ON MAXIMAL SUBSET-SUM-DISTINCT SEQUENCES

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ABSTRACT. Since P. Erdős introduced the concept of "subset-sum-distinctness", lots of mathematicians have been interested in a "dense" set having distinct subset sums. In this paper, we establish a couple of theorems on maximal subset-sum-distinct sequence with respect to the set inclusion.

1. Introduction

A subset-sum-distinct set is defined as a set of positive integers such that no two finite subsets have the same sum. In this case we say briefly that it is "SSD" or it is an "SSD-set". If an increasing sequence of positive integers \( a = \{a_n\}_{n=1}^{\infty} \) is SSD, we say that \( a \) is a subset-sum-distinct sequence or briefly "SSD-sequence".

DEFINITION 1.1. An SSD-sequence \( a \) is called maximal if any set of positive integers containing \( a \) properly cannot be an SSD-sequence.

Stimulated by P. Erdős' open question ([6, p. 114, problem C8]), finite dense SSD-sets have been considered by many mathematicians (see [1], [2], [3], [4], [5], pp. 59–60], [7]). In the next section, we change our point of view to maximal SSD-sequences which are as dense as possible in the sense that no properly larger SSD-sequences exist.

As preliminary knowledge, we observe that SSD-sets are closed under a number of operations:

(i) (Subsets) A subset of an SSD-set is again an SSD-set; hence SSD-sets are also closed under intersection.

(ii) (Dilation) If \( a = \{a_n\}_{n=1}^{\infty} \) is SSD, then so is \( k a = \{ka_n\}_{n=1}^{\infty} \) where \( k \) is a positive integer.

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(iii) (Contractions) Let $a = \{a_n\}_{n=1}^{\infty}$ be SSD and $B = \{b_1, b_2, \cdots, b_k\}$ a finite subset of $a$. Remove every element of $B$ from $a$ and then adjoin $b_1 + b_2 + \cdots + b_k$.

(iv) (Greedy closure) For a given SSD-set $A$, let $b$ be the smallest positive integer such that $b \notin A$ and $A$ with $b$ adjoined is SSD. Call this new set $A'$; let $A' = A$ if there is no such $b$. Repeat this procedure a countably infinite number of times. We call the resulting sequence the greedy closure of $A$. For example, the greedy closure of $\{3, 5, 21\}$ is $\{1, 3, 5, 10, 21, 41, 82, 164, \cdots \}$. Note that every greedy closure of a given SSD-set is a maximal SSD-sequence.

(v) (Translations) If $\{a_1, a_2, \cdots, a_k\}$ is SSD and the integer $K > a_1 + a_2 + \cdots + a_k$, then $\{K + a_1, K + a_2, \cdots, K + a_k\}$ is SSD also. A simple proof appears as Lemma 2.1 in the next section.

(vi) (Large dilation adjunctions) If $A = \{a_1, a_2, \cdots, a_k\}$ and $B$ are SSD, then so is $A$ with $K \cdot B = \{Kb : b \in B\}$ adjoined, provided $K$ is an integer satisfying $K > a_1 + a_2 + \cdots + a_k$.

2. Main Theorems

We start with a lemma that will be used in the proof of Theorem 2.4.

**Lemma 2.1.** If $\{b_1, b_2, b_3, \cdots, b_m\}$ is SSD and $K > b_1 + b_2 + \cdots + b_m$, then so is the set $A := \{K + b_1, K + b_2, K + b_3, \cdots, K + b_m\}$.

**Proof.** Suppose that $A$ is not SSD. Then there are two distinct subsets $I, J$ of $\{1, 2, 3, \cdots, m\}$ such that

$$\sum_{i \in I} (K + b_i) = \sum_{j \in J} (K + b_j).$$

Since $\{b_1, b_2, \cdots, b_m\}$ is SSD, we have $|I| \neq |J|$. So, we may assume that $|J| > |I|$. But then we have

$$K \leq (|J| - |I|)K = \sum_{i \in I} b_i - \sum_{j \in J} b_j \leq b_1 + b_2 + \cdots + b_m < K,$$

a contradiction. \qed

The following theorem gives a delicate sufficient condition for an SSD-sequence to be maximal.
THEOREM 2.2. For an SSD-sequence \(a = \{a_n\}_{n=1}^{\infty}\), suppose there exists \(N\) such that \(a_{N+1} \leq 2a_{N} + 1\) and

\[
\frac{a_{j+1}}{a_j} \leq 3, \quad j \geq N
\]  

