THETA SERIES BY PRIMITIVE ORDERS

Sungtae Jun

Abstract. With the theory of a certain type of orders in a Quaternion algebra, we construct Brandt matrices and theta series. As an application, we calculate the class number of a certain type of orders in a Quaternion algebra with the trace formula of Brandt matrices.

1. Introduction

It is well known that there is a close connection between the theory of orders in Quaternion algebra and modular forms of $\Gamma_0(N)$ [2], [4]. There are three types of orders in Quaternion algebra (See Definition 2.1 below). Among them, two types of orders, so called, special orders were studied in [4]. The remaining type was studied in [1] and [6], in different ways. As a consequence of [6], in this paper we define theta series associated with a certain type of orders in a rational Quaternion algebra. With the results of [6], we obtain a trace formula for the Brandt matrices, which will play a central role in determining the subspace of cusp forms generated by the theta series (See [7]). For an immediate application of trace formula, we obtain an explicit formula for class number of primitive orders.

2. Primitive orders in Quaternion Algebra

Let $Q$ be the rational number field and $Z$ be the ring of integers in $Q$. For a prime $p$ of $Q$, we denote as $Q_p$ the completion of $Q$ at $p$, and for $p < \infty$, denote as $Z_p$ the ring of integers in $Q_p$. Let $A$ be a Quaternion algebra over $Q_p$. A prime $p$ is said to ramify in $A$ if $A_p = A \otimes Q_p$ is


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a division algebra over $Q_p$ (see [8; p154]). Otherwise $A_p$ is isomorphic to Mat(2, $Q_p$) over $Q_p$ and $p$ is said to split in $A$ (see [18; p184]). A lattice on $A$ is a finitely generated $Z$ submodule of $A$ which contains a basis of $A$ over $Q$. Since $Z$ is a principal ideal domain, a lattice is a free $Z$ module of rank 4. An order $M$ of $A$ is a lattice on $A$ which is a subring containing the identity. There is a local-global correspondence for lattices which goes as follows [17; chapterIV]: to a lattice $L$ on $A$, we associate the collection of lattices $L_p = L \otimes_Z Z_p$ of $A_p$, one for each \( p < \infty \). Conversely, if we have a collection of lattices $\{L(p)|p < \infty\}$ on $A_p$, one for each $p < \infty$ and if there exists a lattice $M$ on $A$ such that $L(p) = M_p$ for almost all $p$, i.e. for all but a finite number of $p$, then there exists a unique lattice $L$ on $A$ such that $L(p) = L_p$ for all $p < \infty$. Replacing the word "lattice" by "order" above, we obtain the local-global correspondence for orders. An order of $A$(resp. $A_p$) is said to be maximal if it is not properly contained in any other order of $A$(resp. $A_p$) where $p$ is a finite prime.

**Definition 2.1.** An order $M$ of $A$ is said to be primitive if

1) for all finite ramified primes $p$ of $A$, $M_p$ contains a subring which is $Z_p$ isomorphic to the ring of integers in some quadratic field extension of $Q_p$.

2) for all finite split primes $p$ of $A$, $M_p$ contains a subring which is $Z_p$ isomorphic to the ring of integers in some quadratic field extension of $Q_p$ or isomorphic to $Z_p \oplus Z_p$ in quadratic extension $Q_p \oplus Q_p$ of $Q_p$.

**Remark.** For all ramified primes $p$ of $A$, pritimitive orders $M$ of $A$ were studied by Hijikata, Pizer and Shemanske [4]. Also, for all finite split primes $p$ of $A$, orders $M_p$ of $A_p$ which contain a subring which is $Z_p$-isomorphic to $Z_p \oplus Z_p$ were studied by Hijikata [3].

**2.1** Now let us restrict to the case that really interests us at present. For the remainder of this paper, $A$ will be a rational Quaternion algebra ramified precisely at one finite prime $q$ and $\infty$. Thus $A_\infty = A \otimes Q R$ is Hamilton's Quaternion algebra [10; p343].

If $R$ is an order of $A_p$ which contains $O_L$, the ring of integers in a quadratic field extension $L$ of $Q_p$ for $p \neq q$, then the possibilities for $R$
are:

\[ R = \begin{cases} 
R_{2\nu}(L) = \mathcal{O}_L + \xi P_L^{\nu} & \text{if } L \text{ is unramified} \\
R_{\nu}(L) = \mathcal{O}_L + (1 + \xi)P_L^{\nu-1} & \text{if } L \text{ is ramified} \\
\hat{R}_0(L) = \mathcal{O}_L + (1 - \xi)P_L^{-1} & \text{if } L \text{ is ramified}
\end{cases} \]

for some nonnegative integer \( \nu \) where \( A_p = L + \xi L \) and \( P_L \) is the prime ideal of \( \mathcal{O}_L \) (See [6]).

**Definition 2.2.** Let \( A \) be a rational Quaternion algebra which is ramified precisely at one finite prime \( q \) and \( \infty \). For finite odd primes \( p_1, p_2, \ldots, p_d \neq q \), an order of \( M \) of \( A \) is said to have level \( (q; L(p_1), \nu(p_1); L(p_2), \nu(p_2); \ldots, L(p_d), \nu(p_d)) \) if

i) \( M_q \) is the maximal order of \( A_q \).

ii) for a prime \( p \neq q \), there exists a quadratic field extension \( L(p) \) of \( Q_p \) and a nonnegative integer \( \nu(p) \) (which is even if \( L(p) \) is unramified) such that \( M_p = R_{\nu(p)}(L(p)) \)

iii) \( \nu(p_i) > 0 \) for \( i = 1, 2, \ldots, d \) and \( \nu(p) = 0 \) for \( p \neq q, p_1, \ldots, p_d \) (i.e. \( M_p \) is a maximal order of \( A_p \) if \( p \neq p_1, p_2, \ldots, p_d \)).

**Remark.** For notational convenience, we put \( N' = (q; L(p_1), \nu(p_1); \ldots, L(p_d), \nu(p_d)) \) and \( N = q \prod_{i=1}^{d} p_i^{\nu(p_i)} \) throughout this paper.

**Definition 2.3.** Let \( M \) be an order of level \( N' \) in \( A \). A left \( M \) ideal \( I \) is a lattice on \( A \) such that \( I_p = M_p a_p \) for some \( (a_p \in A_p^\times) \) for all \( p < \infty \). Two left \( M \) ideals \( I \) and \( J \) are said to belong to the same class if \( I = Ja \) for some \( a \in A^\times \). One has the obvious analogous definitions for right \( M \) ideals.

