SOME FIXED POINT THEOREMS FOR CONTRACTIVE AND EXPANSIVE MAPS

GUO-JING JIANG AND SHIN MIN KANG

Abstract. In this paper, fixed point theorems for contractive and expansive maps are established, some of which extend a few results of Das and Debata, Edelstein, Fisher, Leader, Shih and Yeh, and Jungck.

1. Introduction

Let $f$ and $g$ be continuous self maps of a compact metric space $(X, d)$ and let $N$ be the set of positive integers. For $x, y \in X$ and $A, B \subset X$, define

\[ O(x, f) = \{ f^nx \mid n \in N \cup \{0\} \}, \]
\[ O(x, y, f) = O(x, f) \cup O(y, f), \]
\[ O(x, y, f, g) = O(x, y, f) \cup O(x, y, g), \]
\[ \delta(A, B) = \sup \{d(a, b) \mid a \in A, b \in B \}. \]

Let $\delta(A)$ denote the diameter of $A$. Define

\[ C_f = \{ h \mid h : X \to X \text{ and } hf = fh \}, \]
\[ A_f = \{ h \mid h : X \to X \text{ and } h \cap_{n \in N} f^n X = \cap_{n \in N} f^n X \}, \]
\[ H_f = \{ h \mid h : X \to X \text{ and } h \cap_{n \in N} f^n X \subset \cap_{n \in N} f^n X \}. \]

Clearly $C_f$, $A_f$ and $H_f$ are semigroups under composition. Let $F$ and $T$ be families of self maps on $X$. A point $x$ in $X$ is called a fixed point of $F$ if $fx = x$ for $f \in F$, a common fixed point of $F$ and $T$ if $fx = gx = x$ for $f \in F$ and $g \in T$.

Edelstein [2] established the existence of a unique fixed point of a self map $f$ of a compact metric space satisfying the inequality $d(f x, f y) <$
$d(x, y)$. Das and Debata [1], Fisher [3], Leader [7], Shih and Yeh [8] obtained a number of generalizations of this result. Jungck [6] proved two interesting results on fixed point in compact metric spaces, one of which deals with the existence of fixed point of $C_{gf}$ and extends the results of Das and Debata [1], Edelstein [2], Fisher [3], Leader [7], Shih and Yeh [8].

The main purpose of this paper is to extend Jungck’s results to a few much wider classes of maps. In section 2, fixed point theorems are proved by considering a few contractive types conditions for $H_{gf}$, $H_f$ and $H_g$. In section 3, fixed point theorems are established by considering a few expansive types conditions for $H_{gf}$, $C_f$ and $C_g$.

By Proposition 4.1 of Jungck [6] and Proposition 1 of Leader [7], we obtain the following lemmas:

**Lemma 1.1.** Let $f$ be a continuous self map of a compact metric space $(X, d)$. Let $A = \cap_{n \in N} f^n X$. Then
(i) $A$ is a nonempty compact subset of $X$;
(ii) $\{f^n | n \in N \cup \{0\}\} \subset A_f \cap C_f$;
(iii) $C_f \cup A_f \subset H_f$.

**Lemma 1.2.** Let $f$ and $g$ be self maps of a compact metric space $(X, d)$ such that $gf$ is continuous and $f \in A_{gf}$. Then $g \in A_{gf}$.

**Lemma 1.3.** Let $f$ and $g$ be commuting self maps of a compact metric space $(X, d)$ such that $gf$ is continuous. Then $f, g \in A_{gf}$.

**2. Fixed point theorems for $H_{gf}$, $H_f$ and $H_g$**

**Theorem 2.1.** Let $f$ and $g$ be self maps of a compact metric space $(X, d)$ such that $gf$ is continuous and $f \in A_{gf}$. Assume that there exist $S, T \in A_{gf}$ satisfying

\[
(2.1) \quad d(Sx, Ty) < \delta(\cup_{h \in H_{gf}} h O(x, y, f, g))
\]

for $Sx \neq Ty$. Then $f$, $g$, $S$ and $T$ have a unique common fixed point which is a unique fixed point of $H_{gf}$.

**Proof.** Let $A = \cap_{n \in N} (gf)^n X$. It follows from (i) of Lemma 1.1 that $A$ is a nonempty compact subset of $X$. Thus there exist $a, b \in A$ such that $\delta(A) = d(a, b)$. Since $S, T \in A_{gf}$, we can find $x, y \in A$ such that $Sx = a$ and $Ty = b$. By Lemma 1.2 we have $g \in A_{gf}$. Note that
Some fixed point theorems

$f \in A_{gf}$. Then $O(x, y, f, g) \subset A$. We assert that $A$ is a singleton. If not, then $\delta(A) > 0$. Using (2.1),

\[
d(Sx, Ty) < \delta(\cup_{h \in H_{gf}} h O(x, y, f, g)) \\
\leq \delta(\cup_{h \in H_{gf}} h A) \\
\leq \delta(A),
\]

which implies that

\[
0 < \delta(A) = d(Sx, Ty) < \delta(A),
\]

which is impossible. Hence $A$ is a singleton, i.e., $A = \{w\}$ for some $w$ in $X$. This implies that $w$ is a fixed point of $H_{gf}$, in particular, $w$ is a common fixed point of $f$, $g$, $S$ and $T$.

