STRONG CONVERGENCE THEOREMS
FOR LOCALLY PSEUDO-CONTRACTIVE
MAPPINGS IN BANACH SPACES

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Abstract. Let $X$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty bounded open subset of $X$, and $T$ a continuous mapping from the closure of $C$ into $X$ which is locally pseudo-contractive mapping on $C$. We show that if the closed unit ball of $X$ has the fixed point property for nonexpansive self-mappings and $T$ satisfies the following condition: there exists $z \in C$ such that $\|z - T(z)\| < \|z - T(x)\|$ for all $x$ on the boundary of $C$, then the trajectory $t \mapsto z_t \in C$, $t \in [0, 1]$ defined by the equation $z_t = tT(z_t) + (1 - t)z$ is continuous and strongly converges to a fixed point of $T$ as $t \to 1$.

1. Introduction

Let $X$ and $X^*$ be a real Banach space with norm $\| \cdot \|$ and the dual space, respectively, and $C$ a subset of $X$. A mapping $T : C \to X$ is said to be pseudo-contractive ([3], [19]) if for each $u, v \in X$ and $\lambda > 1$

\[
(\lambda - 1)\|u - v\| \leq \|\lambda I - T\|(u) - (\lambda I - T)(v)\|.
\]

Following Kato [14], we can describe an equivalent formulation to this definition. A mapping $T$ from $C$ to $X$ is pseudo-contractive if and only if for every $u, v \in C$ there exists $j \in J(u - v)$ such that

\[
\langle T(u) - T(v), j \rangle \leq \|u - v\|^2;
\]

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where $J : X \to 2^{X^*}$ is the (normalized) duality mapping which is defined by

$$J(x) = \{ x^* \in X^* : \langle x, x^* \rangle = \|x\|^2, \|x\| = \|x^*\| \}.$$ 

Here $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. If (2) holds locally, that is, if for each $x \in C$ has a neighborhood $U$ such that the restriction of $T$ to $U$ is pseudo-contractive.

The pseudo-contractive mappings which are closely more general than the nonexpansive mappings (mappings $T : C \to X$ for which $\|T(x) - T(y)\| \leq \|x - y\|$, $x, y \in C$) are characterized by the fact that a mapping $T : C \to X$ is pseudo-contractive if and only if the mapping $A = I - T$ is accretive on $C$ ([3], [14]). Recent interest in mapping theory for accretive operators, particularly as it relate to existence theorems for nonlinear differential equations, has prompted a corresponding interest in fixed point theory for pseudo-contractive mappings (e.g., [3], [5], [7], [14] $\sim$ [16], [18]).

The purpose of this paper is to continue the discussion concerning the strong convergence of the path $\{z_t\}$, $0 \leq t < 1$ defined by the equation $z_t = tT(z_t) + (1 - t)x$ for $x \in C$. Actually, we prove for a locally pseudo-contractive mapping as well as a locally nonexpansive mapping, that the strong $\lim_{t \to 1^-} z_t$ exists and is a fixed point of $T$ under a certain boundary conditions in a reflexive Banach space with a uniformly Gâteaux differentiable norm. Our results are extensions of results obtained by Bruck et al. [4] and Morales [21] to more general types of spaces. However, our proofs reply mainly on that of given in [4] and [21]. Further, we obtain the strong convergence results in reflexive Banach spaces with a weakly sequentially continuous duality mapping. The study of this type of problems begun over thirty years ago can be found in [2], [12], [13], [17], [23], [27], [29], [31], and [32] among others.

2. Preliminaries

Throughout this paper, we assume $X$ being a Banach space and denote the set of all nonnegative real numbers by $\mathbb{R}^+$. When $\{x_n\}$ is a sequence in $X$, then $x_n \to x$ (resp. $x_n \rightharpoonup x$, $x_n \rightharpoonup x$) will denote strong (resp. weak, weak*) convergence of the sequence $\{x_n\}$ to $x$.

