A NOTE ON WEAKLY PATH-CONNECTED ORTHOMODULAR LATTICES

EUNSOON PARK

ABSTRACT. We show that each orthomodular lattice containing only atomic nonpath-connected blocks is a full subalgebra of an irreducible path-connected orthomodular lattice and there is a path-connected orthomodular lattice $L$ containing a weakly path-connected full subalgebra $C(x)$ for some element $x$ in $L$.

1. Preliminaries

It is known that there exists a weakly path-connected orthomodular lattice with finite sites which is not path-connected and there exists a path-connected orthomodular lattices which contains a nonpath-connected full subalgebra [6].

We will prove that every orthomodular lattice $L$ containing only atomic nonpath-connected blocks is a full subalgebra of an irreducible path-connected orthomodular lattice and there exists a path-connected orthomodular lattice $L$ containing a nonpath-connected full subalgebra $C(x)$ for some $x \in L$.

An orthomodular lattice (abbreviated by OML) $L$ is an ortholattice $L$ which satisfies the orthomodular law: if $x \leq y$, then $y = x \lor (x' \land y)$ $\forall x, y \in L$ [5]. A Boolean algebra $B$ is an ortholattice satisfying the distributive law: $x \lor (y \land z) = (x \lor y) \land (x \lor z)$ $\forall x, y, z \in B$.

A subalgebra of an OML $L$ is a nonempty subset $M$ of $L$ which is closed under the operations $\lor$, $\land$ and $'$. We write $M \leq L$ if $M$ is a subalgebra of $L$. If $M \leq L$ and $a, b \in M$ with $a \leq b$, then the relative interval sublattice $M[a, b] = \{x \in M \mid a \leq x \leq b\}$ is an OML with the

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relative orthocomplementation on \( M[a, b] \) given by \( c'' = (a \lor c') \land b = a \lor (c' \land b) \quad \forall c \in M[a, b] \). In particular, \( L[a, b] \) will be denoted by \([a, b]\) if there is no ambiguity.

The commutator of \( a \) and \( b \) of an OML \( L \) is denoted by \( a \ast b \), and is defined by \( a \ast b = (a \lor b) \land (a \lor b') \land (a' \lor b) \land (a' \lor b') \). For any two elements \( a, b \) of an OML, we say \( a \) commutes with \( b \), in symbols \( a \mathbf{C} b \), if \( a \ast b = 0 \). If \( M \) is a subset of an OML \( L \), the set \( C(M) = \{ x \in L \mid x \mathbf{C} m \quad \forall m \in M \} \) is called the commutant of \( M \) in \( L \) and the set \( \text{Cen}(M) = C(M) \cap M \) is called the center of \( M \). The set \( C(L) \) is called the center of \( L \) and then \( C(L) = \bigcap \{ C(a) \mid a \in L \} \). An OML \( L \) is called irreducible if \( C(L) = \{ 0, 1 \} \), and \( L \) is called reducible if it is not irreducible.

A block of an OML \( L \) is a maximal Boolean subalgebra of \( L \). The set of all blocks of \( L \) is denoted by \( \mathfrak{A}_L \). Note that \( \bigcup \mathfrak{A}_L = L \) and \( \bigcap \mathfrak{A}_L = C(L) \).

For any \( e \) in an OML \( L \), the subalgebra \( S_e = [0, e'] \cup [e, 1] \) is called the (principal) section generated by \( e \). Note that for \( A, B \in \mathfrak{A}_L \), if \( e \in (A \cap B) \) and \( A \cap B = S_e \cap (A \cup B) \), then \( A \cap B = S_e \cap A = S_e \cap B \).

**Definition 1.1.** For blocks \( A, B \) of an OML \( L \) define \( A \updownarrow^w B \) if and only if \( A \cap B = S_e \cap (A \cup B) \) for some \( e \in A \cap B \); \( A \sim B \) if and only if \( A \neq B \) and \( A \cup B \leq L \); \( A \cong B \) if and only if \( A \sim B \) and \( A \cap B \neq C(L) \).

A (weak) path in \( L \) is a finite sequence \( B_0, B_1, ..., B_n \) \((n \geq 0)\) in \( \mathfrak{A}_L \) satisfying \( B_i \sim B_{i+1} \) \((B_i \updownarrow^w B_{i+1})\) whenever \( 0 \leq i < n \). The path is said to join the blocks \( B_0 \) and \( B_n \). A path is said to be proper if and only if \( n = 1 \) or \( B_i \cong B_{i+1} \) holds whenever \( 0 \leq i < n \). A path is called to be strictly proper if and only if \( B_i \cong B_{i+1} \) holds whenever \( 0 \leq i < n \) [1].