If $v$ is another common fixed point of $f$, $g$, $S$ and $T$, then $(gf)^n = v$ for all $n$ in $N$. This implies $v \in A$ and $v = w$. Hence $f$, $g$, $S$ and $T$ have a unique common fixed point $w$. Note that $f$, $g$, $S$ and $T \in A_{gf} \subset H_{gf}$. Therefore $H_{gf}$ has a unique fixed point $w$. This completes the proof. □

**Corollary 2.1.** Let $f$ and $g$ be self maps of a compact metric space $(X, d)$ such that $gf$ is continuous and $f \in A_{gf}$. If $fx \neq gy$ implies

\[
d(fx, gy) < \delta(\cup_{h \in H_{gf}} h O(x, y, f, g)),
\]

then $f$ and $g$ have a unique common fixed point which is a unique fixed point of $H_{gf}$.

**Proof.** Corollary 2.1 follows from Lemma 1.2 and Theorem 2.1. □

**Remark 2.1.** By (iii) of Lemma 1.1 and Lemma 1.3 and Example 3.1 in section 3, it follows that Corollary 2.1 generalizes properly Theorem 4.2 of Jungck [6].

**Corollary 2.2.** Let $f$ be a continuous self map of a compact metric space $(X, d)$. Assume that there exist $S, T \in A_f$ satisfying

\[
d(Sx, Ty) < \delta(\cup_{h \in H_f} h O(x, y, f))
\]

for $Sx \neq Ty$. Then $f$ has a uniformly contractive fixed point which is a unique fixed point of $H_f$. 
PROOF. Take $g = i_X$ (the identity map) in Theorem 2.1. By (ii) of Lemma 1.1, $f \in A_f$. Note that $O(x, y, f, i_X) = O(x, y, f)$. It follows from Theorem 2.1 that $\cap_{n \in N} f^n X = \{w\}$. $w$ is a unique fixed point of $H_f$. By Theorem 1 of Leader [7], we conclude that $f$ has a uniformly contractive fixed point $w$. This completes the proof.

REMARK 2.2. Corollary 4.3 of Jungck [6] is a special case of Corollary 2.2.

THEOREM 2.2. Let $f$ and $g$ be self maps of a compact metric space $(X, d)$ such that $gf$ is continuous. Assume that there exist $S, T \in A_{gf}$ satisfying

\[ d(Sx, Ty) < \delta(\cup_{h \in H_{gf}} \{hx, hy\}) \]

for $Sx \neq Ty$. Then $H_{gf}$ has a unique fixed point.

PROOF. It follows from (ii) of Lemma 1.1 that $gf \in H_{gf}$. The remaining portion of the proof can be derived as in Theorem 2.1.

THEOREM 2.3. Let $f$ and $g$ be self maps of a compact metric space $(X, d)$ such that $gf$ is continuous. Assume that for every compact subset $Y$ of $X$ which contains more than one element and is mapped into itself by $gf$, there exist $S, T \in A_{gf}$ satisfying

\[ d(Sx, Ty) < \delta(Y) \]

for all $x, y$ in $Y$. Then $H_{gf}$ has a unique fixed point.

PROOF. Let $A = \cap_{n \in N} (gf)^n X$. By (i) and (ii) of Lemma 1.1, $A$ is a nonempty compact subset of $X$ and $gf \in A_{gf}$. Suppose that $\delta(A) > 0$. Then there exist $a, b \in A$ such that $\delta(A) = d(a, b)$. Since $SA = A = TA$, we can find $x, y \in A$ such that $Sx = a$ and $Ty = b$. By (2.2), we have

\[ 0 < \delta(A) = d(Sx, Ty) < \delta(A), \]

which is a contradiction. Hence $\delta(A) = 0$, i.e., $A$ is a singleton. The remaining portion of the proof can be derived as in Theorem 2.1. This completes the proof.

