ZERO BASED INVARIANT SUBSPACES AND FRINGE OPERATORS OVER THE BIDISK

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Abstract. Let $M$ be an invariant subspace of $H^2$ over the bidisk. Associated with $M$, we have the fringe operator $F^M_z$ on $M \ominus wM$. It is studied the Fredholmness of $F^M_z$ for (generalized) zero based invariant subspaces $M$. Also $\ker F^M_z$ and $\ker (F^M_z)^*$ are described.

1. Introduction

Let $H^2 = H^2(D^2)$ be the Hardy space over the bidisk $D^2$ with two variables $z, w$. We write $\|f\|$ the Hardy space norm of $f \in H^2$. We denote by $T_z, T_w$ the multiplication operators on $H^2$ by $z, w$. A nonzero closed subspace $M$ of $H^2$ is said to be invariant if $T_zM \subset M$ and $T_wM \subset M$. The structure of invariant subspaces of $H^2$ is fairly complicated and at this moment it seems to be out of reach (see [1, 3, 6, 7]). We have

$$M = \bigoplus_{n=0}^{\infty} w^n(M \ominus wM),$$

so the space $M \ominus wM$ contains many informations of an invariant subspace $M$. In [7], Yang studied the operator $F^M_z$ on $M \ominus wM$ defined by

$$F^M_z f = P_{M \ominus wM}T_z f, \quad f \in M \ominus wM,$$

where $P_A$ is the orthogonal projection from $H^2$ onto $A \subset H^2$, and he called $F^M_z$ the fringe operator of $M$.

Let $N = H^2 \ominus M$. We set

$$\Omega(M) = M \ominus (zM + wM) \quad \text{and} \quad \tilde{\Omega}(N) = N \ominus (T_z^* N + T_w^* N).$$

We have $\Omega(M) \neq \{0\}$.

(1.1) \hspace{1cm} $\Omega(M) = \{f \in M : T_z^* f \in N, T_w^* f \in N\}$

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and
\[(1.2) \quad \tilde{\Omega}(N) = \{ f \in N : T_z f \in M, T_w f \in M \}.\]

It is known that \(\tilde{\Omega}(N)\) may be an empty set. Generally, we do not know whether \(zM + wM\) is closed or not. In [7], Yang pointed out that \(zM + wM\) is closed if and only if \(F^M_z\) has closed range. Let \(H^\infty = H^\infty(\mathbb{D}^2)\) be the space of bounded analytic functions on \(\mathbb{D}^2\) with the supremum norm \(\| \cdot \|_\infty\). In [7], Yang also showed that if there is \(h \in M \cap H^\infty\) satisfying \(h(0, 0) \neq 0\), then \(zM + wM\) is closed and \(\Omega(M) = C \cdot P_M 1\). A bounded linear operator \(T\) on a separable Hilbert space is called Fredholm if \(T\) has closed range, \(\dim \ker T < \infty\) and \(\dim \ker T^* < \infty\) (see [2]). In this case, \(\ind_T = \dim \ker T - \dim \ker T^*\) is called the Fredholm index of \(T\). The Fredholmness is one of the important subjects in operator theory. In [7], Yang pointed out that \(\ker F^M_z = w\tilde{\Omega}(N)\) and \(\ker (F^M_z)^* = \Omega(M)\).

Hence if \(F^M_z\) is Fredholm, then \(\ind_{F^M_z} = \dim \tilde{\Omega}(N) - \dim \Omega(M)\).

We shall study the following questions in this paper.

(Q1) How to prove the closedness of \(zM + wM\)?

(Q2) How to describe the elements in \(\Omega(M)\)?

(Q3) How to describe the elements in \(\tilde{\Omega}(N)\)?

It is difficult to answer these questions completely. In this paper, we study these questions for the zero based invariant subspaces of \(H^2\). Let \(E\) be a nonvoid subset \(\mathbb{D}^2\) and

\[I(E) = \{ f \in H^2 : f = 0 \text{ on } E \}.\]

Then \(I(E)\) is an invariant subspace and \(I(E)\) is called a zero based invariant subspace for \(E\). We may assume that \(I(E) \neq \{0\}\) and

\[E = Z(I(E)) = \{ \lambda \in \mathbb{D}^2 : f(\lambda) = 0 \text{ for every } f \in I(E) \}.\]

In Section 2, we shall study the above questions for \(I(E)\). We shall answer (Q3) for \(M = I(E)\).

Let \(M\) be an invariant subspace of \(H^2\) with \(M \subseteq I(E)\) and \(Z(M) = E\). We write \(\mathbb{N} = \{0, 1, 2, \ldots\}\) and

\[D^n_z D^m_w = \frac{\partial^n}{\partial z^n} \frac{\partial^m}{\partial w^m}, \quad (n, m) \in \mathbb{N}^2,\]

where \(D^0_z D^m_w = D^m_w\), \(D^n_z D^0_w = D^n_z\) and \(D^0_z D^0_w = 1\). For each \(\lambda \in E\), let

\[A_M(\lambda) = \{(n, m) \in \mathbb{N}^2 : (D^n_z D^m_w f)(\lambda) = 0 \text{ for every } f \in M\}.\]

Since \(Z(M) = E\), \((0, 0) \in A_M(\lambda) \subseteq \mathbb{N}^2\) for every \(\lambda \in E\). We have

\[I(E) = \bigcap_{\lambda \in E} \{ f \in H^2 : (D^n_z D^m_w f)(\lambda) = 0 \text{ for every } (n, m) \in A_{I(E)}(\lambda) \}.\]
Let
\[ \widetilde{M} = \bigcap_{\lambda \in E} \{ f \in H^2 : (D_z^m D_w^n f)(\lambda) = 0 \text{ for every } (n, m) \in A_M(\lambda) \}. \]

Then \( \widetilde{M} \) is an invariant subspace. Since \( A_{I(E)}(\lambda) \subset A_M(\lambda) \) for every \( \lambda \in E \), we have that \( M \subset \widetilde{M} \subset I(E) \) and \( E \subset Z(M) \subset Z(M) = E \). Hence \( Z(\widetilde{M}) = E \).

Since \( I(E) = \widetilde{I}(E) \), as a generalization of a zero based invariant subspace \( I(E) \) we assume that \( M = \widetilde{M} \).

Let
\[ M_0 = \bigcap_{\lambda \in E \setminus \{(0,0)\}} \{ f \in H^2 : (D_z^m D_w^n f)(\lambda) = 0 \text{ for every } (n, m) \in A_M(\lambda) \}. \]

Then \( M_0 \) is an invariant subspace, \( M = \widetilde{M} \subset M_0 \), and if \( (0,0) \notin E \), then \( \widetilde{M} = M_0 \). In this paper, \( M_0 \) plays an important role. In Section 3, we shall study questions (Q1), (Q2) and (Q3).

In Section 4, we shall study the special cases. Let \( \Lambda = \{(a, \alpha) : a \in \mathbb{D}\} \). Then \( I(\Lambda) = [z-w] \), where \([L]\) is the smallest invariant subspace containing \( L \subset H^2 \).

Let \( M \) be an invariant subspace satisfying that \( M \subset [z-w] \), \( Z(M) = \Lambda \), \( M = \widetilde{M} \) and \( M_0 = [z-w] \). We shall show that \( F_z^M \) is Fredholm and \( \text{ind } F_z^M = -1 \).

We shall also describe \( \tilde{\Omega}(N) \) and \( \Omega(M) \) completely.

We have a conjecture that if \( \text{dim } \Omega(M) < \infty \), then \( F_z^M \) is Fredholm and \( \text{ind } F_z^M = -1 \). Our results in this paper support that this conjecture is true (see [4, 5, 7, 8, 9, 10, 11]).

2. Zero based invariant subspaces

Let \( M \) be an invariant subspace of \( H^2 \) and \( N = H^2 \ominus M \). In [7], Yang pointed out the following facts.

**Lemma 2.1.** \( \ker F_z^M = w\tilde{\Omega}(N) \) and \( \ker (F_z^M)^* = \Omega(M) \).

**Lemma 2.2.** \( zM + wM \) is closed if and only if \( F_z^M \) has closed range.

**Lemma 2.3.** If there is \( h \in M \cap H^\infty \) satisfying \( h(0,0) \neq 0 \), then \( zM + wM \) is closed and \( \tilde{\Omega}(M) = C \cdot P_M 1 \).

Actually he showed that \( zM + wM = M \cap (zH^2 + wH^2) \) under the assumption in Lemma 2.3. Using the same idea, we have the following.

**Proposition 2.4.** If there is \( h \in M \cap H^\infty \) satisfying \( h(0,0) \neq 0 \), then \( F_z^M \) is Fredholm and \( \text{ind } F_z^M = -1 \).