**Definition 2.4.** The class number of left ideals for any order \( M \) of level \( N' = (q; L(p_1), \nu(p_1); \ldots, L(p_d), \nu(p_d)) \) is the number of distinct classes of such ideals. We denote this class number by \( H(N') \).

**Definition 2.5.** Let \( I \) be a (left or right) \( M \) ideal for some order \( M \) of level \( N' \) in \( A \). The left order of \( I = \{a \in A | aI \subset I\} \) and the right order of \( I = \{a \in A | Ia \in I\} \).

**Definition 2.6.** The norm of an ideal, denoted by \( N(I) \), is the positive rational number which generates the fractional ideal of \( Q \) generated
by \( \{N(a)|a \in I\} \). The conjugate of an ideal \( I \), denoted by \( \overline{I} \), is given by \( \overline{I} = \{ \overline{a} | a \in I \} \). The inverse on an ideal, denoted by \( I^{-1} \), is given by \( I^{-1} = \{ a \in A | IaI \subset I \} \).

**Remark.** Locally, if \( I_p = M_pa_p \) for some \( a_p \in A_p^\times \), then we define \( N(I_p) = N(a_p) \mod Z_p^\times \).

Note: If we have two ideals \( I \) and \( J \) with right order of \( I \) equal to the left order of \( J \), then \( IJ = ( \text{all finite sums } \sum_{i_k \in I, j_k \in J} (i_kj_k) \text{ with } i_k \in I \text{ and } j_k \in J ) \) is an ideal with left order equal to the left order of \( I \) and right order equal to right order of \( J \) (see [16; p210]).

**Proposition 2.7.** Let \( M \) be an order of level \( N' = (q; L(p_1), \nu(p_1); ..., L(p_d), \nu(p_d)) \). Let \( I \) be a left \( M \) ideal with right order \( M' \). Then

i) \( \overline{I} \) is a left \( M' \) ideal with right order \( M \) and \( N(\overline{I}) = N(I) \).

ii) \( II^{-1} = M \) and \( I^{-1}I = M' \).

iii) \( I^{-1} \) is a left \( M' \) ideal with right order \( M \) and \( N(I^{-1}) = N(I)^{-1} \).

**Proof.** i) By Definition 2.5, it is clear that \( \overline{I} \) is a \( Z \) lattice. Furthermore,

\[
(\overline{I})_p = \overline{I} \otimes Z_p = \overline{I}_p = \overline{M_pa_p} \text{ for some } a_p \in A_p \\
= \overline{a_p}M_p = (\overline{a_p}M_p \overline{a_p}^{-1})\overline{a_p} = M'_p \overline{a_p}.
\]

Therefore, \( \overline{I} \) is a left \( M' \) ideal with right order \( M \). \( N(\overline{I}) = N(I) \) follows from \( \{N(\overline{a})|a \in I\} = \{N(a)|a \in I\} \).

ii) The proofs that \( II^{-1} = M \) and \( I^{-1}I = M' \) are given in [16; p192 Theorem 22.7].

iii) \( I^{-1} = \{ a \in A | IaI \subset I \} = \{ x \in A |Ix \subset M \} \) (See [16; p192 (22.6)]). By Definition 2.3, \( I_p = M_pa_p \) for some \( a_p \in A_p \) for each \( p < \infty \). Therefore, \( (I_p)^{-1} = \{ x \in A_p | M_pa_px \subset M_p \} = a_p^{-1}M_p \), which implies \( I_p^{-1} = M'_p a_p^{-1} \) for all \( p < \infty \). Thus we have proven that \( I^{-1} \) is a left \( M' \) ideal with right order \( M \).

For the proof of \( N(I^{-1}) = N(I)^{-1} \), see Theorem 2.4.5 [16; p212]. This completes the proof.
Proposition 2.8. [Pizer] Let $M$ be an order of level $N'$ in $A$. Let $I_1, I_2, ..., I_H$ be a complete set of representatives of all the distinct left $M$ ideal classes. Let $M_j$ be the right order of $I_j, j = 1, 2, ..., H$. Then $I_j^{-1}I_1, ..., I_j^{-1}I_H$ is a complete set of representatives of all distinct left $M_j$ ideal classes (for $i = 1, 2, ..., H$).

Proof. See Proposition 2.13 and Proposition 2.15 [13].

3. Brandt matrices and Theta series

3.1 We now give the connection between modular forms and Quaternion algebras. Let $Q(x)$ be a positive definite integral quadratic form in an even number of $r = 2k$ variables. Integral means that $Q(x) \in \mathbb{Z}$ for all $x \in \mathbb{Z}^r$. Then $Q(x) = \frac{1}{2}x^tTx$ where $x^t = (x_1, x_2, ..., x_r)$ and $T = (a_{ij})$ is a positive definite symmetric matrix with $a_{ij} \in \mathbb{Z}$ and $a_{ii} \equiv 0 \mod 2$. In fact, $T$ is the matrix of the bilinear form $(x, y) = Q(x+y) - Q(x) - Q(y)$. $T$ is called the matrix associated to $Q(x)$.

Definition 3.1. Let $Q(x)$ and $T$ be as above. The level of $Q(x) T$ is the least positive integer $n$ such that $nT^{-1}$ has integer entries with diagonal entries even integers. The discriminant of $Q(x)$ is $(-1)^k \det(T)$.

Proposition 3.2. Let $I$ be a left $M$ ideal for some order $M$ of level $N' = (q; L(p_1), \nu(p_1); ..., L(p_d), \nu(p_d))$ in a positive definite Quaternion algebra $\mathbb{A}$ over $Q$ which is ramified precisely at one finite prime $q$ and $\infty$. Then the quadratic form $N(x)/N(I)$ for $x \in I$ is a positive definite integral quadratic form with level $N$ and discriminant $N^2$ where $N = q \prod_{i=1}^{d} p_i^{\nu(p_i)}$.

Remark. What this means is the following. Let $\epsilon_1, \cdots, \epsilon_4$ be any $\mathbb{Z}$ basis for $I$. Then $Q(x_1, \cdots, x_4) = N(x_1 \epsilon_1 + \cdots + x_4 \epsilon_4)/N(I)$ is a positive definite integral quadratic form with level $N$ and discriminant $N^2$. Since any other $\mathbb{Z}$-basis of $I$ is obtained from $\epsilon_1, \cdots, \epsilon_4$ by operating on $(\epsilon_1, \cdots, \epsilon_4)$ by a matrix $U \in GL(4, \mathbb{Z}) = \{ S \in \text{Mat}_{4 \times 4}(\mathbb{Z}) | \det(S) = \pm 1 \}$, the level and the discriminant are independent of which particular basis we chose.
Proof. Let $Q(x) = N(x)/N(I)$. Since $A_\infty = A \otimes R$ is Hamilton's Quaternion, the norm form is positive definite by [11; p343]. Hence $Q(x)$ is a positive definite form. Next, by the Definition 2.6, $N(I)|N(x)$ for all $x \in I$. This implies $Q(x) = N(x)/N(I)$ is integral.