Let $U = \{x \in X : \|x\| = 1\}$ be the unit sphere of $X$. The norm of $X$ is said to be Gâteaux differentiable (and $X$ is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

(3)
exists for each $x, y$ in $U$. It is said to be uniformly Gâteaux differentiable if each $y \in U$, this limit is attained uniformly for $x \in U$. The norm is said to be Fréchet differentiable if for each $x \in U$, the limit is obtained uniformly for $y \in U$. Finally, the norm is said to be uniformly Fréchet differentiable (and $E$ is said to be uniformly smooth) if the limit in (3) is attained uniformly for $(x, y) \in U \times U$. Since the dual $X^*$ of $X$ is uniformly convex if and only if the norm of $X$ is uniformly Fréchet differentiable, every Banach space with a uniformly convex dual is reflexive and has a uniformly Gâteaux differentiable norm. But there are reflexive Banach spaces with a uniformly Gâteaux differentiable norm that are not even isomorphic to a uniformly smooth Banach space. To see this, consider $X$ to be the direct sum $l^2(l^{p_n})$, the class of all those sequences $x = \{x_n\}$ with $x_n \in l^{p_n}$ and $\|x\| = (\sum_{n<\infty} \|x_n\|^2)^{\frac{1}{2}}$ ([6]). Now if $1 < p_n < \infty$ for all $n \in \mathbb{N}$, where either $\limsup_{n\to\infty} p_n = \infty$ or $\liminf_{n\to\infty} p_n = 1$, then $X$ is a reflexive Banach space ([6]) with a uniformly Gâteaux differentiable norm ([33]), but is not uniformly smooth ([6]) (cf. [26]).

The (normalized) duality mapping $J$ is single valued if and only if $X$ is smooth. It is also well-known that if $X$ has a uniformly Gâteaux differentiable norm, then $J$ is uniformly continuous on bounded subsets of $X$ from the strong topology of $X$ to the weak-star topology of $X^*$. This is explicitly proved in Lemma 2.2 of [28] (also [8]). Suppose that $J$ is single valued. Then $J$ is said to be weakly sequentially continuous if for each $\{x_n\} \subset X$ with $x_n \rightharpoonup x, J(x_n) \rightharpoonup J(x)$ ([10]).

Let $\mu$ be a mean on positive integers $\mathbb{N}$, that is, a continuous linear functional on $c_0$ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that $\mu$ is a mean on $\mathbb{N}$ if and only if

$$\inf\{a_n : n \in \mathbb{N}\} \leq \mu(a) \leq \sup\{a_n : n \in \mathbb{N}\}$$

for every $a = (a_1, a_2, \ldots) \in c_0$. We sometimes use $\mu_n(a_n)$ in place of $\mu(a)$. A mean $\mu$ on $\mathbb{N}$ is called a Banach limit if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, \ldots) \in c_0$. Using the Hahn-Banach theorem, we can prove the existence of a Banach limit. We know that if $\mu$ is a Banach limit, then

$$\liminf_{n \to \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \to \infty} a_n$$
for every \( a = (a_1, a_2, \ldots) \in \ell^\infty \). Let \( \{x_n\} \) be a bounded sequence in \( E \). Then we can define the real valued continuous convex function \( \phi \) on \( E \) by

\[
\phi(z) = \mu_n \|x_n - z\|^2
\]

for each \( z \in E \).

The following lemma which was given in [11, 13, 30] is, in fact, a variant of Lemma 1.2 in [26].

**Lemma 1.** Let \( C \) be a nonempty closed convex subset of a Banach space \( X \) with a uniformly Gâteaux differentiable norm and let \( \{x_n\} \) be a bounded sequence in \( X \). Let \( \mu \) be a Banach limit and \( u \in C \). Then

\[
\mu_n \|x_n - u\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2
\]

if and only if

\[
\mu_n (x - u, J(x_n - u)) \leq 0
\]

for all \( x \in C \), where \( J \) is the duality mapping.

Recall that a closed convex subset \( C \) of \( X \) is said to have the fixed point property for nonexpansive self-mappings if every nonexpansive mapping \( T : C \to C \) has a fixed point, i.e. there is a point \( p \in C \) such that \( T(p) = p \). It is well-known that every bounded closed convex subset of a uniformly smooth Banach space has the fixed point property for nonexpansive self-mapping (cf. [9, p.45]). We also observe that spaces which enjoy the fixed point property for nonexpansive self-mappings are not necessarily spaces with a uniformly Gâteaux differentiable norm. The converse of this fact appears to be unknown as well.