**Proof.** We shall show \( \tilde{\Omega}(N) = \{0\} \). We may assume that \( h(0,0) = 1 \) and write \( h = 1 + zh_1(z) + wh_2 \) for some \( h_1(z), h_2 \in H^\infty \). Let \( f \in \tilde{\Omega}(N) \). We have
\[ f = f(h - zh_1(z) - wh_2) = fh - zfh_1(z) - wfh_2. \]
By (1.2), \(zf \in M\) and \(wf \in M\). So \(zh_1(z) + wh_2 \in M\). Since \(h \in M \cap H^\infty\), we have \(fh \in M\), so by the above we have \(f \in M\). Since \(f \perp M\), we have \(f = 0\). Thus \(\tilde{\Omega}(N) = \{0\}\). By Lemmas 2.1–2.3, we get the assertion. \(\square\)

The following is a well known fact.

**Lemma 2.5.** Let \(M\) be an invariant subspace of \(H^2\). Then \(\tilde{\Omega}(M) \neq \{0\}\). Moreover \(\dim \tilde{\Omega}([f]) = 1\) for every nonzero \(f\) in \(H^2\).

Let \(E\) be a nonvoid subset of \(D^2\). We assume that

\[
I(E) \neq \{0\} \quad \text{and} \quad Z(I(E)) = E.
\]

We write

\[
N(E) = H^2 \ominus I(E).
\]

**Lemma 2.6.** Suppose that \((0, 0) \notin E\). Then \(\tilde{\Omega}(N(E)) = \{0\}\).

*Proof.* Let \(f \in \tilde{\Omega}(N(E))\). By (1.2), \((az + bw)f \in I(E)\) for every \(a, b \in \mathbb{C}\). Since \((0, 0) \notin E\), we have \(f = 0\) on \(E\), so \(f \in I(E)\). Since \(f \perp I(E)\), we get \(f = 0\). \(\square\)

Similarly, we have the following.

**Lemma 2.7.** Suppose that \((0, 0) \in E\) and \(E \neq \{(0, 0)\}\). If \(I(E)\) contains all \(f \in H^2\) satisfying \(f = 0\) on \(E \setminus \{(0, 0)\}\), then \(\tilde{\Omega}(N(E)) = \{0\}\).

*Proof.* Let \(f \in \tilde{\Omega}(N(E))\). By (1.2), \((az + bw)f \in I(E)\) for every \(a, b \in \mathbb{C}\). Then \(f = 0\) on \(E \setminus \{(0, 0)\}\). By the assumption, we have \(f \in I(E)\). Since \(f \perp I(E)\), we get \(f = 0\). \(\square\)

**Proposition 2.8.** Suppose that \((0, 0) \in E\) and \(E \neq \{(0, 0)\}\). If there is \(f \in H^2\) such that \(f = 0\) on \(E \setminus \{(0, 0)\}\) and \(f(0, 0) \neq 0\), then

\[
\tilde{\Omega}(N(E)) = \mathbb{C} \cdot (f - P_{I(E)}f) \neq \{0\}.
\]

*Proof.* Since \(f \notin I(E)\), \(f - P_{I(E)}f \neq 0\) and \(f - P_{I(E)}f \in N(E)\). Since \(f = 0\) on \(E \setminus \{(0, 0)\}\), we have

\[
z(f - P_{I(E)}f), \quad w(f - P_{I(E)}f) \in I(E).
\]

By (1.2), \(f - P_{I(E)}f \in \tilde{\Omega}(N(E))\).

We may assume that \(f(0, 0) = 1\). Let \(g \in \tilde{\Omega}(N(E))\) and \(g \neq 0\). As the proof of Lemma 2.7, \(g = 0\) on \(E \setminus \{(0, 0)\}\) and \(g(0, 0) \neq 0\). We may assume that \(g(0, 0) = 1\). Hence \((f - P_{I(E)}f) - g \in I(E)\). Since \((f - P_{I(E)}f) - g \in \tilde{\Omega}(N(E))\), we get \(g = f - P_{I(E)}f\). \(\square\)

**Example 2.9.** Let \(\alpha \in \mathbb{D}\) with \(\alpha \neq 0\) and

\[
E = \{(0, 0), (0, \alpha), (\alpha, 0), (\alpha, \alpha)\}.
\]

We write \(b_\alpha(z) = (z - \alpha)/(1 - \overline{\alpha}z)\). One may checks that \(I(E) = z b_\alpha(z) H^2 + w b_\alpha(w) H^2\). Let \(f = b_\alpha(z) b_\alpha(w)\). Then \(f(0, 0) = f(\alpha, 0) = f(0, \alpha) = 0\) and
\(f(0,0) = \alpha^2 \neq 0\), so by Proposition 2.8 \(\dim \Omega(N(E)) = 1\). We have \(f \perp I(E)\) and \(\Omega(N(E)) = C \cdot f\). □

In the same way as the one by Yang [7], we may prove the following.

**Theorem 2.10.** Suppose that \((0,0) \in E\) and \(E \neq \{(0,0)\}. \) If there is \(h \in H^\infty\) satisfying \(h = 0\) on \(E \setminus \{(0,0)\}\) and \(h(0,0) \neq 0\), then \(\Omega(I(E)) = C \cdot P(I(E))z + C \cdot P(I(E))w\). Moreover \(\Omega(I(E)) = -1\).

**Proof.** We may assume that \(h(0,0) = 1\). Then there are \(h_1(z)\) and \(h_2\) in \(H^\infty\) such that \(h = 1 + zh_1(z) + wh_2\). We write

\[H_0 = \{ f \in H^2 : f \perp 1, f \perp z, f \perp w \}\.

We shall show that

\[(2.1) \quad zI(E) + wI(E) = I(E) \cap H_0.\]

Let \(f \in I(E) \cap H_0\). We have

\[f = fh - zfh_1(z) - wfh_2.\]

Since \(f \in I(E)\), we have \(zfh_1(z) + wfh_2 \in I(E) + wI(E)\). Since \(H_0 = z^2H^2 + zwH^2 + w^2H^2\), we may write \(f = z^2f_1 + zwf_2 + w^2f_3\) for some \(f_1, f_2, f_3 \in H^2\).

Since \(h = 0\) on \(E \setminus \{(0,0)\}\), we have that \(zfh, wfh, wfh \in I(E)\). Hence

\[fh = z(3f_1 + zfh_1 + zfh + w(f_2h + w(f_3h) \in zI(E) + wI(E),\]

so \(f \in zI(E) + wI(E)\). Thus we get \(I(E) \cap H_0 \subset zI(E) + wI(E)\).

Let \(g \in zI(E) + wI(E)\). Then \(g = zg_1 + wg_2\) for some \(g_1, g_2 \in I(E)\). Since \((0,0) \in E, I(E) \subset zH^2 + wH^2\). Hence for each \(i = 1, 2, g_i = zg_{i,1} + wg_{i,2}\) for some \(g_{i,1}, g_{i,2} \in H^2\). We have

\[g = z^2g_{1,1} + zw(g_{1,2} + g_{2,1}) + w^2g_{2,2} \in H_0.\]

Thus \(zI(E) + wI(E) \subset I(E) \cap H_0\), so we get (2.1). Since \(H_0\) is closed, \(zI(E) + wI(E)\) is closed.

Since \(zh, wh \in I(E)\) and \(h(0,0) = 1\), we have \(P(I(E))z \neq 0\) and \(P(I(E))w \neq 0\).

Let \(g \in I(E) \oplus (C \cdot P(I(E))z + C \cdot P(I(E))w)\). Then \(g \perp 1, g \perp z\) and \(g \perp w\). Hence \(g \in H_0\), so \(g \in I(E) \cap H_0\). Thus by (2.1),

\[I(E) \oplus (C \cdot P(I(E))z + C \cdot P(I(E))w) \subset zI(E) + wI(E)\]

Since \(P(I(E))z, P(I(E))w \perp zI(E) + wI(E)\), we have

\[I(E) = (zI(E) + wI(E)) \oplus (C \cdot P(I(E))z + C \cdot P(I(E))w).\]

Hence

\[\Omega(I(E)) = C \cdot P(I(E))z + C \cdot P(I(E))w.\]

Since \(P(I(E))z \perp wh\) and \(P(I(E))w \not\perp wh\), we have \(C \cdot P(I(E))z \neq C \cdot P(I(E))w\). Hence \(\dim \Omega(I(E)) = 2\).

By Lemmas 2.1, 2.2 and Proposition 2.8, we conclude the assertion. □
Let $\Lambda = \{(a, a) : a \in \mathbb{D}\}$. Then $I(\Lambda) = [z - w]$. It is known that $F_z^{[z-w]}$ is Fredholm and $\text{ind} F_z^{[z-w]} = -1$ (see [7]). The following is a generalization of this fact.

