A NOTE ON A GENERAL MAXIMAL OPERATOR

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1. Introduction

Let \( \mu \) be a positive Borel measure on \( \mathbb{R}^n \) which is positive on cubes. For any cube \( Q \subset \mathbb{R}^n \), a Borel measurable nonnegative function \( \varphi_Q \), supported and positive a.e. with respect to \( \mu \) in \( Q \), is given. We consider a maximal function

\[
M_\mu f(x) = \sup_\varphi \int_\varphi |f|d\mu
\]

where the supremum is taken over all \( \varphi_Q \) such that \( x \in Q \).

This operator was studied in [6], [4], [5] in connection with the Muckenhoupt's \( A_p \)-condition [7], fractional maximal operator and spherical maximal function.

In this note we study some more properties of \( M_\mu \) and some special cases.

Throughout this paper \( Q \) will denote a cube in \( \mathbb{R}^n \) with sides parallel to coordinate axes.

2. A condition related to the two-weight strong-type \((p, q)\) inequality

In this section we first give a necessary condition for the two-weight strong-type \((p, q)\) inequality for \( M_\mu \) when \( p > 1 \) and then we show that it is also a sufficient condition for the two-weight strong-type \((p, \infty)\) inequality, weight.

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inequality for $M_\mu$, restricted to dyadic cubes. The condition is a modification of Sawyer’s condition [8].

Throughout this section $w$ and $\nu$ are positive Borel measure on $\mathbb{R}^n$, positive on cubes.

**Proposition 1.** If $\|M_\mu f\|_{L^p(w)} \leq C\|f\|_{L^p(\nu)}$ for $p > 1$ and $q \geq 1$, then $\mu \ll \nu$ and

$$\left\| M_\mu \left( \varphi_Q^{-1} \left( \frac{d\mu}{d\nu} \right)^{p'-1} \right) \right\|_{L^q(w)} \leq C \left\| \varphi_Q^{-1} \left( \frac{d\mu}{d\nu} \right)^{p'-1} \right\|_{L^p(\nu)} < \infty$$

for all $\varphi_Q$, where $p'$ is the conjugate exponent of $p$.

**Proof.** First suppose it is not true that $\mu \ll \nu$. Then there exists a Borel set $E$ such that $\nu(E) = 0$ but $\mu(E) > 0$. Let $f = \chi_E$. Then $\|f\|_{L^p(\nu)} = 0$ but $M_\mu f(x) > 0$ for all $x \in \mathbb{R}^n$. Therefore, $\|M_\mu f\|_{L^q(w)} > 0$ unless $w = 0$. So, we must have $\mu \ll \nu$.

Now suppose $\left\| \varphi_Q^{-1} \left( \frac{d\mu}{d\nu} \right)^{p'-1} \right\|_{L^p(\nu)} = \infty$ for some $\varphi_Q$. This means

$$\int \varphi_Q^{-1} \left( \frac{d\mu}{d\nu} \right)^{p'-1} d\nu = \infty.$$ 

So, there exists $f_n \in L^p(\nu)$ such that $\|f_n\|_{L^p(\nu)} = 1$ and $\int f_n \varphi_Q \frac{d\mu}{d\nu} d\nu = \int f_n \varphi_Q d\mu \to \infty$ as $n \to \infty$. Since $M_\mu f_n(x) \geq \int f_n \varphi_Q d\mu$ for every $x \in Q$, $\|M_\mu f_n\|_{L^q(w)} \to \infty$ as $n \to \infty$. Since $\|f_n\|_{L^p(\nu)} = 1$ for every $n$, this shows

$$\left\| \varphi_Q^{-1} \left( \frac{d\mu}{d\nu} \right)^{p'-1} \right\|_{L^p(\nu)} < \infty$$

for all $\varphi_Q$.