Finally, to fix the notation, we will denote the closure and boundary of \( C \) by \( \overline{C} \) and \( \partial C \) respectively, and for \( u, v \in X \) we use \( \text{seg}[u, v] \) to denote the segment \( \{tu + (1 - t)v : t \in [0,1]\} \). We will also use \( B(x; r) \) and \( \overline{B}(x; r) \) to stand for the open ball \( \{z \in X : \|x - z\| < r\} \) and the closed ball \( \{z \in X : \|x - z\| \leq r\} \) respectively. We denote the distance between the sets \( A \) and \( B \) by \( \text{dist}(A, B) \), i.e.,

\[
\text{dist}(A, B) = \inf\{\|a - b\| : a \in A, \ b \in B\}.
\]
3. Main results

We begin with the following lemma which was given in [4].

**Lemma 2.** Let $X$ be a Banach space, $C$ a bounded open subset of $X$, and $T : \overline{C} \to X$ a continuous mapping which is locally nonexpansive on $C$. Suppose that for each $z \in C$, there exists a continuous path $t \mapsto z_t$, $0 \leq t < 1$ satisfying

$$z_t = tT(z_t) + (1 - t)z \in C$$

and

$$\text{dist}(\{z_t\}, \partial C) > 0.$$ 

Let $z \in C$, $d = \text{dist}(\{z_t\}, \partial C)$, and $w \in C$ with $\|w - z\| < d$. If $\{w_t\}$ is the path corresponding to $w$, then $\|w_t - z_t\| \leq \|w - z\|$ for all $0 \leq t < 1$.

As in [4], we prepare the following more general proposition for our main results by using Lemma 1 and 2.

**Proposition 1.** Let $X$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm for which the closed unit ball has the fixed point property for nonexpansive self-mappings. Let $C$ be a bounded open subset of $X$ and let $T : \overline{C} \to X$ be a continuous mapping which is locally nonexpansive on $C$. Suppose that for each $z \in C$, there exists a continuous path $t \mapsto z_t$, $0 \leq t < 1$ satisfying

$$z_t = tT(z_t) + (1 - t)z \in C$$

and

$$\text{dist}(\{z_t\}, \partial C) > 0,$$

and that each component of $C$ contains a fixed point of $T$. Then for each $z \in C$, the strong $\lim_{t \to 1^-} z_t$ exists and is a fixed point of $T$.

**Proof.** Since each component of $C$ contains a fixed point of $T$, the set

$$E = \{z \in C : \text{the strong } \lim_{t \to 1^-} z_t \text{ exists}\}$$

is nonempty. So it suffices to show that $E$ is both open and closed in $C$. 
To see that $E$ is closed in $C$, let $\{z^n\} \subset E$ and $z^n \to z \in C$. We can choose $n_0 \in \mathbb{N}$ such that $\|z^n - z\| < \text{dist}(\{z_t\}, \partial C)$ for $n \geq n_0$, and by Lemma 2, $\|z^n_t - z_t\| \leq \|z^n - z\|$ for all $0 \leq t < 1$. Therefore

$$\|z_t - z_t^n\| \leq \|z_t - z_t^n\| + \|z_t^n - z^n\| + \|z^n - z\|$$

$$\leq 2\|z^n - z\| + \|z_t^n - z^n\|$$

for all $n \geq n_0$. Consequently, $\{z_t\}$ is a Cauchy sequence and $z \in E$.

Now we show that $E$ is open in $C$ by using Lemma 1 and a variant patterned after [4] and [27]. Let $z \in E$ and $w = \lim_{t \to 1^-} z_t$. Then $w$ is a fixed point of $T$. Let $d = \text{dist}(\{z_t\}, \partial C)$, and suppose $y \in B(z, \frac{d}{3}) \cap C$.