(2.1)

and for any integer \(n\) with \(1 \leq n \leq a_N\) we have, for some \(I, J \subseteq \{1, 2, 3, \ldots, N\}\),

\[
n = \sum_{i \in I} a_i - \sum_{j \in J} a_j.
\]  

(2.2)

Then \(a\) is maximal. Moreover the constant 3 in (2.1) is the best possible in the sense that if 3 is replaced by any \(\beta > 3\), then the theorem fails to hold.

PROOF. For the maximality of \(a\), it suffices to show that any positive integer \(n\) can be written as in (2.2) with \(I, J \subseteq \{1, 2, 3, \ldots\}\). We show that, for any positive integer \(k\),

\[
\text{(2.3) For all } n \text{ with } 1 \leq n \leq \sum_{j=0}^{k-1} a_{N+j}, \quad n = \sum_{i \in I} a_i - \sum_{j \in J} a_j
\]

for some \(I, J \subseteq \{1, 2, 3, \ldots, N + k - 1\}\). We show first that, for any positive integer \(k\),

\[
\text{(2.4) } a_{N+k} \leq 2 \sum_{j=0}^{k-1} a_{N+j} + 1
\]

by using induction on \(k\). Note that it is true for \(k = 1\) by assumption of the theorem. Suppose (2.4) is true for fixed \(k\). Then

\[
a_{N+k+1} \leq 3a_{N+k} \quad \text{by (2.1)}
\]

\[
= 2a_{N+k} + a_{N+k}
\]

\[
\leq 2a_{N+k} + 2 \sum_{j=0}^{k-1} a_{N+j} + 1
\]

\[
= 2 \sum_{j=0}^{k} a_{N+j} + 1
\]
which proves (2.4) for all \( k \). Let us prove (2.3) by induction on \( k \). If \( k = 1 \), then we have (2.2). Suppose (2.3) is true for fixed \( k \) and \( 1 \leq n \leq \sum_{j=0}^{k-1} a_{N+j} \). If \( 1 \leq n \leq \sum_{j=0}^{k-1} a_{N+j} \), then we can take the same \( I, J \) as in (2.3). If \( n \geq a_{N+k} \), then \( 0 \leq n - a_{N+k} \leq \sum_{j=0}^{k-1} a_{N+j} \). Hence

\[
n - a_{N+k} = \sum_{i \in I} a_i - \sum_{j \in J} a_j
\]

for some \( I, J \subseteq \{1, 2, 3, \ldots , N + k - 1\} \) by the induction hypothesis.

Thus

\[
n = \sum_{i \in I_1} a_i - \sum_{j \in J_1} a_j \quad \text{where} \quad I_1 = I \cup \{N + k\}, \quad J_1 = J.
\]

Now assume that \( \sum_{j=0}^{k-1} a_{N+j} < n < a_{N+k} \). Then

\[
1 \leq a_{N+k} - n \leq a_{N+k} - \sum_{j=0}^{k-1} a_{N+j} - 1 \leq \sum_{j=0}^{k-1} a_{N+j}
\]

where the last inequality comes from (2.4). Hence, by the induction hypothesis,

\[
a_{N+k} - n = \sum_{i \in I} a_i - \sum_{j \in J} a_j
\]

for some \( I, J \subseteq \{1, 2, 3, \ldots , N + k - 1\} \). Thus

\[
n = \sum_{i \in I_2} a_i - \sum_{j \in J_2} a_j \quad \text{where} \quad I_2 = J \cup \{N + k\}, \quad J_2 = I.
\]

For the last statement of the theorem, let \( \beta > 3 \), and choose a positive integer \( m \) large enough so that \( (3^{m+1} + 1)/3^m < \beta \). Then \( a = \{1, 3, 3^2, 3^3, \ldots , 3^m, 3^{m+1} + 1, 3^{m+2}, 3^{m+3}, \ldots \} \) satisfies all the conditions of the theorem with \( N = 1 \) if the constant 3 in (2.1) is replaced by \( \beta \). But \( a \) is not maximal since \( a \cup \{(3^{m+1} + 1)/2\} \) forms an SSD-set. \( \square \)