**Theorem 2.11.** Let $\varphi(z)$ be an inner function with $\varphi(0) = 0$ and $g \in H^\infty$ with $g \neq 0$. Then $F_z^{[\varphi(z)-wg]}$ is Fredholm and $\text{ind} F_z^{[\varphi(z)-wg]} = -1$.

**Proof.** Put $M = [\varphi(z)-wg]$. We shall show that

$$zM + wM = M \cap (z\varphi(z)H^2 + wH^2).$$

Since $M \subset \varphi(z)H^2 + wH^2$, we have

$$zM + wM \subset M \cap (z\varphi(z)H^2 + wH^2).$$

Let $f \in M \cap (z\varphi(z)H^2 + wH^2)$. We may write $f = z\varphi(z)f_1 + w f_2$ for some $f_1, f_2 \in H^2$. Put $h = \varphi(z) - wg$. Then $M = [h]$ and

$$f = (h + wg)f_1 + w f_2 = zhf_1 + w(zgf_1 + f_2).$$

Since $h \in M \cap H^\infty$, we have $hf_1 \in M$. Hence $zhf_1 \in zM$ and

$$w(zgf_1 + f_2) = f - zhf_1 \in M,$$

so there is a sequence of polynomials $\{p_n\}_n$ such that

$$(\varphi(z) - wg)p_n = hp_n \to w(zgf_1 + f_2)$$

in $H^2$ as $n \to \infty$. Putting $w = 0$, we have $\|\varphi(z)p_n(z,0)\| \to 0$, so $\|p_n(z,0)\| \to 0$. Hence

$$\|h(p_n - p_n(z,0)) - w(zgf_1 + f_2)\| \leq \|hp_n - w(zgf_1 + f_2)\| + \|h\|\|p_n(z,0)\| \to 0 \quad \text{as } n \to \infty.$$

Since $p_n - p_n(z,0) = qw_n$ for some polynomial $q_n$, we have

$$h(p_n - p_n(z,0)) = wq_n \in w[h] = wM.$$

Hence $w(zgf_1 + f_2) \in wM$. Therefore by (2.3), $f \in zM + wM$. Thus we get (2.2).

Since $z\varphi(z)H^2 + wH^2$ is closed, by (2.2) $zM + wM$ is closed. By Lemma 2.2, $F_z^M$ has closed range. Let $f \in \Omega(N)$. Then $wf \in M$. Similarly as the last paragraph, we have $wf \in wM$, so $f \in M$. Hence $f = 0$. By Lemma 2.1, we have $\ker F_z^M = \{0\}$. By Lemma 2.5, we have $\dim \Omega(M) = 1$, so by Lemma 2.1 we have $\dim \ker (F_z^M)^* = 1$. Thus we get the assertion.

**Corollary 2.12.** Let $h \in H^\infty$ satisfy $|h(e^{i\theta},0)| > \delta > 0$ for almost every $e^{i\theta} \in \partial \mathbb{D}$. Then $F_z^h$ is Fredholm and $\text{ind} F_z^h = -1$. 
Lemma 3.2. Suppose that $h_1(z), h_2 \in H^\infty$. If $h_1(0) \neq 0$, then by Proposition 2.4 we have the assertion. So we assume that $h_1(0) = 0$. Let $h_1(z) = \varphi(z)f(z)$ be an inner-outer factorization of $h_1(z)$. We have $\varphi(0) = 0$. By the assumption, $f(z)$ is invertible in $H^\infty$. Then we have

$[h] = [f(z)(\varphi(z) + wf^{-1}(z)h_2)] = [\varphi(z) + wf^{-1}(z)h_2]$.

If $h_2 = 0$, then $[h] = \varphi(z)H^2$, so we get the assertion. If $h_2 \neq 0$, then by Theorem 2.11 we get the assertion. □

Example 2.13. By Theorem 2.11, for the following $M$ we have that $F^M_z$ is Fredholm and $\text{ind} F^M_z = -1$;

$M = [z - w], \ M = [(z - w)^2], \ M = [z^2 - w^3]$.

3. Generalizations

Let $M$ be an invariant subspace of $H^2$ satisfying that $M \subset I(E)$ and $Z(M) = E$. We have $A_{I(E)}(\lambda) \subset A_M(\lambda)$ for every $\lambda \in E$,

(3.1) $T^*_z \{0, z^nw^m : (n, m) \in A_M(\lambda) \} \subset \{0, z^nw^m : (n, m) \in A_M(\lambda) \}$

and

(3.2) $T^*_w \{0, z^nw^m : (n, m) \in A_M(\lambda) \} \subset \{0, z^nw^m : (n, m) \in A_M(\lambda) \}$.

We recall that

(3.3) $\tilde{M} = \bigcap_{\lambda \in E} \{ f \in H^2 : (D^n_zD^m_wf)(\lambda) = 0 \text{ for every } (n, m) \in A_M(\lambda) \}$.

Then $M \subset \tilde{M} \subset I(E)$ and $E \subset Z(\tilde{M}) \subset Z(M) = E$. Hence $Z(\tilde{M}) = E$. Since $I(E) = I(E)$, as a generalization of zero based invariant subspaces we assume that

(3.4) $M = \tilde{M}$.

Put $N = H^2 \ominus M$. We shall study about $\tilde{\Omega}(N), \, \Omega(M)$ and the Fredholmness of $F^M_z$ under the above situation.

Lemma 3.1. If $(0, 0) \notin E$, then $\tilde{\Omega}(N) = \{0\}$.

Proof. Let $f \in \tilde{\Omega}(N)$. By (1.2), $(az + bw)f \in M$ for every $a, b \in \mathbb{C}$. Since $(0, 0) \notin E$, $(D^n_zD^m_wf)(\lambda) = 0$ for every $\lambda \in E$ and $(n, m) \in A_M(\lambda)$. By (3.3) and (3.4), we have $f \in M$. Since $M \perp \tilde{\Omega}(N)$, we have $f = 0$. □

Lemma 3.2. Suppose that $M \subset z^nw^mH^2$ for some $(n, m) \in \mathbb{N}^2$ with $(n, m) \neq (0, 0)$. If $f \in \tilde{\Omega}(N)$, then $f \in z^nw^mH^2$.

Proof. Let $f \in \tilde{\Omega}(N)$. Suppose that $f \notin z^nw^mH^2$. Then we may write $f = f_1 \ominus f_2$ for some $f_1 \in z^nw^mH^2$ and $f_2 \in H^2 \ominus z^nw^mH^2$. Since $f_2 \neq 0$, either $zf \notin z^nw^mH^2$ or $wf \notin z^nw^mH^2$. So either $zf \notin M$ or $wf \notin M$. By (1.2), $f \notin \tilde{\Omega}(N)$. This is a contradiction. Thus we get $f \in z^nw^mH^2$. □
Corollary 3.3. Suppose that $M \subset z^nw^mH^2$ for some $(n, m) \in \mathbb{N}^2$ with $(n, m) \neq (0, 0)$. Let $N_1 = H^2 \ominus \overline{z^nw^m}M$. Then $\tilde{\Omega}(N) = z^nw^m\tilde{\Omega}(N_1)$.

By Corollary 3.3, to study $\tilde{\Omega}(N)$ we may assume that $M \nsubseteq zH^2$ and $M \nsubseteq wH^2$.

Lemma 3.4. Suppose that $(0, 0) \in E$, $M \nsubseteq zH^2$ and $M \nsubseteq wH^2$. Then there are $n_1, n_2, \ldots, n_k, m_1, m_2, \ldots, m_k \in \mathbb{N}$ such that $0 \leq n_1 < n_2 < \cdots < n_k$, $0 \leq m_k < m_{k-1} < \cdots < m_1$ and

$$A_M(0, 0) = \bigcup_{j=1}^{k} \{ (n, m) \in \mathbb{N}^2 : 0 \leq n \leq n_j, 0 \leq m \leq m_j \}.$$

Proof. Since $M \nsubseteq zH^2$ and $M \nsubseteq wH^2$, $(0, 0) \notin A_M(0, 0)$ and $(0, m) \notin A_M(0, 0)$ for some $n, m \in \mathbb{N}$. By (3.1) and (3.2), we get the assertion. □

Suppose that $(0, 0) \in E$ and $E \neq \{(0, 0)\}$. Let

$$M_0 = \bigcap_{\lambda \in E \setminus \{(0, 0)\}} \{ f \in H^2 : (D_z^i D_w^m f)(\lambda) = 0 \text{ for every } (n, m) \in A_M(\lambda) \}.$$

Then by (3.3) and (3.4), we have $M \subset M_0$.