Let $\frac{2d}{3} < d_1 < d$. Then the closed ball $\overline{B}(w; d_1) \subset C$ is invariant under $T$. Let $t_n \to 1^-$ and $x_n = y_{t_n}$. Since

$$\|x_n - w\| \leq \|y_{t_n} - z_{t_n}\| + \|z_{t_n} - w\|$$

$$\leq \|z - y\| + \|z_{t_n} - w\|$$

$$< \frac{d}{3} + \|z_{t_n} - w\|,$$

$x_n \in \overline{B}(w; d_1)$ for all $n$ sufficiently large. We also have $x_n - T(x_n) \to 0$ as $n \to \infty$. We now define $\phi : X \to \mathbb{R}^+$ by $\phi(x) = \mu_n \|x_n - x\|^2$. Since $\phi$ is continuous and convex and $X$ is reflexive, $\phi$ attains its infimum over $\overline{B}(w; d_1)$ (cf. [1, p. 79]). Let

$$K = \{u \in \overline{B}(w; d_1) : \phi(u) = \inf \{\phi(x) : x \in \overline{B}(w; d_1)\}\}.$$ 

Then it is clearly that $K$ is nonempty closed bounded and convex. If $u \in K$, then

$$\phi(T(u)) = \mu_n \|x_n - T(u)\|^2$$

$$= \mu_n \|T(x_n) - T(u)\|^2$$

$$\leq \mu_n \|x_n - u\|^2$$

$$= \phi(u),$$

so that $K$ is invariant under $T$. Therefore, due to the assumption, $T$ has a fixed point in $K$. Denote such a fixed point $v$. Since $T$ is nonexpansive on $\overline{B}(w; d_1)$ and $v$ is a fixed point of $T$, we have

$$\langle \frac{1}{t_n} x_n - \left(\frac{1}{t_n} - 1\right)y - v, J(v - x_n) \rangle = \langle T(x_n) - T(v), J(v - x_n) \rangle$$

$$\geq -\|T(x_n) - T(v)\| \|J(v - x_n)\|$$

$$\geq -\|v - x_n\|^2$$

$$= \langle x_n - v, J(v - x_n) \rangle.$$
and hence $\langle (\frac{1}{r_n} - 1)(x_n - y), J(v - x_n) \rangle \geq 0$, where $J$ is the duality mapping. So, we obtain

$$\langle x_n - y, J(x_n - v) \rangle \leq 0$$

for all sufficiently large $n$, and

(4) \hspace{1cm} \mu_n \langle x_n - y, J(x_n - v) \rangle \leq 0.

On the other hand, without loss of generality, we assume that $v - v \notin B(w; d_1)$. But, since $v \in B(w; d_1)$, we can choose $\tau > 0$ such that $v + t(v - v) \in B(w; d_1)$ for all $t \in (0, \tau]$. Then, since it follows from Lemma 1 that

$$\mu_n \langle x - v, J(x_n - v) \rangle \leq 0$$

for all $x \in B(w; d_1)$, we have

(5) \hspace{1cm} \mu_n \langle y - v, J(x_n - v) \rangle \leq 0.

Combining (4) and (5), we obtain

$$\mu_n \|x_n - v\|^2 = \mu_n \langle x_n - v, J(x_n - v) \rangle \leq 0.$$  

Therefore, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to $v$. To complete the proof, let $\{x_{m_k}\}$ be another subsequence of $\{y_t : t \in [0, 1]\}$ such that $x_{m_k} = x_{t_{m_k}}$, $t_{m_k} \rightharpoonup 1^-$ as $k \to \infty$, and $x_{m_k} \to u$, where $T(u) = u$. Then (4) and (5) implies that

$$\langle v - y, J(v - u) \rangle \leq 0$$

and

$$\langle u - y, J(u - v) \rangle \leq 0.$$  

Hence $v = u$ and the strong limit $\lim_{t \to 1^-} y_t$ exists. Therefore $E$ is open in $C$. This completes the proof. \hfill \Box

**Remark 1.** Proposition 1 is an improvement of Proposition 1 of [4].

The following lemma is essentially [21, Theorem 1]. For the sake of completion, we include its proof.
Lemma 3. Let $X$ be a Banach space for which the closed unit ball has the fixed point property for nonexpansive self-mapping. Let $C$ be a bounded open subset of $X$ and $T : C \to X$ a continuous mapping which is locally pseudo-contractive on $C$. Suppose that there exists $z \in C$ such that

$$
(6) \quad \|z - T(z)\| < \|x - T(x)\| \quad \text{for all} \ x \in \partial C.
$$

Then $T$ has a fixed point in $C$.