The inequality

$$\left\| M_\mu \left( \varphi_Q^{-1} \left( \frac{d\mu}{d\nu} \right)^{p'-1} \right) \right\|_{L^q(w)} \leq C \left\| \varphi_Q^{-1} \left( \frac{d\mu}{d\nu} \right)^{p'-1} \right\|_{L^p(\nu)}$$

is obvious if we put $f = \varphi_Q^{-1} \left( \frac{d\mu}{d\nu} \right)^{p'-1}$ in the hypothesis. \(\square\)

Now we write $M_{d,\mu} f$ for $M_\mu f$, restricted to dyadic cubes, that is,

$$M_{d,\mu} f(x) = \sup \int \varphi_Q |f| d\mu,$$

where the sup is taken over all $\varphi_Q$ such that $x \in Q$ and $Q$ is dyadic.

In the following, we restrict ourselves to the case when $\nu \ll \mu$ and to avoid the trivial special cases arising and for the simplicity, we assume that $\varphi_Q > 0$ a.e. on $Q$ with respect to $\nu$.

Throughout this paper, $p'$ denotes the conjugate exponent of $p$. 
PROPOSITION 2. Suppose $\mu \ll \nu$ and for $p > 1$,
\[
\left\| M_{d, \mu} \left( \varphi_Q^{p'-1} \left( \frac{d\mu}{d\nu} \right)^{p'-1} \right) \right\|_{L^\infty(w)} \leq C \left\| \varphi_Q^{p'-1} \left( \frac{d\mu}{d\nu} \right)^{p'-1} \right\|_{L^p(\nu)} < \infty
\]
for all $Q$. Then $\|M_{d, \mu}f\|_{L^\infty(w)} \leq C\|f\|_{L^p(\nu)}$.

Proof. Let $f \in L^p(\nu)$ and fix $\lambda > 0$. Consider $\Omega = \{M_{d, \mu}^{R}f > \lambda\}$, where $M_{d, \mu}^{R}f(x)$ is the $M_{d, \mu}f(x)$ restricted to the dyadic cubes with side length $\leq R$. If $M_{d, \mu}^{R}f(x) > \lambda$, then there exists a dyadic cube $Q_x$ containing $x$ such that side length of $Q_x \leq R$ and $\int \varphi_{Q_x}|f|d\mu > \lambda$. Then we have
\[
\Omega = \bigcup_{x \in \Omega} Q_x.
\]
Let $D = \{Q_x \mid x \in \Omega\}$. Then every cube in $D$ is contained in some maximal cube in $D$ and the maximal cubes are mutually nonoverlapping. Therefore, $\Omega = \bigcup Q_k$, where the $Q_k$'s are maximal cubes in $D$ and so $\hat{Q}_k \cap \hat{Q}_j = \emptyset$ ($\hat{Q}_k$ denotes the interior of $Q_k$) if $k \neq j$ and $\int \varphi_{Q_k}|f|d\mu > \lambda$.

\[
\begin{aligned}
\lambda \leq \int \varphi_{Q_k}|f|d\mu &= \int_{Q_k} |f|\varphi_{Q_k} \frac{d\mu}{d\nu} d\nu \\
&\leq \left( \int_{Q_k} |f|^p d\nu \right)^{\frac{1}{p}} \left( \int_{Q_k} \varphi_{Q_k}^{p'} \left( \frac{d\mu}{d\nu} \right)^{p'} d\nu \right)^{\frac{1}{p'}} \\
&\text{by Hölder's inequality} \\
&= \left( \int_{Q_k} |f|^p d\nu \right)^{\frac{1}{p}} \left\| \varphi_{Q_k}^{p'-1} \left( \frac{d\mu}{d\nu} \right)^{p'-1} \right\|_{L^p(\nu)}^{\frac{p}{p'}} < \infty
\end{aligned}
\]
from the hypothesis. For every $k$,
\[
M_{\mu} \left( \varphi_{Q_k}^{p'-1} \left( \frac{d\mu}{d\nu} \right)^{p'-1} \right) \leq \int \varphi_{Q_k} \varphi_{Q_k}^{p'-1} \left( \frac{d\mu}{d\nu} \right)^{p'-1} d\mu \text{ on } Q_k
\]
\[
= \int \varphi_{Q_k}^{p'} \left( \frac{d\mu}{d\nu} \right)^{p'} d\nu \text{ since } \nu \ll \mu
\]
\[
= \left\| \varphi_{Q_k}^{p'-1} \left( \frac{d\mu}{d\nu} \right)^{p'-1} \right\|_{L^p(\nu)}^p \text{ on } Q_k.
\]
So, since \( w(Q_k) > 0 \),