Lemma 3.5. Suppose that $(0, 0) \in E$ and $E \neq \{(0, 0)\}$. If $M = M_0$, then $\tilde{\Omega}(N) = \emptyset$.

Proof. Let $g \in \tilde{\Omega}(N)$. Then $(az + bw)g \in M$ for every $a, b \in \mathbb{C}$, so $g \in M_0$. By the assumption, we have $g \in M$. Thus we get the assertion. □

We may rewrite $A_M(0, 0)$ as follows:

$$(3.5) \quad A_M(0, 0) = \{ (n, m) \in \mathbb{N}^2 : z^n w^m \perp M \}.$$

Lemma 3.6. Suppose that $(0, 0) \in E$, $E \neq \{(0, 0)\}$, $M \nsubseteq zH^2$ and $M \nsubseteq wH^2$. If $M \neq M_0$, then $\tilde{\Omega}(N) \neq \emptyset$.

Proof. Take $f_0 \in M_0 \ominus M$ with $f_0 \neq 0$. By (3.3) and (3.4), $(D_z^i D_w^m f_0)(0, 0) \neq 0$ for some $(i, j) \in A_M(0, 0)$. Here we use the notations given in Lemma 3.4. Since $z^iw^j \not\perp f_0$, there is $(s, t) \in \mathbb{N}^2$ such that $z^{s+i} w^{t+m} \not\perp z^s w^t f_0$ for some $1 \leq t \leq k$,

$$z^n w^m \perp z^{s+i} w^{t+m} f_0 \quad \text{and} \quad z^n w^m \perp z^{s+i+1} w^{t+m+1} f_0$$

for every $(n, m) \in A_M(0, 0)$. By (3.3) and (3.4), we have $z^s w^t f_0 \notin M$ and $z^{s+i} w^{t+m} f_0, z^{s+i+1} w^{t+m+1} f_0 \in M$. Let $f_1 = z^s w^t f_0 - P_M z^s w^t f_0$. Then $f_1 \in N$ and $f_1 \neq 0$. Moreover we have $zf_1, w f_1 \in M$. By (1.2), we have $f_1 \in \tilde{\Omega}(N)$. □

Proposition 3.7. Suppose that $(0, 0) \in E$ and $E \neq \{(0, 0)\}$. Let $M$ be an invariant subspace of $H^2$ such that $M \nsubseteq I(E), Z(M) = E$ and $M = M_\perp$. Moreover we assume that $M \nsubseteq zH^2$ and $M \nsubseteq wH^2$. Then $\tilde{\Omega}(N) \neq \emptyset$ if and only if $M \nsubseteq M_0$.\]
Proof. The necessity follows from Lemma 3.5. The reverse implication follows from Lemma 3.6.

Under the condition $M \subsetneq M_0$, we shall study about $\dim \tilde{\Omega}(N)$.

**Theorem 3.8.** Suppose that $(0,0) \in E$ and $E \neq \{(0,0)\}$. Let $M$ be an invariant subspace of $H^2$ such that $M \subsetneq I(E)$, $Z(M) = E$, $M \subsetneq M_0$ and $M = \tilde{M}$. Moreover we assume that $M \not\subset zH^2$ and $M \not\subset wH^2$. Let $n_1, n_2, \ldots, n_k, m_1, m_2, \ldots, m_k \in \mathbb{N}$ satisfy the conditions given in Lemma 3.4. Let

$$\Sigma = \{(n_j, m_j) : 1 \leq j \leq k\} \subset A_M(0,0)$$

and

$$M_1 = \{f \in M_0 : f \perp z^n w^m \text{ for every } (n, m) \in A_M(0,0) \setminus \Sigma\}.$$

Then $\tilde{\Omega}(N) = M_1 \ominus M$ and $1 \leq \dim \tilde{\Omega}(N) \leq k$.

**Proof.** Since $M \subsetneq M_0$, there is $f \in M_0 \ominus M$ with $f \neq 0$. Since $M = \tilde{M}$, $f \not\perp z^n w^m$ for some $(i, j) \in A_M(0,0)$. By considering $z^n w^m f$ for $(s, t) \in \mathbb{N}^2$, we have $M \subsetneq M_1 \subset M_0$.

Let $h \in \tilde{\Omega}(N)$. Then $zh, wh \in M$. Since $M = \tilde{M}$, we have $h \in M_0$. For any $(n, m) \in A_M(0,0) \setminus \Sigma$, either $(n+1, m) \in A_M(0,0)$ or $(n, m+1) \in A_M(0,0)$. If $(n+1, m) \in A_M(0,0)$, then $0 = \langle zh, z^n z^m \rangle = \langle h, z^n w^m \rangle$. If $(n, m+1) \in A_M(0,0)$, then $0 = \langle wh, z^n w^m \rangle = \langle h, z^n w^m \rangle$. Hence $h \in M_1$. Thus we get $\tilde{\Omega}(N) \subset M_1 \ominus M$.

Let $f \in M_1 \ominus M$ and $(n, m) \in A_M(0,0)$. Then $f \in M_0$ and $\langle zf, z^n w^m \rangle = \langle f, z^n w^m \rangle = 0$. Hence $zf \in \tilde{M} = M$. Similarly $wf \in M$. Hence $M_1 \ominus M \subset \tilde{\Omega}(N)$. Thus we get the assertion. \qed

**Theorem 3.9.** Suppose that $(0,0) \in E$ and $E \neq \{(0,0)\}$. Let $M$ be an invariant subspace of $H^2$ such that $M \subsetneq I(E)$, $Z(M) = E$ and $M = \tilde{M}$. Moreover we assume that $M \not\subset zH^2$ and $M \not\subset wH^2$. Let $n_1, n_2, \ldots, n_k, m_1, m_2, \ldots, m_k \in \mathbb{N}$ satisfy the conditions given in Lemma 3.4. If $(0,0) \notin Z(M_0)$, then $\dim \tilde{\Omega}(N) = k$.

**Proof.** By the assumption, there is $f_0 \in M_0$ such that $f_0(0,0) = 1$. For each $1 \leq j \leq k$, we have $(z^{n_j} w^{m_j}, z^{n_j} w^{m_j} f_0) \neq 0$. By Lemma 3.4 and (3.5), we have $z^{n_j} w^{m_j} f_0 \notin M$.

Let

$$f_j = z^{n_j} w^{m_j} f_0 - P_M(z^{n_j} w^{m_j} f_0).$$

Then $f_j \in N$ and $f_j \neq 0$. Since $M = \tilde{M}$, it is not so difficult to show that $z f_j, w f_j \in M$ for every $1 \leq j \leq k$. Hence $f_j \in \tilde{\Omega}(N)$ for every $1 \leq j \leq k$. Suppose that $\sum_{j=1}^{k} c_j f_j = 0$ for some $c_1, c_2, \ldots, c_k \in \mathbb{C}$. Since $(n_1, m_1) \in A_M(0,0)$ for every $1 \leq i \leq k$ and $f_0(0,0) = 1$, we have

$$0 = \left\langle \sum_{j=1}^{k} c_j f_j, z^{n_1} w^{m_1} \right\rangle = \left\langle \sum_{j=1}^{k} c_j z^{n_1} w^{m_j} f_0, z^{n_1} w^{m_1} \right\rangle.$$
Example 3.11. Let $n \in \mathbb{N}$ and $\mathbb{C} = f_j = k$. By Theorem 3.8, we get $\dim \Omega(N) = k$. □

We shall show an example satisfying conditions in Theorem 3.9.