We now need to show that $Q(x)$ has level $N$ and discriminant $N^2$. Let $S$ be the matrix associated to $Q(x)$. As the level is a positive integer, we determine the level locally at all primes $< \infty$.

We start to consider the case $p \neq q$ first. By Definition 2.3, $I_p = M_p\beta$ for some $\beta \in A_p^\times$. By 2.1, $M_p = R_{\nu(p)}(L(p))$ for some nonnegative integer $\nu(p)$. Suppose $e_1, e_2, e_3, e_4$ is a basis of $R_\nu$. Then $e_1\beta, e_2\beta, e_3\beta, e_4\beta$ gives a $Z_p$ basis for $I_p$. Since $N(I_p) = N(\beta)$ (see Remark of Definition 2.6),

the $ij$-th entry of $S$ is $Q(e_i\beta + e_j\beta) - Q(e_i\beta) - Q(e_j\beta)$

$$= \frac{1}{N(I_p)}(N(\beta)(N(e_i + e_j) - N(e_i) - N(e_j))$$

$$\equiv N(e_i + e_j) - N(e_i) - N(e_j) = Tr(e, e_j) \mod Z_p^\times.$$

First consider the case, $\nu(p) > 0$. Let $\nu = \begin{cases} \frac{\nu(p)}{2} & \text{if } L(p) \text{ is unramified} \\ \nu(p) - 1 & \text{if } L(p) \text{ is ramified,} \end{cases}$, and $L = L(p)$. Then $R_\nu = \mathcal{O}_L + \xi P_\nu^\omega$. Let $\mathcal{O}_L = Z_p + uZ_p$ for some $u$ in $L$, so that $\mathcal{O}_L$ is the ring of integers in $L$. Now we take $e_1 = 1, e_2 = e_3 = \xi \pi_L^\nu, e_4 = \xi \pi_L^\nu u$ as a $Z_p$ basis of $M_p = R_\nu(L)$. Since $\xi \pi_L^\nu = -\xi \pi_L^\nu$ and $\xi \pi_L^\nu u = -\xi \pi_L^\nu u$ where $\pi_L$ is the prime element of $\mathcal{O}_L$ (See [6]),

$$S = \begin{pmatrix} 2 & Tr(u) & 0 & 0 \\ Tr(u) & 2N(u) & 0 & 0 \\ 0 & 0 & 2N(\pi_L^\nu) & -N(\pi_L^\nu)Tr(u) \\ 0 & 0 & -N(\pi_L^\nu)Tr(u) & 2N(\pi_L^\nu u) \end{pmatrix}.$$

Let $\delta = 4N(u) - Tr(u)^2$. Then

$$S^{-1} = \begin{pmatrix} 2N(u)/\delta & -Tr(u)/\delta & 0 & 0 \\ -Tr(u)/\delta & 2/\delta & 0 & 0 \\ 0 & 0 & 2N(\pi_L^\nu)N(u)/\delta N(\pi_L^\nu)^2 & N(\pi_L^\nu)Tr(u)/\delta N(\pi_L^\nu)^2 \\ 0 & 0 & N(\pi_L^\nu)Tr(u)/\delta N(\pi_L^\nu)^2 & 2N(\pi_L^\nu)/\delta N(\pi_L^\nu)^2 \end{pmatrix}.$$
so the level and the discriminant of $Q(x) = \frac{N(x)}{N(I)}$ are $(4N(u) - \text{Tr}(u^2))^2N(\pi_L^x)$ mod $Z_p^x$ and $(4N(u) - \text{Tr}(u^2))^2N(\pi_L^x)^2$ mod $Z_p^x$, respectively.

If $L(p)$ is an unramified extension field of $Q_p$, then $\nu = \frac{\nu(p)}{2}$ and $\Delta(u)$ is a quadratic nonresidue mod $p$ in $Q_p$, whence $\Delta(u) = -(4N(u) - \text{Tr}(u^2))$ is a unit in $Z_p$. On the other hand, if $L(p)$ is a ramified extension field of $Q_p$, then $\nu = \nu(p) - 1$ and $u = \pi_L$. Hence $\Delta(\pi_L) = -(4N(\pi_L) - \text{Tr}(\pi_L)^2) \equiv p$ mod $Z_p^x$.

In both cases, the level of $Q(x)$ mod $Z_p^x = p^{\nu(p)}$. The discriminant of $Q(x) = \frac{N(x)}{N(I)}$ mod units of $Z_p$ is $\text{disc}(M_p) = \det(\text{Tr}(e_i e_j)) = \det(S) = (4N(u) - \text{Tr}(u^2))^2N(\pi_L^x)^2$. That is, the discriminant of $Q(x)$ mod $Z_p^x = p^{2\nu(p)}$. Thus the level and the discriminant of $Q(x)$ mod units of $Z_p$ are $p^{\nu(p)}$ and $p^{2\nu(p)}$ respectively.

If $\nu(p) = 0$, $M_p$ is a maximal order of $A_p$, in which case the level and discriminant of $\frac{N(x)}{N(I)}$ are both 1 mod units of $Z_p$ (see [14 : Proposition 2.11]).

In the case, $p = q$, the level and discriminant of $\frac{N(x)}{N(I)}$ mod units of $Z_p$, $q$ and $q^2$, have been calculated by A. Pizer[14] and [19].

We conclude that the discriminant of $Q(x)$ is $q^2 \prod_{p \mid p_1 p_2 \cdots p_d} p^{2\nu(p)}$ and the level of $Q(x)$ is $q \prod_{p \mid p_1 p_2 \cdots p_d} p^{\nu(p)}$.

This completes the proof.

