ZERO SCALAR CURVATURE ON OPEN MANIFOLDS

SEONGTAG KIM

ABSTRACT. Let \((M, g)\) be a noncompact complete Riemannian manifold of dimension \(n \geq 3\) with scalar curvature \(S\), which is close to 0. With conditions on a conformal invariant and scalar curvature of \((M, g)\), we show that there exists a conformal metric \(\tilde{g}\), near \(g\), whose scalar curvature \(\tilde{S} = 0\) by gluing solutions of the corresponding partial differential equation on each bounded subsets \(K_i\) with \(\bigcup K_i = M\).

1. Introduction

Let \((M, g)\) be a noncompact complete Riemannian manifold of dimension \(n \geq 3\) with scalar curvature \(S\). In this paper, we look for a conformal metric \(\tilde{g} = u^{4/(n-2)}g\), near \(g\), whose scalar curvature \(\tilde{S} = 0\). This problem is equivalent to finding a smooth positive solution \(u\), which is close to 1, of the following partial differential equation:

\[(A) \quad -c_n \Delta u + Su = 0,\]

where \(c_n = 4(n - 1)/(n - 2)\).

For compact Riemannian manifolds, conformal changes to a constant scalar curvature have been studied separately, based on the sign of \(Q(M, g)\), where

\[Q(M, g) = \inf_{u \in C^\infty_c(M)} \frac{\int_M |\nabla u|^2 + \frac{n-2}{4(n-1)}Su^2 \, dV_g}{\left(\int_M u^{2n/(n-2)} \, dV_g\right)^{(n-2)/n}}.\]

\(Q(M, g)\) is a conformal invariant which can be used for a study of open manifolds. Note that \(Q(M, g) \leq Q(S^n, g_0)\) where \(g_0\) is the standard metric on \(S^n\). Now we state our Theorem.

1991 Mathematics Subject Classification: 53C21, 58G30.
Key words and phrases: scalar curvature, conformal metric.
Author is supported in part by KOSEF96070102013 and BSRI 97-1419 Ministry of Education.
Theorem 1. Let \((M, g)\) be a noncompact complete Riemannian manifold of dimension \(n \geq 3\) with scalar curvature \(S\) and infinite volume. Assume that \(Q(M, g) > 0\) and \(\int_M |S|^{2n/(n+2)} + |S|^{n/2} \, dV_g < \infty\). Then, there exists a conformal metric \(\bar{g} = u^{4/(n-2)} g\) whose scalar curvature is 0. Moreover, \(u\) satisfies the followings:

\[
\begin{align*}
(B) & \quad \int_M |\nabla(u - 1)|^2 + |u - 1|^{2n/(n-2)} \, dV_g < \infty \\
& \text{and} \\
& \int_M |\nabla(u - 1)|^2 + |u - 1|^{2n/(n-2)} \, dV_g \to 0 \text{ as} \\
& \int_M |S|^{2n/(n+2)} + |S|^{n/2} \, dV_g \to 0.
\end{align*}
\]

Note that \(Q(M, g) \geq 0\) is a necessary condition for \((M, g)\) to have a conformal metric with zero scalar curvature (see [3]). When \((M, g)\) is conformal to a subdomain \((K - \Gamma, h)\) of a compact Riemannian manifold \((K, h)\) and \(\Gamma\) is a smooth submanifold of \((K, h)\), \(Q(M, g) > 0\) is a necessary condition for \((M, g)\) to have a complete conformal metric with zero scalar curvature (see [2]). Conformal changes of a metric to zero scalar curvature on a subdomain of a compact Riemannian manifold have been studied by Ma and McOwen [6]. Recently, Li [5] studied conformal changes of a metric to a constant negative scalar curvature on noncompact complete Riemannian manifolds with a lower bound of the first eigenvalue of the conformal Laplacian and a lower bound of scalar curvature. In this paper, we study conformal metrics with zero scalar curvature using integrals of scalar curvature.

2. Proof of main results

First we show the existence of a solution of (A). Using the Fredholm alternative, the following existence of a conformal metric on a smooth bounded domain with nonzero boundary data can be shown (see [3]).

Lemma 1. Let \(K_i\) be a smooth bounded domain with boundary \(\partial K_i\). If \(Q(M, g) > 0\), then there exists a unique positive solution \(u_i\) of (A) with \(u_i = 1\) on \(\partial K_i\).
Assume that there exists a sequence \( \{K_i\} \) of smooth bounded domains with \( K_i \subset K_{i+1} \) and \( \cup K_i = M \). By Lemma 1, there exists a smooth positive solution \( u_i \) of (A) on each \( K_i \) and \( u_i = 1 \) on \( \partial K_i \). We extend the domain of \( u_i \) by defining \( u_i = 1 \) on the outside of \( K_i \) and use the same notation \( u_i \) for this extension.

We construct a positive solution of (A) on \( M \) by gluing solutions \( u_i \) of equation (A) on each \( K_i \). For this, the behavior of \( u_i \) should be studied. Note that \( h_i > -1 \) where \( h_i = u_i - 1 \).

