AN ASYMPTOTIC FORMULA FOR $\exp(\frac{x}{1-x})$

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Abstract. We show that $G(x) = e^{x/(1-x)} - 1$ is the exponential generating function for the labeled digraphs whose weak components are transitive tournaments and derive both a recursive formula and an explicit formula for the number of them on $n$ vertices. Moreover, we investigate the asymptotic behavior for the coefficients of $G(x)$ using Hayman’s method.

1. Introduction

When we know the exponential generating function $G(x)$ for a class of graphs, we can easily derive the exponential generating function $C(x)$ for the corresponding connected graphs using the relation

$$1 + G(x) = e^{C(x)}.$$

This is a well-known technique in graph theory [1].

Let us try in the reverse direction, from $C(x)$ to $G(x)$. The most common power series

$$\frac{1}{1-x} - 1 = x + x^2 + x^3 + x^4 + \cdots$$

is, in some sense, meaningless as an ordinary generating function. However, we can make it meaningful as an exponential generating function. This means that the series

$$C(x) = \frac{1}{1-x} - 1 = \frac{x}{1-x}$$

$$= x + 2! \frac{x^2}{2!} + 3! \frac{x^3}{3!} + 4! \frac{x^4}{4!} + \cdots$$

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could be regarded as the exponential generating function for labeled transitive tournaments in graph theoretical sense [3]. From this fact, we know that the exponential generating function
\[ G(x) = e^{C(x)} - 1 = e^{x/(1-x)} - 1 \]
\[ = x + 3\frac{x^2}{2!} + 13\frac{x^3}{3!} + 73\frac{x^4}{4!} + 501\frac{x^5}{5!} + 4051\frac{x^6}{6!} + \cdots \]
counts labeled digraphs whose weak components are transitive tournaments.

In this paper, we show that a recursive formula for the coefficient \(a_n\) of the term \(x^n/n!\) in \(G(x)\) is
\[ a_n = (2n - 1)a_{n-1} - (n - 1)(n - 2)a_{n-2} \quad \text{for} \quad n \geq 3 \]
with the initial condition \(a_1 = 1\) and \(a_2 = 3\), in two different ways and that an explicit formula for \(a_n\) is
\[ a_n = \sum_{k=0}^{n-1} \binom{n}{k} (n-1)_k, \]
where \((n-1)_k\) means a falling factorial. Moreover, we show that an asymptotics for \(a_n\) is
\[ a_n \sim \frac{2^{n}n^{2n}\exp(-n + \frac{1}{2}\sqrt{4n + 1} - \frac{1}{2})}{(2n + 1 - \sqrt{4n + 1})^n(4n + 1)^{1/4}}. \]

2. Formulas for \(a_n\)

In this section we derive a recursive formula for \(a_n\) in two different ways and next an explicit formula for \(a_n\).

First, differentiating \(y = e^{x/(1-x)} - 1\) and rearranging it, we have a differential equation
\[ (1 - x)^2y' = y + 1, \quad y(0) = 0. \]
Solving this equation for \(y = \sum_{n \geq 1} (a_n/n!)x^n\), we get a recursive formula
\[ a_n = (2n - 1)a_{n-1} - (n - 1)(n - 2)a_{n-2} \quad \text{for} \quad n \geq 3 \]
with the initial condition \(a_1 = 1\) and \(a_2 = 3\), as is evidenced by enumerating labeled digraphs under consideration for small \(n\).
Another method to derive this recursive formula is as follows. Let

$$\sum_{n \geq 0} \frac{a_n x^n}{n!} = \exp \left( \frac{x}{1 - x} \right).$$

Taking the logarithm of both sides of this equation, we have

$$\log \left( \sum_{n \geq 0} \frac{a_n x^n}{n!} \right) = \frac{x}{1 - x}.$$

Differentiating both sides and multiplying through by $x$, we have

$$\frac{\sum_{n \geq 0} n a_n (x^n/n!)}{\sum_{n \geq 0} a_n (x^n/n!)} = \frac{x}{(1 - x)^2}.$$

Clear this equation of fractions. For each $n$, find the coefficients of $x^n/n!$ on both sides of the equation and equate them. Ignoring $a_0 = 1$ from the fact that we do not consider digraphs on zero vertices, we get the same recursive formula.