3.2 Let $M$ be an order of level $N' = (q; L(p_1), \nu(p_1); ..., L(p_d), \nu(p_d))$ in a Quaternion algebra $A$ over $Q$ ramified precisely at one finite prime $q$ and $\infty$. Let $I_1, I_2, ..., I_H$, $H = H(N')$ be representatives of all distinct left $M$ ideal classes. Let $M_j$ be the right order of $I_j$ and $e_j = |U(M_j)|$. We define

$$b_{ij}(n) = \frac{1}{e_j} \sum_{\alpha \in I_j^{-1}I_i \cap N(\alpha) = nN(I_i)/N(I_j)} 1 \quad \text{and} \quad b_{ij}(0) = \frac{1}{e_j}.$$  

Then $b_{ij}(n) = \frac{1}{e_j} \cdot (\text{the number of elements in } I_j^{-1}I_i \text{ whose norms are } nN(I_i)/N(I_j) \text{ for } n > 0)$.

We are now in position to define the Brandt matrices associated with the primitive orders in Quaternion algebra.
**Definition 3.3.** Let the notation be as above. The Brandt matrices for \( n \geq 0 \) are defined by

\[
B(n : N') = (b_{ij}(n)).
\]

Thus \( B(n : N') \) is an \( H \times H \) matrix with \( b_{ij}(n) \) as the \( ij \)-th entry.

**Theorem 3.4.** The entries of the Brandt matrix series,

\[
\Theta(\tau : N') = \sum_{n=0}^{\infty} B(n : N')e^{2\pi i n \tau}
\]

are modular forms of weight 2 on \( \Gamma_0(N) \).

**Proof.** Recall that \( B(n : N') = (b_{ij}(n)) \) where \( b_{ij}(n) \) is just \( \frac{1}{e_j} \) times the number of elements \( \alpha \in I_j^{-1}I_i \) with \( N(\alpha) = nN(I_i)/N(I_j) \) for \( n > 0 \).

Each entry of the Brandt matrix series, \( \Theta(\tau : N') = (\theta_{ij}(\tau)) \), is

\[
\theta_{ij}(\tau) = \sum_{n=0}^{\infty} b_{ij}(n)e^{2\pi i n \tau}
\]

\[
= \frac{1}{e_j} \sum_{x \in I_j^{-1}I_i, N(x) = nN(I_i)/N(I_j)} e^{2\pi i n \tau}
\]

\[
= \frac{1}{e_j} \sum_{x \in I_j^{-1}I_i} e^{2\pi i N(x)N(I_j)/N(I_i)}.
\]

Let \( Q(x) = N(x)N(I_j)/N(I_i) \). Since \( I_j^{-1}I_i \) is a left ideal of \( M_j \), it is a free \( Z \) module of rank 4. So identifying \( I_j^{-1}I_i \) with \( Z^4 \), we have \( \theta_{ij}(\tau) = \frac{1}{e_j} \sum_{x \in Z^4} e^{2\pi i N(x)N(I_j)/N(I_i)} \). By Theorem 20 of [9: VI22] and Proposition 3.2 above, this is a modular form of weight 2 on \( \Gamma_0(N) \). Note that the spherical function with respect to \( Q(x) \) is 1 in the notation of Ogg [9: VI22] and the character associated to \( \theta_{ij}(\tau) \) is 1, since by Proposition 6.12 \( \text{disc}(Q(x)) = N^2 \) and Theorem 20 of [9: VI22] shows that \( \epsilon(d) = \left( \frac{N^2}{d} \right) = 1 \). This completes the proof.

Our final goal is to find the trace formula for the Brandt matrix \( B(n : N') \), which will be the central role in determining the subspace
of modular forms generated by theta series (See [7]). First we need
to determine the mass formula for \( M \) ideals. Let \( M \) be an order of
level \( N' = (q; L(p_1), \nu(p_1); ..., L(p_d), \nu(p_d)) \) on \( A \) and \( I_1, I_2, ..., I_H \) be
representatives of the left \( M \) ideal classes. Recall that the right order of
\( I_i \) is given by \( M_i = \{ a \in A | I_i a \subset I_i \} \).

**Definition 3.5.** Let the notations be as above. The mass formula
for \( M \) ideals where \( M \) is an order of level \( N' = (q; L(p_1), \nu(p_1); ..., L(p_d),
\nu(p_d)) \) is given by

\[
\text{Mass}(M) = 2 \sum_{i=1}^{H} \frac{1}{|U(M_i)|}.
\]

**Theorem 3.6.** Let \( M \) be an order of level
\( N' = (q; L(p_1), \nu(p_1); ..., L(p_d), \nu(p_d)) \) on \( A \). Then

\[
\text{Mass}(M) = \frac{1}{12} (q - 1) \prod_{i=1}^{d} \delta(p_i)
\]

where \( \delta(p_i) = \begin{cases} 
(p_i^2 - p_i)p_i^{\nu(p_i) - 2} & \text{if } L(p_i) \text{ is unramified} \\
(p_i^2 - 1)p_i^{\nu(p_i) - 2} & \text{if } L(p_i) \text{ is ramified and } \nu(p_i) \geq 2 \\
(p_i + 1) & \text{if } L(p_i) \text{ is ramified and } \nu(p_i) = 1
\end{cases}
\]

**Proof.** Let \( M^0 \) be an order of level \( q \) in \( A \) which contains \( M \). Then
as in Proposition 24 and Proposition 25 [12; p685],

\[
\text{Mass}(M) = \text{Mass}(M^0)([U(M^0) : U(M)])
\]

By Eichler[2; p95] \( \text{Mass}(M^0) = \frac{1}{12} (q - 1) \). Thus we need to find \([U(M^0) : U(M)]\).

By Corollary1 [18; p88],

\[
[U(M^0) : U(M)] = \prod_p [U(M^0_p) : U(M_p)].
\]

Since \( M^0_p \) is a maximal order, \( M^0_p = R_0(L(p)) \) and \( M_p = R_{\nu(p)}(L(p)) \).

Suppose \( p \neq p_1, \cdots, p_d \). Then \( M^0_p = M_p \), which implies \([M^0_p : M_p] = 1\). Hence we consider \( p = p_i \) for some \( 1 \leq i \leq d \). In the following
calculations, \([R_i^x : R_{i+1}^x]\) is given in Proposition 2.4 and Proposition 2.7
[6]. If \(L(p)\) is unramified over \(Q_p\), then
\[
[U(M_p^0 : U(M_p)) = [R_0^x : R_2^x] \cdots [R_{\nu(p)-1}^x : R_{\nu(p)}^x] \\
= (p^2 - p)p^2 \cdots p^2 \\
= (p^2 - p)p^{\nu(p)-2}.
\]
If \(L(p)\) is ramified over \(Q_p\) and \(\nu(p) \geq 2\), then
\[
[U(M_p^0 : U(M_p))] = [R_0^x : R_1^x][R_1^x : R_2^x] \cdots [R_{\nu(p)-1}^x : R_{\nu(p)}^x] \\
= (p + 1)(p - 1)pp \cdots p \\
= (p^2 - 1)p^{\nu(p)-2}.
\]
Finally, if \(L(p)\) is ramified over \(Q_p\) and \(\nu(p) = 1\), then
\[
[U(M_p^0 : U(M_p))] = [R_0^x : R_1^x] = p + 1.
\]
Hence
\[
\text{Mass}(M) = \frac{1}{12}(q - 1) \prod_{i=1}^{d} \delta(p_i).
\]
This completes the proof.

