma}) dx
\]
\[
= - \frac{n - \gamma}{\gamma} \int_{B_{R}} |\nabla v|^\gamma dx + \frac{1 - \gamma}{\gamma} R \int_{\partial B_{R}} |\nabla v|^\gamma ds.
\]

The last equality is deduced by the radial symmetry of \(v\). In addition, we get
\[
\int_{B_{R}} f(v)(x \cdot \nabla v) dx = \int_{B_{R}} x \cdot \nabla F(v) dx
\]
\[
= R \int_{\partial B_{R}} F(v) ds - n \int_{B_{R}} F(v) dx
\]
\[
= - n \int_{B_{R}} F(v) dx
\]
by using \(F(v) = 0\) on \(\partial B_{R}\). Thus,
\[
- \frac{n - \gamma}{\gamma} \int_{B_{R}} |\nabla v|^\gamma dx + \frac{1 - \gamma}{\gamma} R \int_{\partial B_{R}} |\nabla v|^\gamma ds = - n \int_{B_{R}} F(v) dx.
\]
Noting $\gamma > 1$, we obtain
\[ \frac{1 - \gamma}{\gamma} R \int_{\partial B_R} |\nabla v|^\gamma ds < 0, \]
and hence
\[ \frac{n - \gamma}{\gamma} \int_{B_R} |\nabla v|^\gamma dx < n \int_{B_R} F(v)dx. \]

On the other hand, multiply the equation in (3.3) by $v$. Integrating by parts, we obtain
\[ \int_{B_R} v f(v)dx = - \int_{B_R} (\Delta_\gamma v) v dx = \int_{\partial B_R} |\nabla v|^\gamma ds. \]
Combining with the result above, we complete the proof easily. □

Proof of Theorem 1.5. (i) In view of Lemma 2.2, it is sufficient to check that either $\lim_{R \to \infty} \inf u(R) = 0$ or $\lim_{R \to \infty} \inf v(R) = 0$, which implies $u \leq v$ or $v \leq u$ by Lemma 2.2. If these were not the case, then $u, v \geq C > 0$ in $\mathbb{R}^n$ by (ii) of Lemma 2.1. Thus, $-\Delta_\gamma u \geq c > 0$ in $\mathbb{R}^n$. Namely,
\[ -R^{1-n}(R^{n-1}|u'|^{-2}u')' \geq c. \]
Multiplying by $R^{n-1}$ and integrating from $0$ to $R$, we see that
\[ (3.2) \quad |u'(R)|^{-2}u'(R) \leq -cR \quad \text{for} \quad R > 0. \]
Noting (i) of Lemma 2.1, we get $-u' > 0$ which implies
\[ |u'(R)|^{-2}u'(R) = -(-u'(R))^{-1}. \]
Combining with (3.2) yields
\[ u'(R) \leq -(cR)^{\frac{1}{n-1}} \quad \text{for} \quad R > 0. \]
Integrating from $r_0$ to $r$, we get
\[ u(r) \leq u(r_0) - \frac{\gamma - 1}{\gamma} c r^{\frac{1}{n-1}}. \]
When $r$ is sufficiently large, $u$ is negative. It is a contradiction.

Moreover, without loss of generality, we assume $u \leq v$. Then from (1.8) it follows that
\[ -\operatorname{div}(|\nabla u|^{-2} \nabla u) \geq u^{p+q} \quad \text{in} \quad \mathbb{R}^n. \]
According to the Liouville type results in [13], we get $u \equiv 0$ in view of $p + q \leq (n-1)\gamma - 1$.

(ii) We look for a solution such that $u = v + 1$. Then system (1.8) becomes equivalent to the single equation
\[ (3.3) \quad -\Delta_\gamma v = f(v) = (1 + v)^p v^q, \quad x \in \mathbb{R}^n. \]
Let $F(t) = \int_0^t f(\tau)d\tau$. We claim that
\[ (3.4) \quad t f(t) - \frac{\gamma}{n-\gamma} F(t) \geq 0, \quad t \geq 0. \]
In fact, in view of (3.3) and integrating by parts, we have

\[ F(t) = \int_0^t (1 + \tau)^p \tau^q d\tau \]

\[ = \frac{t^{q+1}(1 + t)^p}{q + 1} - \int_0^t \frac{\tau^{q+1}}{q + 1} d((1 + \tau)^p) \]

\[ = \frac{1}{q + 1}(1 + t)^p t^{q+1} - \int_0^t p(1 + \tau)^{p-1} \tau^{q+1} d\tau \]

\[ \leq \frac{1}{q + 1} t f(t). \]

Since \( q \geq \frac{n\gamma}{n-\gamma} - 1 \), we obtain

\[ t f(t) \geq (q + 1) F(t) \geq \frac{n\gamma}{n-\gamma} F(t). \]

Therefore, (3.4) is verified.

According to Lemma 3.1, we know the boundary value problem (3.1) does not admit any positive solution by noting (3.4).

Next, consider the following initial value problem

\[
\begin{cases}
-t^{1-n} (t^{n-1} |v'|^{\gamma-2} v')' = f(v), & t > 0, \\
v(0) = 1, v'(0) = 0.
\end{cases}
\]

(3.5)

Clearly, one of the following two cases holds

Case 1: \( v > 0 \), \( v' \leq 0 \) for all \( t > 0 \);

Case 2: \( v \) has the first zero \( R_* \).

We claim that Case 2 does not happen, since this would contradict the above nonexistence statement on the ball \( B_{R_*} \). We conclude that problem (3.5) and hence (3.3), admits a positive entire solution \( v \) which is decaying to zero. More precisely, according to the results in [6] and [7], \( v \) decays fast with the rate \( \frac{n}{n-\gamma} \) when \( q = \frac{n\gamma}{n-\gamma} - 1 \) and slowly with rate \( \frac{\gamma}{q-(\gamma-1)} \) when \( q > \frac{n\gamma}{n-\gamma} - 1 \).

(iii) Let \( (u, v) = (cw, c^{-1}w) \) with \( c \) a positive constant. Then system (1.8) becomes equivalent to

\[ -\Delta_\gamma w = w^{2p-\gamma+1}. \]

According to the existence results in [13], we see that this equation admits positive solutions by virtue of \( 2p-\gamma+1 \geq \frac{n\gamma}{n-\gamma} - 1 \). \( \square \)

References


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