**Corollary 2.3.** If \( a = \{a_n\} \) is an SSD-sequence such that

\[
a_1 = 1 \quad \text{and} \quad \frac{a_{j+1}}{a_j} \leq 3, \quad j \geq 1,
\]

then \( a \) is maximal. In particular, \( \{1, 3, 3^2, 3^3, \ldots \} \) is a maximal SSD-sequence.
PROOF. Apply Theorem 2.2 with $N = 1$. \hfill \Box

As the following theorem shows, it is quite curious that there is no positive lower bound on the reciprocal sums of maximal SSD-sequences.

**Theorem 2.4.** For any $\epsilon > 0$, there exists a maximal SSD-sequence $a = \{a_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \frac{1}{a_n} < \epsilon$.

**Proof.** Let $\epsilon > 0$ be given. First choose an integer $m$ large enough so that

$$\frac{2}{2^m} + \frac{m}{2^m} < \epsilon. \quad (2.5)$$

Then define a sequence $a = \{a_n\}_{n=1}^{\infty}$ by

$$a_n = \begin{cases} 2^m & \text{if } n = 1, \\ 2^m + 2^{n-2} & \text{if } 2 \leq n \leq m + 1, \\ 2^{n-1} & \text{if } n \geq m + 2. \end{cases}$$

We show that $a$ is a maximal SSD-sequence whose reciprocal sum is less than $\epsilon$. For a contradiction, suppose $a$ is not SSD. Split $a$ into two parts $X$ and $Y$ where

$$X := \{a_1, a_{m+2}, a_{m+3}, a_{m+4}, \ldots \} = \{2^m, 2^{m+1}, 2^{m+2}, 2^{m+3}, \ldots \},$$

$$Y := \{a_2, a_3, a_4, \ldots, a_{m+1}\} = \{2^m + 1, 2^m + 2, 2^m + 2^2, \ldots, 2^m + 2^{m-1}\}.$$  

Obviously, $X$ is SSD and, by Lemma 2.1, $Y$ is SSD also. Hence there exist $X_1, X_2 \subseteq X$ and $W_1, W_2 \subseteq \{1, 2, 2^2, \ldots, 2^{m-1}\}$, all disjoint from each other, such that

$$\sum_{x_1 \in X_1} x_1 + \sum_{w_1 \in W_1} (2^m + w_1) = \sum_{x_2 \in X_2} x_2 + \sum_{w_2 \in W_2} (2^m + w_2).$$

Since $2^m$ divides all the elements of $X$, we obtain

$$\sum_{w_1 \in W_1} w_1 - \sum_{w_2 \in W_2} w_2 \equiv 0 \pmod{2^m}. \quad (2.6)$$
But this is impossible because the absolute value of the left side of (2.6) is greater than 0 and less than $2^m$. Thus we obtain the SSD property of $a$. For this choice of $a$, we have

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = \sum_{x \in X} \frac{1}{x} + \sum_{y \in Y} \frac{1}{y} < 2^{m-1} + \frac{m}{2^m} < \epsilon$$

by (2.5). It remains to prove the maximality of $a$. It is enough to show that, for any positive integer $n$, there exist two subsets $U, V$ of $a$ such that

$$(2.7) \quad n = \sum_{u \in U} u - \sum_{v \in V} v.$$ 

Let $n = n_12^m + n_2$ where $n_1, n_2$ are non-negative integers with $0 \leq n_2 \leq 2^m - 1$. Clearly we can find $A \subseteq X$ and $B \subseteq Y$ such that

$$\sum_{b \in B} b = |B|2^m + n_2 \quad \text{and} \quad \sum_{a \in A} a = |n_1 - |B||2^m.$$ 

Thus we obtain (2.7) if we take

$$U = \begin{cases} A \cup B & \text{if } n_1 \geq |B|, \\ B & \text{if } n_1 < |B|, \end{cases} \quad \text{and} \quad V = \begin{cases} \phi & \text{if } n_1 \geq |B|, \\ A & \text{if } n_1 < |B|. \end{cases} \quad \square$$

References


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