3.3 We need to set some notations. Let \(K\) be an imaginary quadratic number field and \(\mathcal{O}\) an order of \(K\). Let \(A\) be a Quaternion algebra over \(Q\) ramified only at \(q\) and \(\infty\) and \(M\) an order of level \(N'\) of \(A\).

Analogously as in the local case, an optimal embedding \(\mathcal{O}/K\) into \(M/A\) is an \(Q\) injective homomorphism \(\varphi\), such that \(\varphi(K) \cap M = \varphi(\mathcal{O})\). Then we denote by \(A(\mathcal{O}, M)\), the number of mod \(U(M)\) equivalence classes of optimal embeddings of \(\mathcal{O}/K\) into \(M/A\). Note that \(A(\mathcal{O}, M)\) depends only on the isomorphism classes of \(\mathcal{O}\) and \(M\). For a prime \(l\), denote by \(C_l(\mathcal{O})\) the number of mod \(U(M_l)\) equivalence classes of optimal embedding of \(\mathcal{O}_l/K_l\) into \(M_l/A_l\) (See 5.2 and Definition 5.1 in [6]). Note that \(C_l(\mathcal{O})\) depends only on \(\mathcal{O}_l\) and the level of \(M_l\).

Let \(M\) be an order of level \(N' = (q; L(p_1), \nu(p_1)) \cdots (L(p_d), \nu(p_d))\) of \(A\). Let \(I_1, I_2, \ldots, I_H\) be a set of representatives of all the left \(M\) ideal classes and \(M_j\) be the right order of \(I_j\) for \(1 \leq j \leq H\).
Theorem 3.6. [Pizer] Let the notation be as above. Then we have

\[ \sum_{i=1}^{H} A(O, M_i) = h(O) \prod_{l|N} C_l(O). \]

where \( h(O) \) is the class number of locally principal \( O \) ideals and the product is over all primes \( l \) dividing \( N \).

Proof. See Theorem 4.8 [15; p192].

Corollary 3.7. [Pizer] In the notation of 3.3, let \( a_i(O) \) denote the number of optimal embeddings of \( O/K \) into \( M_i/A \). Then

\[ \sum_{i=1}^{H} \frac{a_i(O)}{\epsilon_i} = \frac{h(O)}{|U(O)|} \prod_{l|N} C_l(O) \]

where \( \epsilon_i = |U(M_i)|. \)

Proof. See Corollary 4.10 [15; p192].

Theorem 3.8. The trace of Brandt matrix \( B(n : N') \) is

\[ \text{tr}(B(n : N')) = \sum_{s} \sum_{f} \frac{1}{2} b(s, f) \prod_{l|N} c(s, f, l) + \xi(\sqrt{n}) \text{Mass}(M) \]

where \( \xi(\sqrt{n}) = \begin{cases} 1 & \text{if } n \text{ is a perfect square} \\ 0 & \text{otherwise} \end{cases} . \)

The meaning of \( s, f, b(s, f) \) and \( c(s, f, l) \) are as follows.

Let \( s \) run over all integers such that \( s^2 - 4n \) is negative. Hence with some positive integer \( t \) and square free integer \( m \), we can classify \( s^2 - 4n \) by

\[ s^2 - 4n = \begin{cases} t^2m & m \equiv 1 \mod 4 \\ t^24m & m \equiv 2, 3 \mod 4 . \end{cases} \]

For each \( s \), let \( f \) run over all positive divisors of \( t \). Let \( L = Q[x]/(\Phi_s(x)) \) where \( \Phi_s(x) = x^2 - sx + n \) and \( \xi \) is the canonical image of \( x \) in \( L \). Then \( L \)
is an imaginary quadratic number field and $\xi$ generates the order $Z + Z\xi$ of $L$. For each $f$, there is a uniquely determined order $O_f$ containing $Z + Z\xi$ as a submodule of index $f$. Let $\Delta(O_f) = s^2 - 4n/f^2$. Let $h(\Delta(O_f))$ (resp. $\omega(\Delta(O_f))$) denote the number of locally principal $O_f$ ideals (resp. $\frac{1}{2}|U(O_f)|$). Then $b(s, f) = \frac{h(\Delta(O_f))}{\omega(\Delta(O_f))}$.

Let $M$ be an order of level $N'$ of $B$. Then $c(s, f, l)$ is the number of $M_i^\times = (M \otimes Z_l)^\times$ equivalence classes of optimal embeddings of $O_f \otimes Z_l$ into $M \otimes Z_l$. In other words, let $Z + Z\alpha$ be the maximal order of $L$, then $O_f \otimes Z_l = Z_l + Z_l^{l^m \alpha}$ and $(s^2 - 4n)/f^2 \equiv l^m \Delta(\alpha) \mod (Z_l^\times)^2$.

Since $c(s, f, l)$ is the number of $M_i^\times = R_{\nu(l)}(L(l))$ (See 3.3) equivalence classes of optimal embeddings of $l^m \alpha$ into $M_l = R_{\nu(l)}(L(l))$, it is easy to find $c(s, f, l)$ in Theorem 5.19, 5.30, 5.31 and Table 5.28 in [6] or [1] if $s, n$ and $f$ are given.

**Remark.** $h(\Delta(O_f))$ can be expressed in terms of ‘standard’ class number of maximal orders (see Corollary 3.11). It is well known that $w(\Delta(O_f)) = 1$ with two exceptions, $w(-4) = 2$ and $w(-3) = 3$ (see [19; p267]).