Proof. Let $A = I - T$. Then $A$ is continuous on $\overline{C}$ and locally accretive on $C$, which satisfies (6) on $\partial C$. Therefore by Theorem 1 in [21], $A$ has a zero in $C$. Thus $T$ has a fixed point in $C$. \qed

We also need the following Lemma due to Morales [21] for the proof of main results.

Lemma 4 (Lemma 1 of [21]). Let $X$ be a Banach space, $C$ a nonempty bounded open subset of $X$, and $T : C \to X$ a continuous mapping which is locally nonexpansive on $C$. Suppose that there exists $z \in C$ so that

$$
\|z - T(z)\| < \|x - T(x)\| \quad \text{for all} \ x \in \partial C.
$$

Then

(i) there exists a unique continuous path $t \mapsto z_t \in C$, $0 \leq t < 1$, such that $z_t = tT(z_t) + (1 - t)z$;

(ii) the function $h : [0, 1) \to \mathbb{R}$ defined by $h(t) = \|z_t - T(z_t)\|$ is nonincreasing;

(iii) if $\|u - T(u)\| < \rho$ for $u \in C$, then $B(u; \rho) \subset C$, where $\rho = \frac{\|z - T(z)\|}{3}$;

(iv) the set $\{T(z_t) : 0 \leq t < 1\}$ is bounded provided that $\{z_t : 0 \leq t < 1\}$ is a bounded set.

Now, we study the strong convergence of $\{z_t\}$ in a reflexive Banach space with a uniformly Gâteaux differentiable norm. Our first result will be for locally nonexpansive mappings.
THEOREM 1. Let $X$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty bounded open subset of $X$, and $T : C \to X$ a continuous mapping which is locally nonexpansive on $C$. Suppose that the closed unit ball of $X$ has the fixed point property for nonexpansive self-mapping. Suppose that there exists $z \in C$ such that

$$\|z - T(z)\| < \|x - T(x)\|$$

for all $x \in \partial C$.

Then there exists a unique path $t \mapsto z_t \in C$, $0 \leq t < 1$, satisfying

$$z_t = tT(z_t) + (1 - t)z,$$

where the strong $\lim_{t \to 1^-} z_t$ exists and this limit is a fixed point of $T$.

PROOF. We make use of the idea in [21]. We first observe that due to Lemma 3 we may derive that the component of $C$ which contains $z$ has a fixed point of $T$. Since

$$z_t - T(z_t) = (1 - t)(z - T(z_t)),$$

it follows from Lemma 4 (iv) that $\|z_t - T(z_t)\| \to 0$ as $t \to 1^-$. This implies that there exists $\alpha \in (0, 1)$ so that $\|z_t - T(z_t)\| < \rho$ ($\rho$ as in Lemma 4 (iii)) for all $t \in (\alpha, 1)$. Then by Lemma 4 (iii), we have

$$\text{dist}(\{z_t\}, \partial C) > 0.$$ 

Hence the result follows from Proposition 1. \qed

We are ready to establish the main result for a more general class of mappings as a consequence of the above result.

THEOREM 2. Let $X$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty bounded open subset of $X$, and $T : C \to X$ a continuous mapping which is locally pseudo-contractive on $C$. Suppose that the closed unit ball of $X$ has the fixed point property for nonexpansive self-mapping. Suppose that there exists $z \in C$ such that

$$\|z - T(z)\| < \|x - T(x)\|$$

for all $x \in \partial C$.