Let \( A, B \) be two blocks of an OML \( L \). If \( A \sim B \) holds, then there exists a unique element \( e \in A \cap B \) satisfying \( A \cap B = (A \cup B) \cap S_e \) [1]. Using this element \( e \), we say that \( A \) and \( B \) are linked at \( e \) (strongly linked at \( e \)) if \( A \sim B \) \((A \cong B)\), and use the notation \( A \sim_e B \) \((A \cong_e B)\). This element \( e \) is called a vertex of \( L \) and it is the commutator of any \( x \in A \setminus B \) and \( y \in B \setminus A \) [1].

Note that \( A \cong B \) implies \( A \sim B \), and \( A \sim B \) implies \( A \updownarrow^w B \). Some authors, for example Greechie, use the phrase "\( A \) and \( B \) meet in the
section $S_e$" to describe $A \overset{wk}{\sim} B$ [3].

**Definition 1.2.** Let $L$ be an OML, and $A, B \in \mathcal{A}_L$. We will say that $A$ and $B$ are weakly path-connected, path-connected, strictly path-connected in $L$ if $A$ and $B$ are joined by a weak path, a proper path, a strictly proper path, respectively. We will say $A$ and $B$ are nonpath-connected if there is no proper path joining $A$ and $B$, and $L$ is called nonpath-connected if there exist two blocks which are nonpath-connected. An OML $L$ is called weakly path-connected, path-connected, strictly path-connected in $L$ if any two blocks in $L$ are joined by a weak path, a proper path, a strictly proper path, respectively. An OML $L$ is called relatively path-connected if each $[0, x]$ is path-connected for all $x \in L$.

Let $L$ be an OML, and $A, B, C \in \mathcal{A}_L$. If $A$ and $B$ are joined with a strictly proper path $A = B_0 \approx B_1 \approx \ldots \approx B_{m-1} \approx B_m = B$ and if $B$ and $C$ are joined with a strictly proper path $B = C_0 \approx C_1 \approx \ldots \approx C_{n-1} \approx C_n = C$ then $A$ and $C$ are strictly path-connected by the concatenated path $A = B_0 \approx B_1 \approx \ldots \approx B_{m-1} \approx B \approx C_1 \approx \ldots \approx C_{n-1} \approx C_n = C$.

The following propositions are well known.

**Proposition 1.3.** Every finite direct product of path-connected OMLs is path-connected [7].

**Proposition 1.4.** Every infinite direct product of path-connected OMLs containing infinitely many non-Boolean factors is nonpath-connected [6, 8].

### 2. Weakly Path-connected Orthomodular Lattices

A sublattice $M$ of an OML $L$ is said to be a suborthomodular lattice of $L$ in case the restriction of the orthocomplementation on $L$ makes $M$ an OML. A suborthomodular lattice $M$ of an OML $L$ is called subcomplete in case $N \subseteq M$ and $\forall N$ exists as computed in $L$ implies $\forall N$ is in $M$.

In what follows we assume that $(L_1, \leq_1, ^\#)$ and $(L_2, \leq_2, ^+)$ are two disjoint OMLs, that $S^i$ is a proper suborthomodular lattice of $L_i$ ($i = 1, 2$), and that there exists an orthoisomorphism $\theta : S^1 \to S^2$. 
**Definition 2.1.**

1. Let \( L_0 = L_1 \cup L_2 \).
2. Let \( P_1 = \{(x, y) \in L_0 \times L_0 : y = x \theta \} \).
3. Let \( \Delta = \{(x, x) : x \in L_0 \} \).
4. Let \( P \) be the equivalence relation defined by \( P = \Delta \cup P_1 \cup P_1^{-1} \) where \( P_1^{-1} = \{(y, x) : (x, y) \in P_1 \} \).
5. Let \( L = L_0 / P \).
6. For \( i = 1, 2 \), let \( R_i = \{([x], [y]) \in L \times L : \) there exist \( x_i \in [x] \) and \( y_i \in [y] \) such that \( x_i \leq_i y_i \} \).
7. Let \( \leq \) be the relation \((R_1 \cup R_2)^2\).
8. Define \([0]\) to be \([0_1]\) and \([1]\) to be \([1_1]\) where \(0_1 \) and \(1_1 \) are the zero and unit elements of \( L_1 \).
9. Define \(\cdot : L \rightarrow L\) by the following prescription: for \([x] \in L\),
   \[
   [x]' = \begin{cases} 
   [x_1], & \text{if there exists } x_1 \in L_1 \text{ such that } x_1 \in [x], \\
   [x_2^+], & \text{if there exists } x_2 \in L_2 \text{ such that } x_2 \in [x].
   \end{cases}
   \]
10. Two sections \( S^1 \) and \( S^2 \) are said to be corresponding sections of \( L_1 \) and \( L_2 \) in case there exists \( M_i \subset S^i \subset L_i \) (\( i = 1, 2 \)) such that \( M_1 \theta = M_2 \) and \( S^1 = \bigcup \{S_{m^2} : m \in M_1\} \) and \( S^2 = \bigcup \{S_{m^+} : m \in M_2\} \).