**Proof.** Recall that $B(n : N') = (b_{ij}(n))$ where $b_{ij}(n) = \frac{1}{e_i} \sum_{\alpha \in I_{i,j}^{-1}I_i, N(\alpha) = nN(I_i)/N(I_j)} 1$. Then

$$\text{tr}B(n : N') = \sum_{i=1}^{H} b_{ii}(n)$$

$$= \sum_{i=1}^{H} \frac{1}{e_i} \sum_{\alpha \in I_{i}^{-1}I_i, N(\alpha) = nN(I_i)/N(I_i)} 1$$

$$= \sum_{i=1}^{H} \frac{1}{e_i} \sum_{\alpha \in M_i, N(\alpha) = n} 1.$$ 

If $n$ is a perfect square, then $n = a^2$ for some $a \in Z$. Since $M_i$ contains $Z$ for each $i$ and $N(\pm a) = a^2 = n$, then $\sum_{\alpha \in M_i, N(\alpha) = n} 1 = 2$ for each $1 \leq i \leq H$. Hence

$$\sum_{i=1}^{H} \frac{1}{e_i} \sum_{\alpha \in M_i, N(\alpha) = n} 1 = 2 \sum_{i=1}^{H} \frac{1}{e_i} = \text{Mass}(M).$$
Now if \( n \) is not a perfect square in \( Q \), then let \( a_i(s, n) \) denote the number of \( \alpha \in M_i \) with \( \text{tr}(\alpha) = s \), \( N(\alpha) = n \), and with \( x^2 - sx + n \) irreducible over \( Q \). Then \( \sum_{\alpha \in M_i, N(\alpha) = n} 1 = \sum_s a_i(s, n) \) where the sum is over all integers, \( s \) such that \( s^2 - 4n < 0 \).

\[
\sum_{i=1}^H \frac{1}{e_i} \sum_s a_i(s, n) = \sum_{i=1}^H \sum_s \frac{a_i(s, n)}{e_i} = \sum_s \sum_{i=1}^H \frac{a_i(s, n)}{e_i}.
\]

Let \( K = Q[x]/(x^2 - sx + n) \) and let \( x' \) be a root of \( x^2 - sx + n \) in \( K \). Then \( a_i(s, n) \) is equal to the number of isomorphisms \( \phi \) of \( K \) into \( A \) with \( \phi(x') \in M_i \). Let \( O_0 = Z + Zx' \) and \( O_1 \) be an order of \( K \) with \( O_0 \subset O_1 \subset K \). If \( \phi \) is an optimal embedding of \( O_1/K \) into \( M_i/A \), then \( \phi(O_1) = M_i \cap \phi(K) \) and \( x \in O_0 \subset O_1 \) imply \( \phi(x') \in M_i \). Thus every optimal embedding of some order \( O_1, O_0 \subset O_1 \subset K \) into \( M_i/A \) is an isomorphism which is counted in \( a_i(s, n) \). Conversely, if \( \phi : K \to A \) is an isomorphism with \( \phi(x') \in M_i \) then \( M_i \cap \phi(K) = O_1' \) is an order of \( \phi(K) \) containing \( \phi(x') \). Hence \( \phi^{-1}(O_1') \) is an order of \( K \) which contains \( O_0 \) and such that \( \phi \) gives an optimal embedding of \( \phi^{-1}(O_1') \) into \( M_i \). Thus \( a_i(s, n) = \sum_{O_1 \supset O_0} a_i(O_1) \), which we sum over all orders \( O_1 \) of \( K \) which contain \( O_0 \), and \( a_i(O_1) \) is as in Corollary 3.7. Hence we have

\[
\sum_{i=1}^H \frac{a_i(s, n)}{e_i} = \sum_{O_1 \supset O_0} \sum_{i=1}^H \frac{a_i(O_1)}{e_i} = \sum_{O_1 \supset O_0} \frac{h(O_1)}{|U(O_1)|} \prod_{l|N} \epsilon_l(O_1).
\]

by Corollary 3.7.

Now \( \Delta(O_0) = s^2 - 4n \) and \( \Delta(O_1) = (s^2 - 4n)/f^2 \) where \( (s^2 - 4n)/f^2 \equiv 0 \) or \( 1 \) mod 4 and \( f \) is a positive integer. Taking into account the fact that \( K \) must be imaginary quadratic and that an order of \( K \) is uniquely determined by its discriminant, we set \( h(\Delta(O_1)) = h(O_1), \omega(\Delta(O_1)) = \)
\[ \frac{1}{2} |U(\mathcal{O}_1)| \text{ and } c(s, f, l) = c_l(\mathcal{O}_1). \text{ Then} \]
\[ \sum_{s} \sum_{i=1}^{H} \frac{a_i(s, n)}{e_i} = \sum_{s} \sum_{\mathcal{O}_1 \supset \mathcal{O}_0} \frac{h(\mathcal{O}_1)}{|U(\mathcal{O}_1)|} \prod_{l|N} c_l(\mathcal{O}_1) \]
\[ = \sum_{s} \sum_{f} \frac{1}{2} b(s, f) \prod_{l|N} c(s, f, l). \]

Therefore,
\[ \text{tr}(B(n : N')) = \sum_{s} \sum_{f} \frac{1}{2} b(s, f) \prod_{l|N} c(s, f, l) \]
\[ + \xi(\sqrt{n}) \text{Mass}(M). \]

**Lemma 3.9.** Let \( K \) be an imaginary quadratic number field. Let \( \mathcal{O}_K \) be an order of \( K \) of discriminant \( \Delta \) and let \( \mathcal{O}' \) be the suborder of \( \mathcal{O}_K \) of index \( f \). Then
\[ \frac{h(\mathcal{O}'_K)}{\omega(\mathcal{O}'_K)} = \frac{h(\mathcal{O}_K)}{\omega(\mathcal{O}_K)} f \prod_{l|f} (1 - \left\{ \frac{\Delta}{l} \right\} \frac{1}{l}) \]
where \( \left\{ \frac{\Delta}{l} \right\} = \begin{cases} 0 & \text{if } l^2 | \Delta \text{ and } l^{-2} \Delta \equiv 0 \text{ or } 1 \text{ mod } 4 \\ \left( \frac{\Delta}{l} \right) & \text{the Kronecker symbol otherwise} \end{cases} \).

**Proof.** See Lemma 4.16 [15; p197]

**Corollary 3.10.** Let \( K \) be an imaginary quadratic number field. Let \( \mathcal{O} \) be the maximal order of \( K \) and \( \mathcal{O}' \) a suborder of index \( f \). Then
\[ \frac{h(\mathcal{O}'_K)}{\omega(\mathcal{O}'_K)} = \frac{h(\mathcal{O}_K)}{\omega(\mathcal{O}_K)} f \prod_{l|f} (1 - \left\{ \frac{K}{l} \right\} \frac{1}{l}) \]
where
\[ \left( \frac{K}{l} \right) = \begin{cases} 1 & \text{if } l \text{ splits in } K \\ 0 & \text{if } l \text{ ramifies in } K \\ -1 & \text{if } l \text{ remains prime in } K \end{cases} \]
is the Kronecker symbol. Note that \( h(\mathcal{O}_K) \) is the standard class number of \( K \).
PROOF. See Corollary 4.17 [15; p.197].