Example 3.10. For $\alpha \in \mathbb{D}$, let $b_\alpha(z) = (z - \alpha)/(1 - \bar{\alpha}z)$. For each $\ell \geq 1$, let

$$M = b_\alpha(z)b_\alpha(w) \sum_{j=0}^{\ell} z^{\ell-j}w^j H^2$$

and $E = Z(M)$. Then

$$E = (\{\alpha\} \times \mathbb{D}) \cup (\mathbb{D} \times \{\alpha\}) \cup \{(0, 0)\},$$

$M \subseteq I(E)$, $M \nsubseteq zH^2$, $M \nsubseteq wH^2$ and $M = \bar{M}$. Moreover we have that $M_0 = b_\alpha(z)b_\alpha(w)H^2$, $Z(M_0) = (\{\alpha\} \times \mathbb{D}) \cup (\mathbb{D} \times \{\alpha\})$ and

$$A_M(0, 0) = \bigcup_{i=1}^{\ell} \{(i - 1, 0), (i - 1, 1), \ldots, (i - 1, \ell - i)\}.$$

So in Lemma 3.4, we have

$$(n_1, m_1) = (0, \ell - 1), (n_2, m_2) = (1, \ell - 2), \ldots, (n_\ell, m_\ell) = (\ell - 1, 0)$$

and $k = \ell$. By Theorem 3.8, we have $\dim \Omega(N) = \ell$. □

Example 3.11. Let $M = [z(z - w), w(z - w)]$. Then we have $M_0 = [z - w]$ and $Z(M) = Z(M_0) = \{(a, a) : a \in \mathbb{D}\}$, $\bar{M} = M$ and $M_0 \ominus M = \mathbb{C} \cdot (z - w)$. Hence $\Omega(N) = \mathbb{C} \cdot (z - w)$ and $\dim \Omega(N) = 1$. Moreover

$$A_M(0, 0) = \{(0, 0), (0, 1), (1, 0)\},$$

so in Lemma 3.4 we have $(n_1, m_1) = (0, 1), (n_2, m_2) = (1, 0)$ and $k = 2$. Hence $\dim \Omega(N) = 1 < 2 = k$. □

In Theorem 3.8, we have $\dim \Omega(N) \leq k$. In Example 3.11, we showed an example of $M$ satisfying $\dim \Omega(N) < k$. In Theorem 3.9, if $(0, 0) \notin Z(M_0)$, then $\dim \Omega(N) = k$. In the following, we shall show an example of $M$ satisfying that $(0, 0) \in Z(M_0)$ and $\dim \Omega(N) = k$.

Example 3.12. Let

$$M = \{f \in [z - w] : f \perp z, z^2, w, z\bar{w}, z^2w, w^2, w^3\}.$$ 

Then $M_0 = [z - w]$ and

$$A_M(0, 0) = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (2, 0), (2, 1)\}.$$

Note that $(n_1, m_1) = (0, 3), (n_2, m_2) = (2, 1)$ and $k = 2$ in Lemma 3.4. Moreover

$$M = [z(z^2 - w^2), z^3(z - w), z^2w(z - w), zw^2(z - w), w^3(z - w)]$$
Proof. Since \((3\) Then \[\sum_{i}\Gamma.\]

We have
\[M_1 \cap M = \mathbb{C} \cdot w(z^2 - w^3) \cap \mathbb{C} \cdot (z^3 - z^2 w + zw^2 - w^3).\]

Then by Theorem 3.8, \(\dim \Omega(N) = 2 = k.\)

Suppose that \((0, 0) \in E\) and \(E \neq \{(0, 0)\}.\) Let \(M\) be an invariant subspace of \(H^2\) such that \(M \not\subseteq I(E), Z(M) = E\) and \(M = \overline{M}.\) Moreover we assume that \(M \not\subseteq zH^2\) and \(M \not\subseteq wH^2.\) To describe \(\Omega(M),\) we set
\[B_M(0, 0) = \mathbb{N}^2 \setminus A_M(0, 0).\]

Let \(n_1, n_2, \ldots, n_k, m_1, m_2, \ldots, m_k \in \mathbb{N}\) satisfy the conditions given in Lemma 3.4. Put
\[(s_1, t_1) = (0, m_1 + 1), \quad (s_2, t_2) = (n_1 + 1, m_2 + 1), \quad \ldots,
(s_k, t_k) = (n_k - 1, m_k + 1), \quad (s_{k+1}, t_{k+1}) = (n_k + 1, 0).
\]

Then \(0 = s_1 < s_2 < \cdots < s_{k+1}, 0 = t_{k+1} < t_k < \cdots < t_1\) and
\[(3.6) \quad B_M(0, 0) = \bigcup_{j=1}^{k+1} \{ (s_j + n, t_j + m) : (n, m) \in \mathbb{N}^2 \}.
\]

Let \(1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_q\) be the integers such that for each \(1 \leq i \leq q\) there is \(1 \leq j \leq k + 1\) satisfying \(s_j + t_j = \sigma_i\) and
\[\{(s_j, t_j) : 1 \leq j \leq k + 1\} = \bigcup_{i=1}^{q} \{ (s_j, t_j) : 1 \leq j \leq k + 1, s_j + t_j = \sigma_i \}.
\]

Set \(\Gamma = \{(s_j, t_j) : 1 \leq j \leq k + 1\}\) and
\[(3.7) \quad \Gamma_i = \{(s_j, t_j) : 1 \leq j \leq k + 1, s_j + t_j = \sigma_i\}.
\]

Then \(\sum_{i=1}^{q} \# \Gamma_i = \# \Gamma = k + 1,\) where \(\# \Gamma\) denotes the number of elements in \(\Gamma.\)

Lemma 3.13. \(P_M z^{s_j} w^{t_j} \neq 0\) and \(P_M z^{s_j} w^{t_j} \in \Omega(M)\) for every \(1 \leq j \leq k + 1.\)

Proof. Since \((s_j, t_j) \notin A_M(0, 0),\) we have \(z^{s_j} w^{t_j} \notin M.\) Then \(P_M z^{s_j} w^{t_j} \neq 0,\)
\[z^{s_j} w^{t_j} = P_M z^{s_j} w^{t_j} \oplus (z^{s_j} w^{t_j} - P_M z^{s_j} w^{t_j})\]
and \(z^{s_j} w^{t_j} - P_M z^{s_j} w^{t_j} \in N.\) Since \(T_z z^{s_j} w^{t_j}, T_w z^{s_j} w^{t_j} \in N,\) by (1.1) we have
\(P_M z^{s_j} w^{t_j} \in \Omega(M).\)

\(\square\)

Corollary 3.14. \(\dim \sum_{j=1}^{k+1} \mathbb{C} \cdot P_M z^{s_j} w^{t_j} \leq \dim \Omega(M).\)
Example 3.15. Let
\[ M = \left[ z(z^3 + z^2w + zw^2 + w^3), w(z^3 + z^2w + zw^2 + w^3) \right]. \]
Then \( M = \tilde{M}, M \not\subset zH^2 \) and \( M \not\subset wH^2 \). We have
\[ B_M(0,0) = \bigcup_{j=0}^{4} ((4 - j, j) + \mathbb{N}^2) \]
and \( k = 4 \). We also have
\[ \sum_{j=0}^{4} \mathbb{C} \cdot P_M z^{4-j} w^j = \mathbb{C} \cdot z(z^3 + z^2w + zw^2 + w^3) + \mathbb{C} \cdot w(z^3 + z^2w + zw^2 + w^3) = \Omega(M) \]
and
\[ \tilde{\Omega}(N) = \mathbb{C} \cdot (z^3 + z^2w + zw^2 + w^3). \]

Theorem 3.16. Suppose that \((0,0) \in E \) and \( E \neq \{(0,0)\} \). Let \( M \) be an invariant subspace of \( H^2 \) such that \( M \subsetneq I(E) \), \( Z(M) = E \) and \( M = \tilde{M} \). Moreover we assume that \( M \not\subset zH^2 \) and \( M \not\subset wH^2 \). If there is \( h \in M_0 \cap H^\infty \) satisfying \( h(0,0) \neq 0 \), then \( F_z^M \) is Fredholm and \( \text{ind} F_z^M = -1 \).

Proof. First, we shall show that
\[ zM + wM = M \cap \sum_{j=1}^{k+1} z^{s_j} w^{t_j} (zH^2 + wH^2). \]

Let \( s_1, s_2, \ldots, s_{k+1}, t_1, t_2, \ldots, t_{k+1} \in \mathbb{N} \) satisfy the conditions given above Lemma 3.13. Since \( M \subset \sum_{j=1}^{k+1} z^{s_j} w^{t_j} H^2 \), we have
\[ zM + wM \subset M \cap \sum_{j=1}^{k+1} z^{s_j} w^{t_j} (zH^2 + wH^2). \]

Let
\[ f \in M \cap \sum_{j=1}^{k+1} z^{s_j} w^{t_j} (zH^2 + wH^2). \]

We may assume that \( h(0,0) = 1 \) and write \( h = 1 + zh_1(z) + wh_2 \) for some \( h_1(z), h_2 \in H^\infty \). Then
\[ f = fh - zf h_1(z) - wh_2. \]
Since \( f \in M \), we have \( zf h_1(z) + wh_2 \in zM + wM \). We may also write
\[ f = \sum_{j=1}^{k+1} z^{s_j} w^{t_j} (zf_j + wg_j), \quad f_j, g_j \in H^2. \]
We have
\[ fh = z \left( \sum_{j=1}^{k+1} z^{s_j} w^{t_j} f_j h \right) + w \left( \sum_{j=1}^{k+1} z^{s_j} w^{t_j} g_j h \right). \]

Since \( h \in M_0 \cap H^\infty \), we have \( f_j h, g_j h \in M_0 \). By (3.6), we have
\[ \sum_{j=1}^{k+1} z^{s_j} w^{t_j} f_j h, \sum_{j=1}^{k+1} z^{s_j} w^{t_j} g_j h \perp z^n w^m \]
for every \((n, m) \in A_M(0, 0)\). Since \( M = \tilde{M} \), we get
\[ \sum_{j=1}^{k+1} z^{s_j} w^{t_j} f_j h, \sum_{j=1}^{k+1} z^{s_j} w^{t_j} g_j h \in M. \]

Hence \( fh \in zM + wM \), so \( f \in zM + wM \) and
\[ M \cap \sum_{j=1}^{k+1} z^{s_j} w^{t_j} (zH^2 + wH^2) \subset zM + wM. \]

Thus we get (3.8).