\[
\left\| M_{\mu} \left( \varphi Q_k^{p'-1} \left( \frac{d\mu}{dv} \right)^{p'-1} \right) \right\|_{L^\infty(w)} \leq \left\| \varphi Q_k^{p'-1} \left( \frac{d\mu}{dv} \right)^{p'-1} \right\|_{L^p(\nu)}^p.
\]

Since

\[
\left\| M_{\mu} \left( \varphi Q_k^{p'-1} \left( \frac{d\mu}{dv} \right)^{p'-1} \right) \right\|_{L^\infty(w)} \leq C \left\| \varphi Q_k^{p'-1} \left( \frac{d\mu}{dv} \right)^{p'-1} \right\|_{L^p(\nu)} < \infty,
\]

\[
\left\| \varphi Q_k^{p'-1} \left( \frac{d\mu}{dv} \right)^{p'-1} \right\|_{L^p(\nu)}^p \leq C \left\| \varphi Q_k^{p'-1} \left( \frac{d\mu}{dv} \right)^{p'-1} \right\|_{L^p(\nu)}
\]

for every \( Q_k \).

Therefore, since \( \left\| \varphi Q_k^{p'-1} \left( \frac{d\mu}{dv} \right)^{p'-1} \right\|_{L^p(\nu)} \neq 0 \),

\[
\left\| \varphi Q_k^{p'-1} \left( \frac{d\mu}{dv} \right)^{p'-1} \right\|_{L^p(\nu)}^{p-1} \leq C \text{ when } p > 1
\]

\[
\leq \frac{C}{\lambda} \left\| f \right\|_{L^p(\nu)} \left\| \varphi Q_k^{p'-1} \left( \frac{d\mu}{dv} \right)^{p'-1} \right\|_{L^p(\nu)}^{\frac{p}{p'}}
\]

by (1). Since \( 0 < \left\| \varphi Q_k^{p'-1} \left( \frac{d\mu}{dv} \right)^{p'-1} \right\|_{L^p(\nu)} < \infty \), we have \( 1 \leq \frac{C}{\lambda} \left\| f \right\|_{L^p(\nu)} \), i.e., \( \lambda \leq C \left\| f \right\|_{L^p(\nu)} \).

Since \( \Omega = \bigcup Q_k \), this implies \( \left\| M_{d,\mu}^R f \right\|_{L^\infty(w)} \leq C \left\| f \right\|_{L^p(\nu)} \). \( R \) is arbitrary. Therefore, we have \( \left\| M_{d,\mu} f \right\|_{L^\infty(w)} \leq C \left\| f \right\|_{L^p(\nu)} \). \( \square \)

For any cube \( Q \), let \( Q^d \) denote the smallest dyadic cube containing \( Q \). Suppose there exist positive constants \( C_1 \) and \( C_2 \), depending only on the measures s.t.

\[
(2) \quad C_1 \varphi Q^d \leq \varphi Q \leq C_2 \varphi Q^d \text{ on } Q.
\]

Then for any cube \( Q \) containing \( x \)

\[
\int \varphi_Q |f| d\mu \leq C_2 \int \varphi Q^d |f| d\mu \leq C_2 M_{d,\mu} f(x)
\]

Therefore,

\[
M_{\mu} f(x) \leq C_2 M_{d,\mu} f(x)
\]

Thus we have
Proposition 3. Suppose (2) holds and assume \( \mu \ll \nu \) and for \( p > 1 \) and for all \( \varphi_Q \)

\[
(3) \quad \left\| M_{\mu} \left( \varphi_Q^{p'-1} \left( \frac{d\mu}{d\nu} \right)^{p'-1} \right) \right\|_{L^\infty(w)} \leq C \left\| \varphi_Q^{p'-1} \left( \frac{d\mu}{d\nu} \right)^{p'-1} \right\|_{L^p(\nu)} < \infty.
\]

Then \( \| M_{\mu} f \|_{L^\infty(w)} \leq C \| f \|_{L^p(\nu)} \).

