NOTES ON SOME IDENTITIES INVOLVING
THE RIEMANN ZETA FUNCTION

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ABSTRACT. We first review Ramaswami's and Apostol's identities involving the Zeta function in a rather detailed manner. We then present corrected, or generalized formulas, or a different method of proof for some of them. We also give closed-form evaluation of some series involving the Riemann Zeta function by an integral representation of $\zeta(s)$ and Apostol's identities given here.

1. Introduction and Preliminaries

The Riemann Zeta function $\zeta(s)$ is defined by

\begin{equation}
\zeta(s) := \begin{cases} 
\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\
(1-2^{1-s})^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0; s \neq 1).
\end{cases}
\end{equation}

The Riemann Zeta function $\zeta(s)$ can, except for a simple pole at $s = 1$ with its residue 1, be continued analytically to the whole complex $s$-plane by means of a familiar contour integral representation (cf. Whittaker and Watson [9, p. 266]) or many other known integral representations (cf. Erdélyi et al. [5, p. 33]).

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The Hurwitz (or generalized) Zeta function $\zeta(s, a)$ is defined by

$$
\zeta(s, a) := \sum_{k=0}^{\infty} \frac{1}{(k + a)^s}
$$

(1.2)

$$
(\Re(s) > 1; \ a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}; \mathbb{Z}^- := \{-1, -2, -3, \ldots\}),
$$

which can, just as $\zeta(s)$, be continued analytically to the whole complex $s$-plane except for a simple pole at $s = 1$ (with its residue 1). Clearly, we have

(1.3) $\zeta(s, 1) = \zeta(s) = (2^s - 1)^{-1} \zeta(s, \frac{1}{2})$ and $\zeta(s, 2) = \zeta(s) - 1$.

Ramaswami [6] presented numerous interesting recursion formulas which can be employed to get the analytic continuation of Riemann Zeta function $\zeta(s)$ over the whole $s$-plane. Apostol [1] also gave some interesting formulas involving the Riemann Zeta function some of which are generalizations of the above mentioned Ramaswami’s identities. Here we are aiming at reviewing their results in a rather detailed manner and then presenting corrected, or generalized formulas, or a different method of proof for some of them. We also give closed-form evaluation of some series involving the Riemann Zeta function, the subject of which has an over two-century history [7], by using an integral representation of $\zeta(s)$ and Apostol’s identities given here.

For these, we first recall the Bernoulli polynomials $B_n(x)$ defined by the generating function:

(1.4)

$$
\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} \quad (|z| < 2\pi).
$$

The numbers $B_n := B_n(0)$ are called the Bernoulli numbers generated by

(1.5)

$$
\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \quad (|z| < 2\pi).
$$

The Bernoulli numbers and polynomials satisfy lots of interesting and useful relations, among other things, the following are listed here:

(1.6)

$$
B_n(x + 1) - B_n(x) = nx^{n-1} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \ \mathbb{N} := \{1, 2, 3, \ldots\}),
$$
which yields

\begin{equation}
B_n(0) = B_n(1) \quad (n \in \mathbb{N} \setminus \{1\});
\end{equation}

\begin{equation}
\sum_{k=1}^{m} k^n = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1} \quad (m, n \in \mathbb{N}).
\end{equation}

If \( n \in \mathbb{N} \), denote \( n = p_1^{a_1} \cdots p_k^{a_k} \) into its prime factorization. Then the Möbius function \( \mu \) is defined as follows (see [2, pp. 24-25]):

\begin{equation}
\mu(n) := \begin{cases} 
1 & \text{if } n = 1, \\
(-1)^k & \text{if } a_1 = \cdots = a_k = 1, \\
0 & \text{otherwise.}
\end{cases}
\end{equation}

A remarkably simple formula for the divisor sum \( \sum_{d|n} \mu(d) \) is given here: For \( n \in \mathbb{N} \), we have

\begin{equation}
\sum_{d|n} \mu(d) = \left[ \frac{1}{n} \right] = \begin{cases} 
1 & \text{if } n = 1, \\
0 & \text{if } n > 1,
\end{cases}
\end{equation}

where \([x]\) denotes the greatest integer \( \leq x \).