**Claim 1.** \( \int_M |h_i|^{2n/(n-2)} \, dV_g \) is bounded.

**Proof.** Note that \( h_i \) satisfies the following equation

\[
-c_n \Delta h_i + Sh_i = -S \quad \text{on } K_i.
\]

From the given condition of Theorem 1,

\[
Q(M, g) \left( \int_{K_i} |h_i|^{2n/(n-2)} \, dV_g \right)^{(n-2)/n} \leq \int_{K_i} (-c_n \Delta h_i + Sh_i) h_i \, dV_g \]

\[
= \int_{K_i} -Sh_i \, dV_g
\]

\[
\leq \left( \int_{K_i} |S|^{2n/(n+2)} \, dV_g \right)^{(n+2)/2n} \left( \int_{K_i} |h_i|^{2n/(n-2)} \, dV_g \right)^{(n-2)/2n}.
\]

Therefore, we have

\[
Q(M, g) \left( \int_{K_i} |h_i|^{2n/(n-2)} \, dV_g \right)^{(n-2)/2n} \leq \left( \int_{K_i} |S|^{2n/(n+2)} \, dV_g \right)^{(n+2)/2n} \leq \left( \int_M |S|^{2n/(n+2)} \, dV_g \right)^{(n+2)/2n}.
\]

Claim 1 is proved. \( \square \)

Following Aviles and McOwen [1], we show that there is a uniform bound for \( u_i \) on each compact subset of \( M \).
CLAIM 2. For each given compact set $X \subset M$, there exists a constant $C_0$ such that

$$\max_{x \in X} u_i(x) \leq C_0,$$

where $C_0$ is a constant independent of $i$.

PROOF. Since $X$ is compact, there exist $R > 0$ and a finite number of balls $B_R(y_1) \cdots B_R(y_N)$ which cover $X$ with $y_k \in X$ for $k = 1 \cdots N$. Let $W = \bigcup_{k=1}^{k=N} B_{2R}(y_k)$ and $Y$ be smooth bounded domains with $W \subset Y$. Since $u_i$ satisfies $-c_n \Delta u_i + S u_i = 0$ on $Y$ for large $i$,

$$\sup_{x \in X} u_i(x) \leq \sup_{x \in B_R(y_k)} u_i(x) \leq CR^{-n/p} ||u_i||_{L^p(B_{2R}(y_k))}$$

for some $k \in \{1, \cdots, N\}$, where $p > 1$ and $C$ depends only on $n$, $p$ and $Y$ (see [4]). By Claim 1 and Minkowski's inequality,

$$||u_i||_{L^p(B_{2R}(y_k))} = ||h_i + 1||_{L^p(B_{2R}(y_k))} \leq ||h_i||_{L^p(B_{2R}(y_k))} + |W|^{(n-2)/2n}$$

$$\leq ||h_i||_{L^p(M)} + |W|^{(n-2)/2n}$$

for $p = 2n/(n-2)$, where $|W|$ is a Riemannian volume of $W$. Therefore, there exists a uniform bound on $\sup_{x \in X} u_i(x)$ for large $i$. □

Using the standard elliptic estimates (see [1]) and Claim 2, we have a convergent subsequence $\{u_{i_k}\}$ which converges to $u$ in $C^{2,\alpha}$ on each compact subset. By the maximum principle, there exists a nonnegative solution $u$ for (A).

Next we give a uniform bound on $\int_M |\nabla h| \cdot dV_g$.

CLAIM 3. $\int_M |\nabla h| \cdot dV_g$ is bounded.
PROOF. Using (2), (3) and Hölder inequality,

\[
\int_{K_i} c_n |\nabla h_i|^2 \, dV_g = \int_{K_i} -c_n h_i \Delta h_i \, dV_g = \int_{K_i} -Sh_i - Sh_i^2 \, dV_g \\
\leq \int_{K_i} |Sh_i| + |Sh_i^2| \, dV_g \\
\leq \int_{K_i} |Sh_i| \, dV_g + \left( \int_{K_i} |S|^{n/2} \, dV_g \right)^{2/n} \left( \int_{K_i} |h_i|^{2n/(n-2)} \, dV_g \right)^{(n-2)/n} \\
\leq \left( \int_{K_i} |S|^{2n/(n+2)} \, dV_g \right)^{(n+2)/n} / Q(M,g) \\
+ \left( \int_{K_i} |S|^{n/2} \, dV_g \right)^{2/n} \left( \int_{K_i} |S|^{2n/(n+2)} \, dV_g \right)^{(n+2)/n} / Q(M,g)^2 \\
\leq \left( \int_{M} |S|^{2n/(n+2)} \, dV_g \right)^{(n+2)/n} / Q(M,g) \\
+ \left( \int_{M} |S|^{n/2} \, dV_g \right)^{2/n} \left( \int_{M} |S|^{2n/(n+2)} \, dV_g \right)^{(n+2)/n} / Q(M,g)^2.
\]

We conclude that \( \int_M |\nabla h_i|^2 \, dV_g < \infty \). \( \square \)

From Claim 1, Claim 3 and \( u_i = 1 + h_i \to u \), we have (B) in Theorem 1. Since volume of \((M,g)\) is infinite and (B) holds, \( u \) can not be identically zero. By the maximum principle, we have a positive solution of (A). The last part of Theorem 1 comes from the estimates of Claim 1 and Claim 3.

References


Department of Mathematics  
Sungkyunkwan University  
Suwon 440-746, Korea  
*E-mail*: stkim@yurim.skku.ac.kr