It is not difficult to see that \( \sum_{j=1}^{k+1} z^{s_j} w^{t_j} (zH^2 + wH^2) \) is closed, so \( zM + wM \) is closed.

By Theorem 3.9, we have \( \dim \tilde{\Omega}(N) = k \). By Lemma 3.13, we also have \( P_M z^{s_j} w^{t_j} \neq 0 \) and
\[ \sum_{j=1}^{k+1} c \cdot P_M z^{s_j} w^{t_j} \subset \Omega(M). \]

Suppose that \( \sum_{j=1}^{k+1} c_j P_M z^{s_j} w^{t_j} = 0 \) for some \( \{c_j\}_{j=1}^{k+1} \subset \mathbb{C} \). Since \( h \in M_0 \), we have \( z^{s_j} w^{t_j} h \in M = M \) for every \( 1 \leq j \leq k + 1 \). Since \( h(0, 0) = 1 \), for each \( 1 \leq i \leq k + 1 \) we have
\[ 0 = \left\langle \sum_{j=1}^{k+1} c_j P_M z^{s_j} w^{t_j}, z^{s_i} w^{t_i} h \right\rangle = \sum_{j=1}^{k+1} c_j \left\langle z^{s_j} w^{t_j}, z^{s_i} w^{t_i} h \right\rangle = c_i. \]

Hence \( \{P_M z^{s_j} w^{t_j}\}_{j=1}^{k+1} \) is linearly independent, so by Corollary 3.14 \( k + 1 \leq \dim \Omega(M) \).

To show \( k + 1 = \dim \Omega(M) \), let \( f \in \Omega(M) \) satisfy \( f \perp P_M z^{s_j} w^{t_j} \) for every \( 1 \leq j \leq k + 1 \). Then \( f \perp z^{s_j} w^{t_j} \) for every \( 1 \leq j \leq k + 1 \). Since \( f \perp z^n w^m \) for every \((n, m) \in A_M(0, 0)\), we have
\[ f \in M \cap \sum_{j=1}^{k+1} z^{s_j} w^{t_j} (zH^2 + wH^2). \]

By (3.8), we have \( f \in zM + wM \), so \( f = 0 \). Thus we get the assertion. \( \square \)
For each positive integer $Z$, there are invariant subspaces $M$ of $H^2$ satisfying $M \subset M_0 = [z - w]$, $Z(M) = \Lambda$, $M \subset M_0 = [z - w]$ and $M = \tilde{M}$. Moreover we assume that $M \not\subset zH^2$ and $M \not\subset wH^2$. Since $M_0 = [z - w]$ and $M = \tilde{M}$, we have

$$M = \{ f \in [z - w] : f \perp z^n w^m \text{ for every } (n, m) \in A_M(0, 0) \}.$$  

For each positive integer $n$, let

$$[z - w]_n = \sum_{j=0}^{n-1} C \cdot (z^{n-j} w^j - w^n).$$

Then

$$[z - w] = \bigoplus_{n=1}^{\infty} [z - w]_n.$$  

Let

$$\mathcal{L}_n = \sum_{j=0}^{n} C \cdot z^{n-j} w^j.$$  

Then $[z - w]_n \subset \mathcal{L}_n$. We note that $P_{\mathcal{L}_n} f = P_{[z - w]_n} f$ for every $f \in [z - w]$.

Since $M_0 = [z - w]$, $A_M((a, a)) = \{(0, 0)\}$ for every $a \in \mathbb{D} \setminus \{0\}$. By Lemma 3.4, there are $n_1, n_2, \ldots, n_k, m_1, m_2, \ldots, m_k \in \mathbb{N}$ satisfying that $0 \leq n_1 < n_2 < \cdots < n_k$, $0 \leq m_k < m_{k-1} < \cdots < m_1$ and

$$A_M(0, 0) = \bigcup_{j=1}^{k} \{(n, m) \in \mathbb{N}^2 : 0 \leq n \leq n_j, 0 \leq m \leq m_j \}.$$  

Since $Z(M) = \Lambda$ and $M \not\subset M_0 = [z - w]$, we have $A_M(0, 0) \neq \{(0, 0)\}$, so $n_j + m_j \geq 1$ for every $1 \leq j \leq k$. Hence there are integers $1 \leq \ell_1 < \ell_2 < \cdots < \ell_p$ such that for each $1 \leq i \leq p$ there is $1 \leq j \leq k$ satisfying $n_j + m_j = \ell_i$ and

$$\Sigma = \bigcup_{i=1}^{p} \{(n_j, m_j) : 1 \leq j \leq k, n_j + m_j = \ell_i \}.$$  

Set

$$\Sigma_i = \{(n_j, m_j) : 1 \leq j \leq k, n_j + m_j = \ell_i \}.$$  

Then $\Sigma_i \neq \emptyset$ and $\Sigma_i \cap \Sigma_j = \emptyset$ for $i \neq j$. We have $\sum_{i=1}^{p} \# \Sigma_i = \# \Sigma = k$. Let

$$\Sigma^c = \bigoplus_{(n, m) \in \Sigma} C \cdot z^n w^m \text{ and } \Sigma^c_i = \bigoplus_{(n, m) \in \Sigma_i} C \cdot z^n w^m.$$  

Recall that $B_M(0, 0) = \mathbb{N}^2 \setminus A_M(0, 0)$ and

$$(s_1, t_1) = (0, m_1 + 1), \ (s_2, t_2) = (n_1 + 1, m_2 + 1), \ \ldots, \ (s_k, t_k) = (n_{k-1} + 1, m_k + 1), \ (s_{k+1}, t_{k+1}) = (n_k + 1, 0).$$
Then by (4.3),

\[(4.4)\quad B_M(0,0) = \bigcup_{j=1}^{k+1}((s_j, t_j) + \mathbb{N}^2).\]

Let \(1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_q\) be the integers such that for each \(1 \leq i \leq q\) there is \(1 \leq j \leq k + 1\) satisfying \(s_j + t_j = \sigma_i\) and

\[\{(s_j, t_j) : 1 \leq j \leq k + 1\} = \bigcup_{i=1}^{q}\{(s_j, t_j) : 1 \leq j \leq k + 1, s_j + t_j = \sigma_i\}.\]

Then by (4.3), we have

\[\sum s_i \leq \sum_{i=1}^{k+1} s_i.\]

Let \(B_f\) be an invariant subspace of \(H^2\) such that \(z^{s_1}w^{t_1} \notin f\). Since \(f \in [z - w]\), by (4.1) and (4.2)

\[M \ni \mathcal{P}_{[z-w],s_1+t_1}f = \sum_{j=0}^{s_1+t_1-1} c_j(z^{s_1+t_1-1}w^j - w^{s_1+t_1}) \neq 0.\]

This shows (iii). \(\square\)

**Theorem 4.2.** Let \(M\) be an invariant subspace of \(H^2\) with \(M \subseteq [z-w]\) such that \(Z(M) = \Lambda, M \subset M_0 = [z-w]\) and \(M = \tilde{M}\). Moreover we assume that \(M \not\subseteq zH^2\) and \(M \not\subseteq wH^2\). Let \(n_1, n_2, \ldots, n_k, m_1, m_2, \ldots, m_k \in \mathbb{N}\) satisfy the conditions given in Lemma 3.4. Then \(\max\{k - 1, 1\} \leq \dim \tilde{M}(N) \leq k\).