Example. Let \( \varphi_Q(x) = \mu(Q)^{\frac{n}{\alpha}} \chi_Q \), where \( 0 \leq \alpha < n \). Then 
\( M_{\mu} f(x) = \sup_{x \in Q} \mu(Q)^{\frac{n}{\alpha}} \int_Q |f| d\mu \) is the weighted fractional maximal operator. If \( \mu \) satisfies the doubling condition, then for every cube \( Q \)

\[
\varphi_Q^\alpha \leq \varphi_Q \leq C_{\mu,n} \varphi_Q^\alpha \text{ on } Q,
\]

where \( C_{\mu,n} \) is a constant depending only on \( \mu \) and the dimension \( n \). Therefore, in this case Proposition 3 holds and (3) reduces to the Sawyer’s condition [8]. So we will put the Sawyer’s theorem as a corollary.

Corollary. [8] Suppose \( \mu \) satisfies the doubling condition and \( p > 1 \). If \( 0 \leq \alpha < n \), define 
\( M_{\mu,\alpha} f(x) = \sup_{x \in Q} \mu(Q)^{\frac{n}{\alpha}} \int_Q |f| d\mu \).

Then \( \| M_{\mu,\alpha} f(x) \|_{L^\infty(w)} \leq C \| f \|_{L^p(\nu)} \) for all \( f \in L^p(\nu) \) if and only if \( \mu \ll \nu \) and

\[
\| \chi_Q M_{\mu,\alpha}(\chi_Q \left( \frac{d\mu}{d\nu} \right)^{p'-1}) \|_{L^\infty(w)} \leq C \| \chi_Q \left( \frac{d\mu}{d\nu} \right)^{p'-1} \|_{L^p(\nu)} < \infty
\]

for all cubes \( Q \subset \mathbb{R}^n \).

3. \( L^{p,q} \) norm inequality for the Hardy-Littlewood maximal operator

In this section we consider the special case when \( w \) and \( \nu \) are equal weights and \( \varphi_Q \) is specifically given.

Let \( d\mu = u(x)dx \) where \( u(x) \) is a function s.t. \( 0 < u < \infty \) a.e. with respect to the Lebesgue measure on \( \mathbb{R}^n \).

We’ll first give some definitions in [2].

Definition 1. The nonincreasing rearrangement \( g^*_\mu(t) \) of a function \( g \) with respect to the measure \( \mu \) is defined as

\[
g^*_\mu(t) = \inf \{ s : \mu(\{ x : |g(x)| > s \}) \leq t \}
\]
DEFINITION 2. $L^{p,q}$ is the collection of all functions $g$ with $\|g\|_{p,q;\mu} < \infty$, where

$$\|g\|_{p,q} = \|g\|_{p,q;\mu} = \begin{cases} \left( \frac{q}{p} \int_0^\infty (t^{\frac{1}{p}} g_{\mu}^*(t))^{q \frac{dt}{t}} \right)^{\frac{1}{q}}, & 1 \leq p < \infty, \ 1 \leq q < \infty \\ \sup_{t>0} t^{\frac{1}{p}} g_{\mu}^*(t), & 1 \leq p < \infty, \ q = \infty. \end{cases}$$

If $\varphi_Q(x) = \frac{1}{|Q|} \frac{x_Q(x)}{u(x)}$, then $M_\mu f(x)$ becomes the ordinary Hardy-Littlewood maximal function $Mf(x)$ of $f$. Here $|Q|$ denotes the Lebesgue measure of $Q$.

Let $dw = w(x)dx$ and $\Phi(t) = \sup_Q \{w(Q)\varphi_{Q,\mu}^*(w(Q)t)\}$.

Then we have $\Phi \in L^{p',1}$ implies that $\|Mf\|_{L^p(w)} \leq C\|f\|_{L^p(\mu)}$. (For the proof, we refer to [6].)