**Theorem 2.2.** Let \( S^1 \) and \( S^2 \) be corresponding sections of \( L_1 \) and \( L_2 \). Let \( L_i \) be complete and let \( S^i \) be subcomplete (\( i = 1, 2 \)). Then \( L \) is a complete OML [3].

**Definition 2.3.** An OML \( L \) is said to be obtained by pasting two OMLs \( L_1 \) and \( L_2 \) along the sections \( S^1 \) and \( S^2 \) if all the conditions of 2.2 are satisfied, and we write \( L = P(L_1, L_2; S^1, S^2; \theta) \).

Let \( X = \{a_1, a_2, a_3, \ldots\} \), and let \( \wp(X) \) be the power set of \( X \). Then the Boolean algebra \( B \) consists of all finite and cofinite elements of the power set \( \wp(X) \) of \( X \) is denoted by \( B = \langle a_1, a_2, a_3, \ldots \rangle \). The pasting of two disjoint OMLs \( L_1 \) and \( L_2 \) along the principal sections \( S_{c_1} \leq L_1 \) and \( S_{c_2} \leq L_2 \) generated by \( c_1, c_2 \) respectively is denoted by \( L = P(L_1, L_2; S_{c_1}, S_{c_2}; \theta) \) (see definition 2.3). We may omit the isomorphism \( \theta \) if there is no difficulty.

Let \( L \) be an OML. A subalgebra \( S \) of \( L \) is said to be a full subalgebra if every block of \( S \) is a block of \( L \). Note that each \( C(x) \) is a full subalgebra of \( L \) for all \( x \in L \) since \( \mathcal{A}_{C(x)} = \{B \in \mathcal{A}_L | x \in B\} \).
Theorem 2.4. If \( L \) is an OML such that each pair of nonpath-connected blocks \( A, B \) of \( L \) have atoms \( a \in A \) and \( b \in B \), then \( L \) is a full subalgebra of an irreducible path-connected OML.

Proof. Let \( S \) be the set of all nonpath-connected pairs of blocks of the given OML \( L \), and let \( \{A, B\} \in S \). Then, for all \( \{A, B\} \in S \), \( A \neq B \) and there exist two atoms \( a, b \) such that \( a \in A \) and \( b \in B \) by the given hypothesis (it may be that \( a = b \)). Let \( C = \langle a, c, d \rangle \) with \( c \neq d \) and \( c, d \not\in L \). Let \( L_1 = P(L, C; S_a^L, S_c^C) \). Then \( L_1 \) is an OML by theorem 2.2. Let \( D = \langle d, e, f \rangle \) with \( e \neq f \) and \( e, f \not\in L_1 \). Let \( L_2 = P(L_1, D; S_d^{L_1}, S_d^D) \). Then \( L_2 \) is an OML by theorem 2.2. Let \( E = \langle b, g, h \rangle \) with \( g \neq h \) and \( g, h \not\in L_2 \). Let \( L_3 = P(L_2, E; S_b^{L_2}, S_b^E) \). Then \( L_3 \) is an OML by theorem 2.2. Let \( F = \langle h, m, n \rangle \) with \( m \neq n \) and \( m, n \not\in L_3 \). Let \( L_4 = P(L_3, F; S_h^{L_3}, S_h^F) \). Then \( L_4 \) is an OML by theorem 2.2. Let \( G = \langle f, p, n \rangle \) with \( p \neq f \), \( p \neq n \) and \( p \not\in L_4 \). Let \( L_5 = L_4 \cup G \) where the operations and ordering are the union of those in \( L_5 \) and \( G \). Then \( L_5 \) is an OML since \( x \vee y = 1 \) where \( x \in G \), \( y \in \bigcup (A_{L_4} \setminus \{C, D, E, F\}) \) and \( x \vee z \) and \( z \vee \) where \( z \in L_5 \forall z \in C \cup D \cup E \cup F \).

Moreover, \( A \approx_{a'} C \approx_{d'} D \approx_{f'} G \approx_{n'} F \approx_{h'} E \approx_{p'} B \) since \( C(L_5) = D \cap F = \{0, 1\}, a \in A \cap C, d \in D \cap E \), \( f \in D \cap G \), \( n \in G \cap F \), \( h \in F \cap E \), \( p \in E \cap B \).