**Proof.** Let \(f \in \tilde{M}(N)\). By (1.2), \(zf, wf \in M \subset [z-w]\), so \(f \in [z-w]\). Recall that

\[M_1 = \{f \in [z-w]: f \perp z^n w^m\ \text{for every} \ (n, m) \in A_M(0,0) \setminus \Sigma\}.\]

Then we have \(f \in M_1\). Hence \(\tilde{M}(N) \subset M_1\). Since \(z M_1 \subset M\) and \(w M_1 \subset M\), we have

\[\tilde{M}(N) = M_1 \cap M.\]

We have

\[M = \bigoplus_{n=1}^{\infty} M \cap [z-w]_n\quad \text{and}\quad M_1 = \bigoplus_{n=1}^{\infty} M_1 \cap [z-w]_n,\]
so
\[\tilde{\Omega}(N) = \bigoplus_{i=1}^{p} \tilde{\Omega}(N) \cap [z - w]_{\ell_i}.\]

Hence
\[\dim \tilde{\Omega}(N) = \sum_{i=1}^{p} \dim \tilde{\Omega}(N) \cap [z - w]_{\ell_i}.\]

(4.5)

For \(2 \leq i \leq p\), there is \((s, t) \in B_M(0, 0)\) such that \(s + t = \ell_i\). Let
\[K_i = \{(s, t) \in B_M(0, 0) : s + t = \ell_i\}.\]

By Lemma 4.1(iii), we have \(#K_i \geq 2\). For each \((n_j, m_j) \in \Sigma_i\), let
\[f_j = z^{n_j}w^{m_j} - \frac{1}{\#K_i} \sum_{(s, t) \in K_i} z^s w^t \in [z - w]_{\ell_i}.\]

It is not difficult to see that
\[f_j \in M_1 \ominus M = \tilde{\Omega}(N), \quad (n_j, m_j) \in \Sigma_i,\]
so
\[\tilde{\Omega}(N) \cap [z - w]_{\ell_i} = \sum_{(n_j, m_j) \in \Sigma_i} C \cdot f_j.\]

Hence
\[\dim \tilde{\Omega}(N) \cap [z - w]_{\ell_i} = #\Sigma_i, \quad 2 \leq i \leq p.\]

We consider two cases for \(i = 1\).

Case 1. Suppose that there is \((s, t) \in B_M(0, 0)\) such that \(s + t = \ell_1\). Similarly as above, we have \(\dim \tilde{\Omega}(N) \cap [z - w]_{\ell_1} = #\Sigma_1\). Hence in this case, by (4.5) we have
\[\dim \tilde{\Omega}(N) = \sum_{i=1}^{p} #\Sigma_i = #\Sigma = k.\]

Case 2. Suppose that \(\{(s, t) \in B_M(0, 0) : s + t = \ell_1\} = \emptyset\). In this case, take \((n_0, m_0) \in \Sigma_1\). Then
\[\tilde{\Omega}(N) \cap [z - w]_{\ell_1} = \sum_{(n, m) \in \Sigma_1} C \cdot (z^n w^m - z^{n_0} w^{m_0}),\]
so
\[\dim \tilde{\Omega}(N) \cap [z - w]_{\ell_1} = #\Sigma_1 - 1.\]

Hence
\[\dim \tilde{\Omega}(N) = \dim \tilde{\Omega}(N) \cap [z - w]_{\ell_1} + \sum_{i=2}^{p} \dim \tilde{\Omega}(N) \cap [z - w]_{\ell_i}\]
\[= #\Sigma_1 - 1 + \sum_{i=2}^{p} #\Sigma_i = k - 1.\]
3.13. Since $M$ every $(n, m)$ Let Corollary 4.4.
Lemma 4.3.
Lemma 4.5.

Proof. By Example 2.13, $F_{M}^{[z-w]}$ is Fredholm and ind $F_{M}^{[z-w]} = -1$. By Lemma 3.4, dim $([z-w] \cap M) < \infty$. Then by Lemma 4.3, we get the assertion.

In the proof of Theorem 4.2, we described the elements in $\tilde{\Omega}(N)$. By Lemma 2.1 and Corollary 4.4, we have dim $\Omega(M) = \dim \tilde{\Omega}(N) + 1$. We shall describe the elements in $\Omega(M)$. We shall use the same notations given above Lemma 3.13. Since $M \subset [z-w]$, we have $2 \leq \sigma_1$. We note that $n + m \geq \sigma_1$ for every $(n, m) \in B_M(0,0)$. Moreover if $(n, m) \in B_M(0,0)$ and $n + m = \sigma_1$, then $(n, m) \in \Gamma_1$.

Lemma 4.5. 
(i) $\# \Gamma_1 \geq 2$ and if $(n, m) \in B_M(0,0)$, then $n + m = \sigma_1$ if and only if $(n, m) \in \Gamma_1$.
(ii) $\dim \sum_{(s_j, t_j) \in \Gamma_1} C \cdot P_M z^{s_j} w^{t_j} = \# \Gamma_1 - 1$.
(iii) For each $2 \leq i \leq q$, we have $\dim \sum_{(s_j, t_j) \in \Gamma_i} C \cdot P_M z^{s_j} w^{t_j} = \# \Gamma_i$.

Proof. (i) By Lemma 4.1(ii) and (iii), we have $\# \Gamma_1 \geq 2$. The second assertion is already pointed out above Lemma 4.5.

(ii) Take $(s_{j_0}, t_{j_0}) \in \Gamma_1$. Since $M = M$, for $(s, t) \in \Gamma_1$ we have $z^{s_j} w^{t_j} - z^{s_{j_0}} w^{t_{j_0}} \in M$ and $\sum_{(s, t) \in \Gamma_1} C \cdot (z^{s_j} w^{t_j} - z^{s_{j_0}} w^{t_{j_0}}) \subset M$.

By (i), $z^{s_j} w^{t_j} \perp M \oplus \sum_{(s, t) \in \Gamma_1} C \cdot (z^{s_j} w^{t_j} - z^{s_{j_0}} w^{t_{j_0}})$ for every $(s_j, t_j) \in \Gamma_1$. Hence $\sum_{(s_j, t_j) \in \Gamma_1} C \cdot P_M z^{s_j} w^{t_j} \subset \sum_{(s, t) \in \Gamma_1} C \cdot (z^{s_j} w^{t_j} - z^{s_{j_0}} w^{t_{j_0}})$. 

By Theorem 3.8, $1 \leq \dim \tilde{\Omega}(N) \leq k$. Thus we get the assertion. □
Let \( g \in \left( \sum_{(s,t) \in \Gamma_1} \mathbb{C} \cdot (z^s w^t - z^{s_0} w^{t_0}) \right) \oplus \left( \sum_{(s,t) \in \Gamma_1} \mathbb{C} \cdot P_M z^{s_j} w^{t_j} \right) \).

Then \( g \perp z^{s_j} w^{t_j} \) for every \((s_j, t_j) \in \Gamma_1\), so \( g = 0 \). Hence

\[
\sum_{(s,t) \in \Gamma_1} \mathbb{C} \cdot P_M z^{s_j} w^{t_j} = \sum_{(s,t) \in \Gamma_1} \mathbb{C} \cdot (z^s w^t - z^{s_0} w^{t_0}).
\]

Therefore we get (ii).

(iii) Since \( 2 \leq i \), there is \((s, t) \in B_M(0,0) \setminus \Gamma \) such that \( s + t = \sigma_i \). Let

\[
\tilde{\Gamma}_i = \{(s,t) \in B_M(0,0) : s + t = \sigma_i \}.
\]

Then \( \Gamma_i \subset \tilde{\Gamma}_i \). Take \((s_0, t_0) \in \tilde{\Gamma}_i \setminus \Gamma_i \). Since \( M = \tilde{M} \), for \((s, t) \in \tilde{\Gamma}_i \) we have

\[
z^s w^t - z^{s_0} w^{t_0} \in M
\]

for every \((s_j, t_j) \in \Gamma_i \). Hence

\[
\sum_{(s,t) \in \Gamma_i} \mathbb{C} \cdot P_M z^{s_j} w^{t_j} \subset \sum_{(s,t) \in \Gamma_i} \mathbb{C} \cdot (z^s w^t - z^{s_0} w^{t_0}) \subset M.
\]

Let

\[
h \in \left( \sum_{(s,t) \in \Gamma_i} \mathbb{C} \cdot (z^s w^t - z^{s_0} w^{t_0}) \right) \oplus \left( \sum_{(s,t) \in \Gamma_i} \mathbb{C} \cdot P_M z^{s_j} w^{t_j} \right).
\]

Then \( h \perp z^{s_j} w^{t_j} \) for every \((s_j, t_j) \in \Gamma_i \). Hence

\[
h \in \sum_{(s,t) \in \tilde{\Gamma}_i \setminus \Gamma_i} \mathbb{C} \cdot (z^s w^t - z^{s_0} w^{t_0}).
\]

This shows that

\[
\sum_{(s,t) \in \Gamma_i} \mathbb{C} \cdot P_M z^{s_j} w^{t_j} = \left( \sum_{(s,t) \in \tilde{\Gamma}_i} \mathbb{C} \cdot (z^s w^t - z^{s_0} w^{t_0}) \right) \oplus \left( \sum_{(s,t) \in \tilde{\Gamma}_i \setminus \Gamma_i} \mathbb{C} \cdot (z^s w^t - z^{s_0} w^{t_0}) \right).
\]

Hence

\[
\dim \sum_{(s,t) \in \Gamma_i} \mathbb{C} \cdot P_M z^{s_j} w^{t_j} = (#\tilde{\Gamma}_i - 1) - (#(\tilde{\Gamma}_i \setminus \Gamma_i) - 1) = #\Gamma_i.
\]

We note that

\[
z^{s_j} w^{t_j} - \frac{1}{\#(\Gamma_i \setminus \Gamma_i)} \sum_{(s,t) \in \Gamma_i \setminus \Gamma_i} z^s w^t \in \mathbb{C} \cdot P_M z^{s_j} w^{t_j}, \quad (s_j, t_j) \in \Gamma_i.
\]
Theorem 4.6. Let $M$ be an invariant subspace of $H^2$ with $M \supseteq [z - w]$ such that $Z(M) = \Lambda$, $M \subset M_0 = [z - w]$ and $M = \tilde{M}$. Moreover we assume that $M \nsubseteq zH^2$ and $M \nsubseteq wH^2$. Let $n_1, n_2, \ldots, n_k, m_1, m_2, \ldots, m_k \in \mathbb{N}$ satisfy the conditions given in Lemma 3.4 and $\ell_1 = \min_{1 \leq j \leq k} n_j + m_j$. Then we have the following.