3.4 Let $L$ and $L'$ be two quadratic extensions of $\mathbb{Q}_p$ contained in $A_p$. By an embedding we mean an injective $\mathbb{Q}_p$ (or $\mathbb{Z}_p$) homomorphism.

Assume that $L \subset B$ and let $\mathcal{O}'$ be an order of $L'$. We say that $\mathcal{O}'$ is embeddable in $R_\nu(L)$ if there exists an embedding $\phi$ of $L'$ into $B$ such that $\phi(\mathcal{O}') \subset R_\nu(L)$.

DEFINITION 3.11. Define $\mu(L, L')$ to be the nonnegative integer or $\infty$ characterized by the property: $\mathcal{O}_{L'}$ is embeddable in $R_\nu(L)$ if and only if $\nu \leq \mu(L, L')$.

Obviously, $\mu(L, L')$ exists and depends only on discriminants of $L$ and $L'$. Also if discriminants of $L$ and $L'$ are equal, then $\mu(L, L') = \mu(L', L) = \infty$. For the details, see [6].

THEOREM 3.12. Let $A$ be a rational Quaternion algebra ramified precisely at one finite prime $q$ and $\infty$ and $M$ be an order of $A$ of level $N' = (q; L(p_1), \nu(p_1); \ldots; L(p_d), \nu(p_d))$ where $2 \mid \prod_{i=1}^d p_i$. Then the class number of an order $M$ is

$$H(N') = \text{Mass}(M) + \frac{1}{4} \left(1 - \left(\frac{-d}{q}\right)\right) \prod_{\nu|N} C(l)$$

$$+ \frac{1}{3} \left(1 - \left(\frac{-3}{q}\right)\right) \prod_{\nu|N} C'(l),$$

where $N = q \prod_{i=1}^d p_i^{\nu(p_i)}$.

$$C(l) = \begin{cases} 2 & \text{if } \mu(Q_l(\sqrt{-1}), L(l)) = 1 \text{ and } \nu(l) = 1 \\ 2 & \text{if } \mu(Q_l(\sqrt{-1}), L(l)) = \infty \\ 0 & \text{otherwise} \end{cases}.$$
\[ C'(l) = \begin{cases} 
  c(1, 1, l) & \text{if } l \neq 3 \\
  0 & \text{if } l = 3, \quad \mu = 0 \\
  1 & \text{if } l = 3, \quad \mu = 2 \text{ and } \nu(3) = 1 \\
  2 & \text{if } l = 3, \quad \mu = 2 \text{ and } \nu(3) = 2 \\
  0 & \text{if } l = 3, \quad \mu = 2 \text{ and } \nu(3) \geq 3 \\
  1 & \text{if } l = 3, \quad \mu = \infty \text{ and } \nu(3) = 1 \\
  2 & \text{if } l = 3, \quad \mu = \infty \text{ and } \nu(3) = 2 \\
  6 & \text{if } l = 3, \quad \mu = \infty \text{ and } \nu(3) \geq 3 
\end{cases} \]

and

\[ c(1, 1, l) = \begin{cases} 
  2 & \mu(\Omega_l(\sqrt{-3}), L(l)) = 1 \text{ and } \nu(l) = 1 \\
  2 & \mu(\Omega_l(\sqrt{-3}), L(l)) = \infty \\
  0 & \text{otherwise} 
\end{cases} \]

Here the product is over all distinct primes \( l \) dividing \( \frac{N}{q} \) and \( \left( \frac{\cdot}{\cdot} \right) \) is the Kronecker symbol. In particular, \( \left( \frac{-3}{3} \right) = \left( \frac{-4}{2} \right) = 0 \) and \( \left( \frac{-3}{2} \right) = -1 \). Also, \( \mu = \mu(L(3), Q_3(\sqrt{-3})) \).

**Proof.** From the definition of the Brandt matrix, we see that \( H(N') = tr(B(1 : N')) \) (see Remark 2.25 [14]). Let us calculate \( tr(B(1 : N')) \). By Theorem 3.9, if \( M \) is an order of level \( N' \), then

\[ tr(B(1 : N')) = \sum_s \sum_f \frac{1}{2} b(s, f) \prod_{l \mid N} c(s, f, l) + \text{Mass}(M). \]

Here, we need to explain \( b(s, f) \) and \( c(s, f, l) \) first. Let \( \eta \) be a canonical image of \( x \) in \( Q[x]/(x^2 + sx + 1) \). Then for each \( f \), there is uniquely determined order \( \mathcal{O}_f \) containing \( Z + Z\eta \) as a submodule of index \( f \). Let \( h(\mathcal{O}_f)(w(\mathcal{O}_f)) \) denote the number of locally principal \( \mathcal{O}_f \) ideals (resp. \( \frac{1}{2|U(\mathcal{O}_f)|} \)). Then \( b(s, f) = \frac{h(\mathcal{O}_f)}{w(\mathcal{O}_f)} \) Also \( c(s, f, l) \) is the number of \( M_l^x = R_{\nu(l)}(L(l)) \) (see Definition 2.1) equivalence classes of optimal embeddings of \( l^m\alpha \) into \( M_l = R_{\nu(l)}(L(l)) \) where \( Z + Z\alpha \) is the maximal order of \( Q[x]/(x^2 + sx + 1) \) and \( \mathcal{O}_f \otimes Z_l = Z_l + Z_l l^m\alpha \).
As \( Q[x]/(x^2+sx+1) \) is a quadratic imaginary number field, \( s^2-4 < 0 \). Hence, there are three choices for \( s \). Namely, \( s = 0 \) or \( 1 \) and \(-1\). However, since \( Q[x]/(x^2+x+1) \cong Q[x]/(x^2-x+1) \cong Q(\sqrt{-3}) \), it suffices to consider only the cases, \( s = 0 \) and \( 1 \).

**i) case \( s = 0 \).** (i.e. \( s^2 - 4n = -4 \)).

Let \( K = Q[x]/(x^2+1) \cong Q(\sqrt{-1}) \). Then \( Z + Z\sqrt{-1} \) is the maximal order of \( K \). So \( f = 1 \). Let \( \mathcal{O} = Z + Z\sqrt{-1} \) for convenience.