Then there exists a unique path $t \mapsto z_t \in C$, $0 \leq t < 1$, satisfying

$$(7) \quad z_t = tT(z_t) + (1 - t)z,$$

where the strong $\lim_{t \to 1^-} z_t$ exists and this limit is a fixed point of $T$. 

Proof. We also follow the arguments developed in [21]. Let \( F(x) = 2x - Tx \). Then by Proposition 2 of [24], \( F^{-1} \) exists on \( F(C) \), and by the assumption on \( T \), \( F^{-1} \) is also locally nonexpansive on \( F(C) \). Now since it is not known whether \( F \) is one-to-one on \( \partial C \), we will redefine the domain of \( T \) to assure that \( F \) is also invertible on the boundary of its domain. Due to Theorem 2 of [19], we may choose \( w \in C \) such that

\[
\|w - T(w)\| < \|z - T(z)\|. 
\]

We now define the set

\[ C_0 = \{ x \in C : \|x - T(x)\| < \|z - T(z)\| \}. \]

Then \( \partial C_0 \subset C \), and

\[
\|w - T(w)\| < \|x - T(x)\| \quad \text{for} \quad x \in \partial C_0. 
\]

This means that the path \( t \mapsto w_t \) exists and is uniquely defined on \((0, 1)\) (Lemma 3 of [19]). Also by Assertion 1 of [15], we know that \( \text{seg}[w, F(w)] \subset F(C_0) \). Therefore \( F^{-1} \) is nonexpansive on \( \text{seg}[w, F(w)] \), and

\[
\|w - F^{-1}(w)\| \leq \|w - F(w)\| < \|x - T(x)\| \quad \text{for} \quad x \in \partial C_0. 
\]

Since \( \partial F(C_0) = \partial F(C_0) \), for each \( y \in \partial F(C_0) \) there exists \( x \in \partial C_0 \) so that \( y = F(x) \) and

\[
\|w - F^{-1}(w)\| < \|y - F^{-1}(y)\|. 
\]

Consequently, by Theorem 1, there exists a unique path \( t \mapsto u_t \in F(C_0), \ t \in [0, 1] \), satisfying the equation

\[
u_t = tF^{-1}(u_t) + (1 - t)w,
\]

where the strong limit \( \lim_{t \to 1^-} u_t \) exists and is a fixed point of \( F^{-1} \). By the uniqueness of the path \( t \mapsto w_t, \ F^{-1}(u_t) = w_s, \) where \( s = \frac{1}{2 - t} \), and hence the strong limit \( \lim_{t \to 1^-} w_t \) exists. Finally, to prove that the path described by (7) strongly converges to a fixed point of \( T \), we can follow the arguments of the proof of Theorem 1 of [24]. \( \square \)

As the immediate consequences of Theorem 1 and 2, we have following:
COROLLARY 1 ([21, Theorem 3]). Let $X^*$ be a uniformly convex Banach space, $C$ a nonempty bounded open subset of $X$, and $T : \overline{C} \to X$ a continuous mapping which is locally nonexpansive on $C$. Suppose that there exists $z \in C$ such that

$$\|z - T(z)\| < \|x - T(x)\|$$

for all $x \in \partial C$.

Then there exists a unique path $t \mapsto z_t \in C$, $0 \leq t < 1$, satisfying

$$z_t = tT(z_t) + (1 - t)z,$$

where the strong $\lim_{t \to 1^-} z_t$ exists and this limit is a fixed point of $T$.

COROLLARY 2 ([21, Theorem 4]). Let $X^*$ be a uniformly convex Banach space, $C$ a nonempty bounded open subset of $X$, and $T : \overline{C} \to X$ a continuous mapping which is locally pseudo-contractive on $C$. Suppose that there exists $z \in C$ such that

$$\|z - T(z)\| < \|x - T(x)\|$$

for all $x \in \partial C$.

Then there exists a unique path $t \mapsto z_t \in C$, $0 \leq t < 1$, satisfying

$$z_t = tTz_t + (1 - t)z,$$

where the strong $\lim_{t \to 1^-} z_t$ exists and this limit is a fixed point of $T$.

REMARK 2. Theorem 1 is also an extension of Theorem 1 of [4] and Theorem 2 is an improvement of Theorem 1 of [20].