The Pochhammer symbol (or the shifted factorial) \( (\alpha)_n \) is defined, for any complex number \( \alpha \), by

\begin{equation}
(\alpha)_n = \begin{cases} 
\alpha(\alpha+1) \cdots (\alpha+n-1) & \text{if } n \in \mathbb{N}, \\
1 & \text{if } n = 0.
\end{cases}
\end{equation}

The generalized binomial expansion is also recalled:

\begin{equation}
(1 - z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^n \quad (|z| < 1).
\end{equation}

2. Identities involving the Zeta functions

We begin with recalling three of Ramaswami’s identities [6] which can be employed to get the analytic continuation of Riemann Zeta function \( \zeta(s) \) over the whole \( s \)-plane:

\begin{equation}
\zeta(s) (1 - 2^{1-s}) = \sum_{n=1}^{\infty} \frac{(s)_n}{n!} \zeta(n + s) 2^{-n-s},
\end{equation}
\[ (2.2) \quad \zeta(s) \left(1 - 3^{1-s}\right) = 1 + 2 \sum_{n=1}^{\infty} \frac{(s)_{2n}}{(2n)!} \zeta(2n + s) 3^{-2n-s}, \]

\[ (2.3) \quad \zeta(s) \left(1 - 2^{-s} - 3^{-s} - 6^{-s}\right) = 1 + 2 \sum_{n=1}^{\infty} \frac{(s)_{2n}}{(2n)!} \zeta(2n + s) 6^{-2n-s}. \]

It follows from the definition (1.2) of the Hurwitz Zeta function \( \zeta(s, a) \) that

\[ (2.4) \quad \frac{\partial}{\partial a} \zeta(s, a) = -s \zeta(s + 1, a), \]

\[ (2.5) \quad \zeta(s, a) - \zeta(s, a + k) = \sum_{n=0}^{k-1} (a + n)^{-s} \quad (k \in \mathbb{N}), \]

which, for \( k = 1 \), yields

\[ (2.6) \quad \zeta(s, a) - \zeta(s, a + 1) = a^{-s}. \]

By grouping integers modulo \( k \), it is easy to obtain the multiplication formula for \( \zeta(s, a) \):

\[ (2.7) \quad \zeta(s, ka) = k^{-s} \sum_{n=0}^{k-1} \zeta\left(s, a + \frac{n}{k}\right) \quad (k \in \mathbb{N}), \]

which, upon setting \( a = \frac{1}{k} \), gives

\[ (2.8) \quad \zeta(s) = k^{-s} \sum_{m=1}^{k} \zeta\left(s, \frac{m}{k}\right) \quad (k \in \mathbb{N}). \]

We may use the generalized binomial expansion (1.12) to obtain the Taylor expansion of \( \zeta(s, a + 1) \) in the neighborhood of \( a = 0 \) as follows:

\[
\zeta(s, a + 1) = \sum_{k=1}^{\infty} k^{-s} \left(1 + \frac{a}{k}\right)^{-s} = \sum_{k=1}^{\infty} k^{-s} \sum_{n=0}^{\infty} \frac{(s)_n}{n!} \left(-\frac{a}{k}\right)^n
= \sum_{n=0}^{\infty} \frac{(s)_n}{n!} (-a)^n \zeta(n+s) \zeta(n)\right)\]

\[
= \sum_{n=0}^{\infty} \frac{(s)_n}{n!} \zeta(n+s) (-a)^n,
\]
the exchange of the summations being guaranteed by the absolute convergence of the involved double series. We thus have

\begin{equation}
\zeta(s, a + 1) = \sum_{n=0}^{\infty} \frac{(s)_n}{n!} \zeta(n + s) (-a)^n \quad (|a| < 1; \ \Re(s) > 1),
\end{equation}

which plays a key role in the work of Apostol [1] who noted (2.4) to get (2.9) and may hold for all \( s \) except \( s = 1 \) by the analytic continuation.