Now we need to find \( b(0, 1) \) of \( \mathcal{O} \).

By [23; p267], the class number of \( \mathcal{O} \) is 1 and the number of units in \( \mathcal{O} \) is 4. That is, \( h(\mathcal{O}) = 1 \) and \( w(\mathcal{O}) = \frac{1}{2} |U(\mathcal{O})| = 2 \).

Hence \( b(0, 1) = \frac{h(\mathcal{O})}{w(\mathcal{O})} = \frac{1}{2} \).

Next we need to calculate \( c(s, f, l) \) for \( l|N \).

First, if \( l = q \), then \( c(0, 1, q) = (1 - \left(\frac{-4}{q}\right)) \) is given in Proposition 6 [4; p102].

Second, consider \( l|q \). \( \mathcal{O}_1 \otimes Z_l = (Z + Z\sqrt{-1}) \otimes Z_l = Z_l + Z_l\sqrt{-1} \).

\( \Delta(\sqrt{-1}) = -4 \) implies that \( Z_l + Z_l\sqrt{-1} \cong Z_l \oplus Z_l \) or \( Z_l + Z_l\sqrt{-1} \) is the ring of integers in a field \( Q_l(\sqrt{-1}) \).

If \( Z_l + Z_l\sqrt{-1} \cong Z_l \oplus Z_l \), then since \( L(l) \) is a field, by Theorem 3.10 in [6] \( \mu Q_l(\sqrt{-1}), L(l) = 0 \) or 1. By Theorem 5.30 and 5.31 in [6], \( c(0, 1, l) \), the number of \( M \) of \( R_{\nu(l)}(L(l)) \) equivalence classes of optimal embeddings of \( \sqrt{-1} \) into \( M = R_{\nu(l)}(L(l)) \) is 2 if \( L(l) \) is ramified and \( \nu(l) = 1 \), i.e. \( \mu Q_l(\sqrt{-1}), L(l) = 1 \) and \( \nu(l) = 1 \). Otherwise, by Theorem 5.19 and Table 5.28 in [6] \( c(0, 1, l) = 0 \). If, on the other hand, \( Z_l + Z_l\sqrt{-1} \) is the ring of integers in a field \( Q_l(\sqrt{-1}) \), then since \( 2 \nmid \frac{N}{q}, l \nmid \Delta(\sqrt{-1}) = -4 \).

So \( Q_l(\sqrt{-1}) \) is unramified. By Theorem 5.19 in [6], \( c(0, 1, l) = 2 \) if \( L(l) \) is unramified, that is \( \mu Q_l(\sqrt{-1}), L(l) = \infty \). Otherwise, by Theorem 5.19 and Table 5.28 in [6] \( c(0, 1, l) = 0 \).

Hence

\[
c(0, 1, l) = \begin{cases} 
2 & \text{if } \mu Q_l(\sqrt{-1}), L(l) = 1 \text{ and } \nu(l) = 1 \\
2 & \text{if } \mu Q_l(\sqrt{-1}), L(l) = \infty \\
0 & \text{otherwise}
\end{cases}
\]
ii) case \( s = 1 \). (i.e. \( s^2 - 4n = -3 \)).

Let \( K = Q[x]/(x^2 + x + 1) = Q(\sqrt{-3}) \). Then \( Z + Z\sqrt{-3} \) is the maximal order of \( K \). Hence, \( f = 1 \). Let \( \mathcal{O} = Z + Z\sqrt{-3} \) for convenience.

The class number of \( \mathcal{O} \) is 1 and the number of units in \( \mathcal{O} \) is 6 (see [19; p267]). Hence \( b(1, 1) = \frac{h(\mathcal{O})}{\omega(\mathcal{O})} = \frac{1}{3} \) and we obtain \( c(1, 1, 1) \) as in the theorem by the table 5.28 in [6].

Again, we need to calculate \( c(s, f, l) \) for \( l|N \).

First, if \( l = q \), then \( c(1, 1, q) = (1 - \frac{-3}{q}) \) was calculated by Eichler [2; p102].

Second, if \( l|\frac{N}{q} \) and \( l \neq 3 \), then \( c(1, 1, l) \) is the number of \( M_l^\times = R_{\nu(l)}(L(l)) \) equivalence classes of optimal embeddings of \( \sqrt{-3} \) into \( M_l = R_{\nu(l)}(L(l)) \).

Since \( \Delta(\sqrt{-3}) = -12 \), \( Q_l(\sqrt{-3}) \) is either unramified or isomorphic to \( Q_l \oplus Q_l \).

Analogous to the case i), by Theorem 5.19, 5.30, 5.31 and Table 5.28 in [6], \( c(1, 1, l) \) is calculated as in the theorem.

Finally, if \( l|\frac{N}{q} \) and \( l = 3 \), since \( \Delta(\sqrt{-3}) = -12 = -3 \cdot 4 \), \( Q_l(\sqrt{-3}) \) is ramified. By table 5.28 and Theorem 5.19 in [6],

\[
c(1, 1, 3) = \begin{cases} 
0 & \text{if } \mu = 0 \\
1 & \text{if } \mu = 2 \text{ and } \nu(3) = 1 \\
2 & \text{if } \mu = 2 \text{ and } \nu(3) = 2 \\
0 & \text{if } \mu = 2 \text{ and } \nu(3) \geq 3 \\
1 & \text{if } \mu = \infty \text{ and } \nu(3) = 1 \\
2 & \text{if } \mu = \infty \text{ and } \nu(3) = 2 \\
6 & \text{if } \mu = \infty \text{ and } \nu(3) \geq 3 
\end{cases}
\]

where \( \mu = \mu(L(3), Q_3(\sqrt{-3})) \) (see Definition 3.3).
Combining i) and ii), we obtain that

\[
\sum_s \frac{1}{2} \sum_f b(s, f) \prod_{l \mid N} c(s, f, l) = \frac{1}{2} b(0, 1) \prod_{l \mid N} c(0, 1, l) \\
+ \frac{1}{2} b(1, 1) \prod_{l \mid N} c(1, 1, l) + \frac{1}{2} b(-1, 1) \prod_{l \mid N} c(-1, 1, l) \\
= \frac{1}{4} (1 - (-\frac{4}{q})) \prod_{l \mid \frac{N}{q}} C(l) + \frac{1}{3} (1 - (-\frac{3}{q})) \prod_{l \mid \frac{N}{q}} C'(l).
\]

References


Department of Applied Math.
Konkuk Univ. 322
Choongju 380-701, Korea