We add pairwise disjoint paths \( C_\alpha \approx_{d_\alpha} D_\alpha \approx_{f_\alpha} G_\alpha \approx_{n_\alpha} F_\alpha \approx_{h_\alpha} E_\alpha \) with \( A_\alpha \approx_{a_\alpha} C_\alpha \) and \( E_\alpha \approx_{p_\alpha} B_\alpha \) to \( L \) for each nonpath-connected pair of blocks \( \{A_\alpha, B_\alpha\} \in S \) by the similar process which is given in the first part of this proof, where \( d_\alpha, f_\alpha, n_\alpha, h_\alpha \) are distinct atoms not in \( L \) and that \( d_\alpha \neq d_\beta \), \( f_\alpha \neq f_\beta \), \( n_\alpha \neq n_\beta \) and \( h_\alpha \neq h_\beta \) for all two distinct pairs of blocks \( \{A_\alpha, B_\alpha\}, \{A_\beta, B_\beta\} \) of \( S \). Then the resulting OML \( \Gamma \) contains at least one path between each \( \{A_\alpha, B_\alpha\} \in S \) and \( L \) is a subalgebra of \( \Gamma \). Also, \( \forall \alpha \neq \beta \) any distinct blocks \( U, V \) in \( \Gamma \) with \( U \in \{C_\alpha, D_\alpha, G_\alpha, F_\alpha, E_\alpha\} \) and \( V \in \{C_\beta, D_\beta, G_\beta, F_\beta, E_\beta\} \) are path-connected by a concatenated path since each pair of blocks \( \{U, A_\alpha\}, \{A_\alpha, B_\alpha\}, \{B_\beta, V\} \) are joined with strictly proper paths. Thus \( \Gamma \) is path-connected. Furthermore, \( \Gamma \) is irreducible and \( L \) is a full subalgebra of \( \Gamma \). This completes the proof.

The following corollary follows.
COROLLARY 2.5. Every OML $L$ containing only atomic nonpath-connected blocks is a full subalgebra of an irreducible path-connected OML.

An OML $L$ is called the horizontal sum of a family $(L_i)_{i \in I}$ (denoted by $\circ(L_i)_{i \in I}$) of at least two subalgebras, if $\bigcup L_i = L$, and $L_i \cap L_j = \{0, 1\}$ whenever $i \neq j$, and one of the following equivalent conditions is satisfied:

1. if $x \in L_i \setminus L_j$ and $y \in L_j \setminus L_i$, then $x \lor y = 1$;
2. every block of $L$ belongs to some $L_i$;
3. if $S_i$ is a subalgebra of $L_i$, then $\bigcup S_i$ is a subalgebra of $L$ [2].

THEOREM 2.6. There exists a path-connected orthomodular lattice $L$ containing a nonpath-connected full subalgebra $C(x)$ for some $x \in L$.

PROOF. Let $MO2$ be the horizontal sum of two Boolean algebras $2^2$ and $2^2$ with four elements. Let $L = \prod_{n \geq 1} L_n$ where $L_n = MO2 \quad \forall n \geq 1$. Let $A_n, B_n$ be the two distinct blocks in $L_n$, $A = \prod_{n \geq 1} A_n$ and $B = \prod_{n \geq 1} B_n$. Then $A$ and $B$ are nonpath-connected by proposition 1.4 and $L$ does not contain a nonatomic block since each block of $L$ is an infinite direct product of $2^2$. Let $\Gamma$ be the path-connected extention of $L$ which is constructed by the same method in theorem 2.4. Then $L$ is a full subalgebra of $\Gamma$. Choose $x = (y, 0, 0, ... \in L \subseteq \Gamma$ where $y \in A_1$ and $y \notin \{0, 1\}$, and $0 \in A_n$, $\forall n \geq 2$. Then $C(x) = \{A_1 \times (\prod_{n \geq 2} L_n)\} \cup \{B \mid x \in B \in (A_\Gamma \setminus A_L)\}$ is a full subalgebra of $\Gamma$, but $C(x)$ is not path-connected since two blocks $A$ and $C = A_1 \times (\prod_{n \geq 2} B_n)$ are nonpath-connected in $\Gamma$ by proposition 1.4. \hfill $\square$

The following corollary follows by the given OML $L = \prod_{n \geq 1} L_n$ in theorem 2.6 which is nonpath-connected by proposition 1.4 and weakly path-connected by our constructive method. Hence $C(x)$ satisfies the following conclusion.

COROLLARY 2.7. There is a path-connected OML with a weakly path-connected full subalgebra $C(x)$ for some $x \in L$.

References


Department of Mathematics
Soongsil University
Seoul 156-743, Korea