By taking \( a = -\frac{h}{k} \) in (2.9), \( 0 \leq h \leq k - 1 \), and summing on \( h \), Apostol [1] obtained

\begin{equation}
\sum_{h=0}^{k-1} \zeta \left( s, 1 - \frac{h}{k} \right) = \sum_{n=0}^{\infty} \frac{(s)_n}{n!} \zeta(n + s) k^{-n} \sum_{h=0}^{k-1} h^n,
\end{equation}

which, upon employing (2.8) and (1.8), immediately leads to the Apostol’s identity [1, p. 240]:

\begin{equation}
\zeta(s) \left( 1 - k^{1-s} \right) = \sum_{n=1}^{\infty} \frac{(s)_n \zeta(n + s)}{n! k^{n+s}} \frac{B_{n+1}(k) - B_{n+1}}{n+1} \quad (k \in \mathbb{N}),
\end{equation}

which, for \( k = 2 \), reduces to (2.1).

Similarly, Apostol [1] obtained the following identities:

\begin{equation}
\zeta(s) \left( 1 - k^{1-s} \right) = \sum_{h=1}^{k-1} h^{-s} + \sum_{n=1}^{\infty} (-1)^n \frac{(s)_n \zeta(n + s)}{n! k^{n+s}} \frac{B_{n+1}(k) - B_{n+1}}{n+1} \quad (k \in \mathbb{N}),
\end{equation}

where the empty sum (as usual, in what follows) is understood to be nil;

\begin{equation}
\zeta(s) \left( 1 - k^{1-s} \right) = \frac{1}{2} \sum_{h=1}^{k-1} h^{-s} + \sum_{n=1}^{\infty} \frac{(s)_{2n} \zeta(2n + s)}{(2n)! k^{2n+s}} \frac{B_{2n+1}(k) - B_{2n+1}}{2n+1} \quad (k \in \mathbb{N});
\end{equation}

\begin{equation}
\sum_{h=1}^{k-1} h^{-s} = \sum_{n=1}^{\infty} \frac{(s)_{2n-1} \zeta(2n - 1 + s)}{(2n - 1)! k^{2n-1+s}} \frac{B_{2n}(k) - B_{2n}}{n} \quad (k \in \mathbb{N});
\end{equation}
\[
\zeta(s) \sum_{d|k} \mu(d) d^{-s} = \phi(-s, k) + 2 \sum_{n=0}^{\infty} \frac{(s)_{2n} \zeta(2n + s)}{(2n)! k^{2n+s}} \phi(2n, k) \quad (k \in \mathbb{N} \setminus \{1, 2\}),
\]

where
\[
\phi(\alpha, k) := \sum_{\substack{1 \leq h \leq k/2 \\(h,k)=1}} h^\alpha
\]
is the sum of the \(\alpha\)th powers of those integers not exceeding \(k/2\) which are relatively prime to \(k\), and \(\mu\) denotes the Môbius function defined by (1.9). The special cases of (2.14) when \(k = 3\) and \(k = 6\) reduce to (2.2) and (2.3), respectively.

By setting \(a = \pm \frac{k}{2}\) in (2.9), \(1 \leq h \leq \frac{k}{2}, k \geq 2\), summing on \(h\), and combining the two resulting identities, denoting by \(f(\alpha, k)\) the sum
\[
f(\alpha, k) := \sum_{1 \leq h \leq k/2} h^\alpha,
\]
we obtain
\[
(1 - k^{-s}) \zeta(s) + \frac{1 + (-1)^k}{2k^s} (2^s - 1) \zeta(s) = f(-s, k) + 2 \sum_{n=0}^{\infty} \frac{(s)_{2n} \zeta(2n + s)}{(2n)! k^{2n+s}} f(2n, k) \quad (k \in \mathbb{N} \setminus \{1\}),
\]
which is a corrected version of Apostol’s identity [1, Eq. (15)] and, when \(k = 2\), reduces to (2.1).