(i) Suppose that $s + t \neq \ell_1$ for any $(s, t) \in B_M(0, 0)$. Then

$$\Omega(M) = \sum_{(s,t) \in \Gamma} \mathbb{C} \cdot P_M z^s w^t$$

and $\dim \Omega(M) = k$.

(ii) Suppose that there is $(s, t) \in B_M(0, 0)$ such that $s + t = \ell_1$. Let $g = \sum_{(s,t) \in \Gamma_1} z^s w^t (z - w) \in M$.

Then

$$\Omega(M) = \mathbb{C} \cdot g \oplus \sum_{(s,t) \in \Gamma} \mathbb{C} \cdot P_M z^s w^t$$

and $\dim \Omega(M) = k + 1$.

Proof. (i) By the proof of Theorem 4.2, we have $\dim \tilde{\Omega}(N) = k - 1$. By Lemma 2.1 and Corollary 4.4, we have $\dim \Omega(M) = k$. By Lemma 3.13,

$$\sum_{(s,t) \in \Gamma} \mathbb{C} \cdot P_M z^s w^t \subset \Omega(M)$$

and

$$\dim \sum_{(s,t) \in \Gamma} \mathbb{C} \cdot P_M z^s w^t = \sum_{i=1}^q \dim \sum_{(s,t) \in \Gamma_i} \mathbb{C} \cdot P_M z^s w^t$$

$$= \#\Gamma_1 - 1 + \sum_{i=2}^q \#\Gamma_i \quad \text{by Lemma 4.5}$$

$$= \#\Gamma - 1 = k + 1 - 1 = k.$$ 

Thus we get (i).

(ii) In this case, by the proof of Theorem 4.2 we have $\dim \tilde{\Omega}(N) = k$, so $\dim \Omega(M) = k + 1$. In the same way as the one in (i), we have

$$\sum_{(s,t) \in \Gamma} \mathbb{C} \cdot P_M z^s w^t \subset \Omega(M)$$

and

$$\dim \sum_{(s,t) \in \Gamma} \mathbb{C} \cdot P_M z^s w^t = k.$$ 

By Lemma 4.5(i), $\#\Gamma_1 \geq 2$. Put

$$\Gamma_1 = \{(s_{j_1}, t_{j_1}), (s_{j_2}, t_{j_2}), \ldots, (s_{j_n}, t_{j_n})\} \subset B_M(0, 0),$$

(4.6)
where \(0 \leq s_{j_1} < s_{j_2} < \cdots < s_{j_n}\) and \(\gamma \geq 2\). We have \(\sigma_1 \leq s + t\) for every \((s, t) \in B_M(0, 0)\), and for \((s, t) \in B_M(0, 0)\), \(\sigma_1 = s + t\) if and only if \((s, t) \in \Gamma_1\). If \(s_{j_{n+1}} - s_{j_n} = 1\), then \((s_{j_n}, t_{j_n} - 1) \in \Sigma\). Hence
\[
el_1 \leq s_{j_n} + t_{j_n} - 1 = \sigma_1 - 1 < \sigma_1 \leq s + t
\]
for every \((s, t) \in B_M(0, 0)\). This contradicts with the assumption of (ii). Hence \(s_{j_{n+1}} - s_{j_n} = t_{j_n} - t_{j_n+1} \geq 2\) for every \(1 \leq n \leq \gamma - 1\). This shows that \((s_{j_n} + 1, t_{j_n} - 1) \in A_M(0, 0)\) for every \(1 \leq n \leq \gamma - 1\) and \((s_{j_n} - 1, t_{j_n} + 1) \in A_M(0, 0)\) for every \(2 \leq n \leq \gamma\). If \(s_{j_1} \geq 1\), then we have \((s_{j_1} - 1, t_{j_1} + 1) \in A_M(0, 0)\). For, if \((s_{j_1} - 1, t_{j_1} + 1) \in B_M(0, 0)\), then \((s_{j_1} - 1, t_{j_1} + 1) \in \Gamma_1\) and this contradicts with (4.6). Similarly if \(t_{j_n} \geq 1\), then \((s_{j_n} + 1, t_{j_n} - 1) \in A_M(0, 0)\).

Let
\[
g = \sum_{n=1}^{\gamma} z^{s_{jn}} w^{t_{jn}} (z - w) \in M.
\]
We have
\[
P_M T_w^* g = P_M \left( \sum_{n=1}^{\gamma} \left( z^{s_{jn}-1} w^{t_{jn}+1} \right) + \left( \sum_{n=1}^{\gamma} z^{s_{jn}} w^{t_{jn}} \right) \right)
\]
\[
= P_M \left( \sum_{n=1}^{\gamma} z^{s_{jn}} w^{t_{jn}} \right).
\]
Since
\[
M \cap \left( C \cdot z^{s_1} \oplus C \cdot z^{s_1-1} w \oplus \cdots \oplus C \cdot w^{s_1} \right) = \sum_{n=2}^{\gamma} C \cdot (z^{s_{j_1} w^{t_{j_1}}} - z^{s_{jn}} w^{t_{jn}}),
\]
we have
\[
P_M \left( \sum_{n=1}^{\gamma} z^{s_{jn}} w^{t_{jn}} \right) = 0.
\]
Hence \(P_M T_w^* g = 0\). Similarly \(P_M T_w g = 0\). Thus by (1.1), we get \(g \in \Omega(M)\).

Since \(g \perp z^s w^t\), we have \(g \perp P_M z^s w^t\) for every \((s, t) \in \Gamma\). Hence
\[
C \cdot g \oplus \sum_{(s, t) \in \Gamma} C \cdot P_M z^s w^t \subset \Omega(M)
\]
and
\[
\dim \left( C \cdot g \oplus \sum_{(s, t) \in \Gamma} C \cdot P_M z^s w^t \right) = k + 1.
\]
Thus we get
\[
\Omega(M) = C \cdot g \oplus \sum_{(s, t) \in \Gamma} C \cdot P_M z^s w^t.
\]

We shall give an example satisfying \(M \neq \overline{M}\).
Example 4.7. Let 
\[ M = [z^2 - w^2, z^3(z - w), z^2w(z - w), zw^2(z - w), w^3(z - w)]. \]
Then \( M_0 = [z - w] \), \( A_M(0,0) = \{(0,0), (0,1), (1,0), (1,1)\} \) and 
\[ \tilde{M} = \{f \in [z - w] : f \perp z, f \perp zw, f \perp w\}. \]
We have \( zw(z - w) \in \tilde{M} \) and \( zw(z - w) \notin M \), so \( M \neq \tilde{M} \). We have \( \Sigma = \{(1,1)\} \), so \( M_1 = [z(z - w), w(z - w)] \). We have \( z^2 - 2zw + w^2 \in M_1 \oplus M \) and \( z(z^2 - 2zw + w^2) \notin M \). Hence \( M_1 \oplus M \notin \Omega(N) \) and compare with the assertion of Theorem 3.8. By calculation, we have 
\[ \Omega(N) = \mathbb{C} \cdot ((z^3 + zw^2) - (z^2w + w^3)) \]
and 
\[ \Omega(M) = \mathbb{C} \cdot (z^2 - w^2) + \mathbb{C} \cdot (2z^4 - 3z^3w + 2z^2w^2 - 3zw^3 + 2w^4). \]
By Example 2.13 and Lemma 4.3, \( F^M_z \) is Fredholm and \( \text{ind} F^M_z = -1. \)

References


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