To find $a_n$ itself, we regard the function $e^{z/(1-z)}$ as a complex function. Let

$$\exp \left( \frac{z}{1 - z} \right) = \sum_{n \geq 0} \frac{a_n}{n!} z^n.$$

Then, by Cauchy’s formula, we have

$$\frac{a_n}{n!} = \frac{1}{2\pi i} \int_C \frac{e^{z/(1-z)}}{z^{n+1}} \frac{dz}{1 + w}.$$

$$= \frac{1}{2\pi i} \int_T \frac{e^w}{w^{n+1}} (1 + w)^{n-1} \, dw.$$

$$= \frac{1}{2\pi i} \times 2\pi i \times \text{Res} \left[ \frac{e^w}{w^{n+1}} (1 + w)^{n-1}; 0 \right].$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{(n-k)!},$$

$$= \binom{n}{n-1} \frac{1}{(n-1)!}.$$
where \( z/(1 - z) = w, C : re^{i\theta} \) with \( 0 < r < 1 \) and \( 0 \leq \theta \leq 2\pi \), and \( \Gamma \) is the circle corresponding to \( C \). Therefore, we have

\[
a_n = \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{n!}{(n-k)!} \\
= \sum_{k=0}^{n-1} \binom{n}{k} (n-1)_k \quad \text{for} \quad n \geq 1.
\]

**Theorem 1.** Let \( a_n \) be the number of labeled digraphs of order \( n \) whose weak components are transitive tournaments. Then

1. \( a_n = (2n-1)a_{n-1} - (n-1)(n-2)a_{n-2} \) for \( n \geq 3 \) with the initial condition \( a_1 = 1 \) and \( a_2 = 3 \).
2. \( a_n = \sum_{k=0}^{n-1} \binom{n}{k} (n-1)_k \).

**3. Asymptotics for \( a_n \)**

In this section we want to investigate the asymptotic behavior for the coefficients of \( 1 + G(z) = \exp(\frac{z}{1-z}) \), that is, to find a simple function of \( n \) that affords a good approximation to the values of our coefficients when \( n \) is large.

To do this, we let \( f(z) = \exp(\frac{z}{1-z}) \) and apply Hayman’s method for this \( f(z) \). Now, we introduce the admissibility for Hayman’s method and the method itself.

**Definition 2.** [2, 4] Let \( f(z) = \sum_{n \geq 0} a_n z^n \) be regular in \( |z| < R \), where \( 0 < R \leq \infty \). Next define two auxiliary functions

\[
a(r) = r \frac{f'(r)}{f(r)}
\]

and

\[
b(r) = r a'(r).
\]

We say that \( f(z) \) is admissible in \( |z| < R \) if

(a) there exists an \( R_0 < R \) such that \( f(r) > 0 \) for \( R_0 < r < R \),
(b) there exists a function \( \delta(r) \) defined for \( R_0 < r < R \) such that \( 0 < \delta(r) < \pi \) for those \( r \), and such that uniformly for \( |\theta| \leq \delta(r) \), we have

\[
f(re^{i\theta}) \sim f(r)e^{i\theta a(r) - \frac{1}{2} \theta^2 b(r)} \quad \text{as} \quad r \to R,
\]
(c) uniformly for $δ(r) \leq |θ| \leq π$, we have

$$f(re^{iθ}) = \frac{o(f(r))}{\sqrt{b(r)}} \quad \text{as} \quad r \to R,$$

(d) we have $b(r) \to \infty$ as $r \to R$.

**Lemma 3.** ([2]) Suppose that $f(z) = \sum_{n \geq 0} a_n z^n$ is regular in $|z| < 1$, positive in some range $R_0 < z < 1$, and that there exist constants $0 < α, 0 < β < 1$, and a positive function $C(r), 0 < r < 1$, satisfying

$$C''(r) \quad \text{as} \quad r \to 1,$$

and such that

$$\log f(z) \sim C(|z|)(1 - z)^{-α} \quad \text{as} \quad z \to 1,$$

uniformly for $|\arg z| \leq β(1 - r)$.

Suppose further that for $r$ sufficiently near $1$, we have

$$|f(re^{iθ})| \leq |f(re^{iβ(1 - r)})| \quad \text{for} \quad β(1 - r) \leq |θ| \leq π.$$

Then $f(z)$ is admissible in $|z| < 1$.

**Lemma 4.** ([2, 4]) Let $f(z) = \sum_{n \geq 0} a_n z^n$ be an admissible function in $|z| < R$ and let the function $a(r)$ be positive increasing in some range $r_0 \leq r < R$. Let $r_n$ be the positive real root of the equation $a(r_n) = n$ for each $n = 1, 2, 3, \ldots$ such that $r_0 < r_n < R$. Then

$$a_n \sim \frac{f(r_n)}{r_n^α \sqrt{2πb(r_n)}}.$$

**Lemma 5.** Let

$$f(z) = \exp\left(\frac{z}{1 - z}\right) = \sum_{n \geq 0} \frac{a_n}{n!} z^n.$$

Then $f(z)$ is admissible in $|z| < 1$.