Combining (2.12) and (2.13), Apostol [1] noted that the cases of \(k = 2\) and \(k = 3\) reduce to (2.1) and (2.2), respectively. However, it may not be easy to get (2.2) from (2.12) and (2.13). In fact, setting \(k = 3\) in (2.12) and (2.13) with (1.6) and (1.7), we obtain
\[
\zeta(s) \left(1 - 3^{1-s}\right) = \sum_{n=1}^{\infty} \frac{(s)_n \zeta(n + s)}{n! 3^{n+s}} \left(1 + 2^n\right),
\]
which can be obtained directly by setting \(k = 3\) in (2.10) with (1.6) and (1.7). The Ramaswami’s identity (2.2) here has been proved as a special case of Apostol’s identity (2.14) when \(k = 3\). So it may be interesting
to compare (2.2) with (2.16), and to prove their resulting series identity directly.

We conclude this note by presenting formulas for closed-form evaluation of series involving the Zeta function, the subject of which has an over two-century history as noted in Srivastava [7] and has been an object of many mathematicians' concern since then (see, e.g., [3], [4], [8]). First recall an integral representation for \( \zeta(s) \) (cf. [5, p. 33]):

\[
(2.17) \quad \zeta(s) = \frac{1}{2} + \frac{1}{s - 1} + 2 \int_0^\infty \frac{\sin(s \arctan t)}{e^{2\pi t} - 1} \frac{dt}{(1 + t^2)^{-\frac{s}{2}}}.
\]

whose integral part holds for all finite values of \( s \).

By combining (2.17) with (2.10) and (2.11), we obtain

\[
(2.18) \quad \sum_{n=2}^\infty \frac{\zeta(n)}{k^n} \frac{B_n(k) - B_n}{n} = \log k \quad (k \in \mathbb{N});
\]

\[
(2.19) \quad \sum_{n=2}^\infty (-1)^n \frac{\zeta(n)}{k^n} \frac{B_n(k) - B_n}{n} = H_{k-1} - \log k \quad (k \in \mathbb{N}),
\]

where \( H_k \) denotes the harmonic numbers defined by

\[
(2.20) \quad H_k := \sum_{j=1}^k \frac{1}{j} \quad (k \in \mathbb{N}) \quad \text{and} \quad H_0 = 0.
\]

Adding and subtracting two formulas (2.18) and (2.19), we get

\[
(2.21) \quad \sum_{n=1}^\infty \frac{\zeta(2n+1)}{k^{2n+1}} \frac{B_{2n+1}(k)}{2n+1} = \log k - \frac{1}{2} H_{k-1} \quad (k \in \mathbb{N});
\]

\[
(2.22) \quad \sum_{n=1}^\infty \frac{\zeta(2n)}{k^{2n}} \frac{B_{2n}(k) - B_{2n}}{n} = H_{k-1} \quad (k \in \mathbb{N}).
\]

Some special cases of (2.18) and (2.19) are listed here:

\[
(2.23) \quad \sum_{n=2}^\infty \frac{\zeta(n)}{2^n} = \log 2,
\]
which is recorded in [8, Eq. 3.4(44)];

\[ \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{2^n} = 1 - \log 2, \]

which is also recorded in [8, Eq. 3.4(43)];

\[ \sum_{n=2}^{\infty} \frac{\zeta(n)}{3^n} (1 + 2^{n-1}) = \log 3; \]

\[ \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{3^n} (1 + 2^{n-1}) = \frac{3}{2} - \log 3. \]

References


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