We now consider this Hardy-Littlewood maximal operator $Mf$ for the single weight problem, i.e., when $w = \mu$. Throughout this section, the norms are all with respect to the measure $d\mu = u(x)dx$.

DEFINITION 3. [7] We say $u \in A_p$ if

$$\left( \int_Q u(x)dx \right) \left( \int_Q u(x)^{-\frac{1}{p}-1}dx \right)^{p-1} \leq C|Q|^p \quad \text{if} \ 1 < p < \infty$$

$$\int_Q u(x)dx \leq C|Q| \ \text{ess inf}_{x \in I} \ u(x) \quad \text{if} \ p = 1,$$

for any cube $Q$, where $C$ is a constant independent of $Q$.

DEFINITION 4. [1] Suppose either $1 < p < \infty$ and $1 \leq q \leq \infty$ or $p = q = 1$. A nonnegative, locally integrable function $u(x)$ is in $A(p, q)$ if there exists a constant $C$ such that for any cube $Q$,

$$\|\chi_Q\|_{p,q} \|\chi_Qu^{-1}\|_{p',q'} \leq C|Q|.$$

We note that $u \in A(p, p)$ if and only if $u \in A_p$.

We now list some theorems in [1] as lemmas:

LEMMA 1. [1] If either $1 < p < \infty$ and $1 \leq q \leq \infty$ or $p = q = 1$, then $\|Mf\|_{p,\infty} \leq C\|f\|_{p,q}$ implies $u \in A(p, q)$. 
LEMMA 2. [1] If $1 \leq q \leq p < \infty$, then $u \in A(p, q)$ implies $\|Mf\|_{p, \infty} \leq C\|f\|_{p,q}$.

LEMMA 3. [1] If $1 < p < \infty$ and $1 < q \leq \infty$, then $u \in A(p, q)$ implies $\|Mf\|_{p,s} \leq C\|f\|_{p,s}$ for $1 \leq s \leq \infty$.

LEMMA 4. [1] If either $1 < p < \infty$ and $1 < q \leq \infty$ or $p = q = 1$, then $u \in A(p, q)$ if and only if $\|Mf\|_{p,\infty} \leq C\|f\|_{p,q}$.

Using the above Lemmas we are able to see the following propositions.

PROPOSITION 4. If either $1 < p < \infty$ and $1 \leq q \leq \infty$ or $p = q = 1$, then $\|Mf\|_{p,q} \leq C\|f\|_{p,q}$ implies $u \in A(p, q)$.

Proof. $\|Mf\|_{p,\infty} \leq \|Mf\|_{p,q}$ for all $1 \leq p, q \leq \infty$.
So from Lemma 1 it holds. □

From Lemmas 1 & 3 and from the fact $\|Mf\|_{p,\infty} \leq \|Mf\|_{p,q}$ for every $1 \leq q \leq \infty$, we have,

PROPOSITION 5. For $1 < p < \infty$ and $1 < q \leq \infty$, we have $\|Mf\|_{p,q} \leq C\|f\|_{p,q}$ if and only if $u \in A(p, q)$.

In [3] we have the following as a theorem.

"For $1 \leq p \leq q < \infty$ and $1 \leq r \leq \infty$, if $\mu$ is a doubling measure, then $\|Mf\|_{q,\infty;\mu} \leq B\|f\|_{p,r;\nu}$ if and only if $\Phi \in L^{p',r'}(0, \infty)$".

From this fact, we know that for any $1 \leq p < \infty$ and $1 \leq q \leq \infty$, if $\mu$ is a doubling measure, then $\|Mf\|_{p,\infty;\mu} \leq B\|f\|_{p,q;\nu}$ if and only if $\Phi \in L^{p',q'}(0, \infty)$.

Therefore from Lemma 4, we can see the following relationship between $A(p, q)$ condition and $\Phi$.

PROPOSITION 6. If either $1 < p < \infty$ and $1 \leq q \leq \infty$ or $p = q = 1$ and if $\mu$ is a doubling measure, then $u \in A(p, q)$ if and only if $\Phi \in L^{p',q'}(0, \infty)$.

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References


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