**Proof.** Since $f(z) = \exp\left(\frac{z}{1 - z}\right) = e^{-1} \cdot \exp\left(\frac{1}{1 - z}\right)$, it suffices to show that $g(z) = \exp\left(\frac{1}{1 - z}\right)$ is admissible in $|z| < 1$ [2]. To do this, we apply Lemma 3 for $g(z)$.

We note that $g(z)$ is regular in $|z| < 1$ and that $g(r)$ is positive for $0 < r < 1$. Let us take $α = 1, β$ any number in between 0 and 1, and $C(r) = 1$ for $0 < r < 1$. Then, clearly, the conditions (3.1) and (3.2) are satisfied.
We want to check the condition (3.3). Since
\[
\left| \frac{g(re^{i\theta})}{g(re^{i\beta(1-r)})} \right| = \left| \frac{\exp(1/(1-re^{i\theta}))}{\exp(1/(1-re^{i\beta(1-r)}))} \right| = \left| \exp \left( \frac{1}{1-re^{i\theta}} - \frac{1}{1-re^{i\beta(1-r)}} \right) \right|,
\]
it is enough to show that
\[
\Re \left( \frac{1}{1-re^{i\theta}} - \frac{1}{1-re^{i\beta(1-r)}} \right) \leq 0
\]
for \( \beta(1-r) \leq |\theta| \leq \pi \) and \( r \) sufficiently near 1. Actually, we have
\[
\Re \left( \frac{1}{1-re^{i\theta}} - \frac{1}{1-re^{i\beta(1-r)}} \right) = \frac{r(1-r)(1+r)(\cos \theta - \cos \beta(1-r))}{(1-2r \cos \theta + r^2)(1-2r \cos \beta(1-r) + r^2)} \leq 0
\]
for \( \beta(1-r) \leq |\theta| \leq \pi \) and \( r \) sufficiently near 1. Therefore, the condition (3.3) is satisfied.

Now we want to state an asymptotic for the coefficient \( a_n \) in \( f(z) = \exp(\frac{z}{1-z}) = \sum_{n \geq 0} \frac{a_n}{n!} z^n \).

**Theorem 6.** Let \( a_n \) be the number of labeled digraphs of order \( n \) whose weak components are transitive tournaments. Then
\[
a_n \sim \frac{2^n n^{2n} \exp(-n + \frac{1}{2} \sqrt{4n + 1} - \frac{1}{2})}{(2n + 1 - \sqrt{4n + 1})^n (4n + 1)^{1/4}}.
\]

**Proof.** Since we already showed in Lemma 5 that \( f(z) \) is an admissible function in \( |z| < 1 \), we may apply Lemma 4 for \( f(z) \).

First, we note that \( f(z) \) is regular in \( |z| < 1 \), and have
\[
a(r) = \frac{r}{(1-r)^2},
\]
\[
b(r) = \frac{r(1+r)}{(1-r)^3}.
\]

Since \( a(r) \) is positive increasing for \(-1 \leq r < 1\), we let \( r_n \) be the solution of the equation \( a(r_n) = n \) for positive integer \( n \) such that \( 0 < r_n < 1 \). In this case, the equation is
\[
\frac{r_n}{(1-r_n)^2} = n
\]
and thus our solution is
\[ r_n = 1 + \frac{1}{2n} - \sqrt{\frac{1}{n} + \frac{1}{4n^2}}. \]

Therefore, we have
\[ f(r_n) = \exp \left( \frac{1}{2} \sqrt{4n + 1} - \frac{1}{2} \right) \]

and
\[ b(r_n) = \frac{4n + 1 - \sqrt{4n + 1}}{\sqrt{4n + 1} - 1} \sim n \sqrt{4n + 1}. \]

Using the formula in Lemma 4, we have
\[ a_n \sim \frac{(2n)^n \exp \left( \frac{1}{2} \sqrt{4n + 1} - \frac{1}{2} \right)}{(2n + 1 - \sqrt{4n + 1})^n \sqrt{2\pi n \sqrt{4n + 1}}}. \]

Finally, using Stirling’s formula, we have
\[ (3.4) \quad a_n \sim \frac{2^n n^{2n} \exp \left( -n + \frac{1}{2} \sqrt{4n + 1} - \frac{1}{2} \right)}{(2n + 1 - \sqrt{4n + 1})^n (4n + 1)^{1/4}}. \]

The last column of the following table shows the speed of convergence for our estimator.

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<th>(a_n)</th>
<th>((3.4)/a_n)</th>
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References


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