We next obtain convergences of the path $\{z_t\}$ for spaces with a weakly sequentially continuous duality mapping.

THEOREM 3. Let $X$ be a reflexive Banach space with a weakly sequentially continuous duality mapping, $C$ a nonempty bounded open subset of $X$, and $T : \overline{C} \to X$ a continuous mapping which is locally nonexpansive on $C$. Suppose that there exists $z \in C$ such that

$$\|z - T(z)\| < \|x - T(x)\|$$

for all $x \in \partial C$.

Then there exists a unique path $t \mapsto z_t \in C$, $0 \leq t < 1$, satisfying

$$z_t = tT(z_t) + (1 - t)z,$$

where the strong $\lim_{t \to 1^-} z_t$ exists and this limit is a fixed point of $T$. 
PROOF. We first observe that due to Lemma 1 of [10], the duality mapping \( J \) is single-valued. Now let \( x_n = z_{t_n} \) for \( t_n \to 1^- \) as \( n \to \infty \) and let \( \{x_{n_k}\} \) be a subsequence of \( \{x_n\} \) such that \( x_{n_k} \rightharpoonup v \). Since \( z_t - T(z_t) = (1 - t)(z - T(z_t)) \), we also have from Lemma 4 (iv) that

\[
\|z_t - T(z_t)\| \to 0 \quad \text{as} \quad t \to 1^-.
\]

So, we have

\[
\limsup_{k \to \infty} \|x_{n_k} - T(v)\| \leq \limsup_{k \to \infty} \{\|x_{n_k} - T(x_{n_k})\| + \|T(x_{n_k}) - T(v)\|\}
\]
\[
\leq \limsup_{k \to \infty} \|x_{n_k} - v\|.
\]

If \( T(v) \neq v \), from Theorem 1 in [10], we have

\[
\limsup_{k \to \infty} \|x_{n_k} - v\| < \limsup_{k \to \infty} \|x_{n_k} - T(v)\|
\]
\[
\leq \limsup_{k \to \infty} \|x_{n_k} - v\|.
\]

This is a contradiction. Hence \( v \) is a fixed point of \( T \). Since \( T \) is nonexpansive, by the same argument in proof of Proposition 1, we have

\[
\langle z - x_{n_k}, J(x_{n_k} - v) \rangle \geq 0.
\]

Thus it follows that

\[
\|x_{n_k} - v\|^2 \leq \langle x_{n_k} - v, J(x_{n_k} - v) \rangle + \langle z - x_{n_k}, J(x_{n_k} - v) \rangle
\]
\[
= \langle z - v, J(x_{n_k} - v) \rangle.
\]

Since \( x_{n_k} \rightharpoonup v \) and \( J \) is weakly sequentially continuous, we have \( x_{n_k} \rightharpoonup v \). We now follow the last argument given in the proof of Proposition 1, to conclude that \( \{z_t\} \) converges strongly to \( v \) in \( C \). This completes the proof. \( \square \)

We also have the following result as a consequence of Theorem 3.
THEOREM 4. Let $X$ be a reflexive Banach space with a weakly sequentially continuous duality mapping, $C$ a nonempty bounded open subset of $X$, and $T : C \to X$ a continuous mapping which is locally pseudo-contractive on $C$. Suppose that there exists $z \in C$ such that

$$
\|z - T(z)\| < \|x - T(x)\| \quad \text{for all } x \in \partial C.
$$

Then there exists a unique path $t \mapsto z_t \in C$, $0 \leq t < 1$, satisfying

$$
z_t = tT(z_t) + (1 - t)z,
$$

where the strong limit $\lim_{t \to 1^-} z_t$ exists and this limit is a fixed point of $T$.

PROOF. The result follows from the proof of Theorem 2. \qed

REMARK 3. Theorem 4 is a local version of Theorem 3 in [23].

REMARK 4. In the case that $T : C \to X$ is a closed mapping which is continuous and locally pseudo-contractive as well as locally nonexpansive on $C$ in the above results, we can also obtain the same results using the methods developed in [22] and [25].

References


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