THEOREM 2.4. Let $f$ and $g$ be continuous self maps of a compact metric space $(X, d)$. Assume that there exist $S \in A_f$ and $T \in A_g$ satisfying

\[ d(Sx, Ty) < \delta(\cup_{u \in H_f} uO(x, f), \cup_{v \in H_g} vO(y, g)) \]

for $Sx \neq Ty$. Then $f$, $g$, $S$ and $T$ have a unique common fixed point which is a unique common fixed point of $H_f$ and $H_g$. 
Some fixed point theorems

PROOF. Let $A = \cap_{n \in \mathbb{N}} f^n X$ and $B = \cap_{n \in \mathbb{N}} g^n X$. By (i) and (ii) of Lemma 1.1, $A$ and $B$ are nonempty compact subsets of $X$ and $fA = A$, $gB = B$. Thus there exists $a \in A$ and $b \in B$ such that $\delta(A, B) = d(a, b)$. Note that $SA = A$ and $TB = B$. Then there exist $x \in A$ and $y \in B$ such that $Sx = a$ and $Ty = b$. Suppose that $a \neq b$. Then by (2.3),

$$d(a, b) = d(Sx, Ty)$$
$$< \delta(\cup_{u \in H_f} u \circ O(x, f), \cup_{v \in H_g} v \circ O(y, g))$$
$$\leq \delta(\cup_{u \in H_f} u A, \cup_{v \in H_g} v B)$$
$$\leq \delta(A, B) = d(a, b),$$

which is a contradiction. Therefore $a = b$ and $\delta(A, B) = 0$. This implies $A = B = \{w\}$, say. Clearly $w$ is a common fixed point of $H_f$ and $H_g$. Since every common fixed point of $f$ and $S$ belongs to $A = \{w\}$ and $f w = S w = w$, so $f$ and $S$ have a unique common fixed point $w$. Similarly $w$ is also a unique common fixed point of $g$ and $T$. Thus $w$ is a unique common fixed point of $H_f$ and $H_g$. This completes the proof. \qed

3. Nonunique fixed points

THEOREM 3.1. Let $f$ and $g$ be continuous self maps of a compact metric space $(X, d)$ satisfying $f \in A_{gf}$. If $f x \neq g y$ implies

$$d(f x, g y) > \inf \{d(u x, f u x), d(u y, f u y), d(u x, g u x),$$
$$d(u y, g u y), d(h x, h y) | u \in H_{gf}, h \in C_f \cap C_g\},$$

(3.1)

then at least one of $f$ or $g$ has a fixed point.

PROOF. Let $A = \cap_{n \in \mathbb{N}} (g f)^n X$. By (i) of Lemma 1.1, $A$ is a non-empty compact subset of $X$. It follows from Lemma 1.2 that $g \in A_{gf}$. By the continuity of $f$ and $g$ and compactness of $A$, there exist $a, b \in A$ such that

$$d(a, f a) \leq d(x, f x) \quad \text{and} \quad d(b, g b) \leq d(x, g x)$$

(3.2)

for all $x \in A$. We assume without loss of generality that

$$d(a, f a) \leq d(b, g b)$$

(3.3)
Note that \( gA = A \). Then there exists a point \( w \in A \) such that \( gw = a \). Suppose that \( a \neq fa \), i.e., \( fa \neq gw \). By (3.1), (3.2) and (3.3) we have
\[
\begin{align*}
d(fa, gw) &> \inf\{d(ua, fua), d(uw, fw), d(ua, gua), \\
d(uw, gwu), d(ha, hw) \mid u \in H_{gf}, h \in C_f \cap C_g\}
\geq \inf\{d(a, fa), d(b, gb), d(hgw, hw) \mid u \in H_{gf}, h \in C_f \cap C_g\}
= \inf\{d(a, fa), d(ghw, hw) \mid u \in H_{gf}, h \in C_f \cap C_g\}
= d(a, fa),
\end{align*}
\]
which implies that
\[
d(a, fa) = d(fa, gw) > d(a, fa),\]
which is impossible. Hence \( a = fa \). This completes the proof. \( \square \)

**Remark 3.1.** The following example demonstrates that Theorem 3.1 is more general than Theorem 4.4 of Jungck [6].

**Example 3.1.** Let \( X = \{1, 2, 5\} \) and \( d(x, y) = |x - y| \). Define \( f, g : X \to X \) by
\[
\begin{align*}
f1 &= f2 = g1 = 1 \quad \text{and} \quad f5 = g2 = g5 = 2.
\end{align*}
\]
Then \( f \) and \( g \) are self maps of a compact metric space \( (X, d) \) such that \( gf \) is continuous and \( \cap_{n \in N}(gf)^nX = \{1\} = f \cap_{n \in N}(gf)^nX \). It is now a simple matter to show that
\[
\begin{align*}
0 &= \inf\{d(ux, fux), d(uy, fuy), d(ux, gux), \\
d(uy, guy), d(hx, hy) \mid u \in H_{gf}, h \in C_f \cap C_g\}
< d(fx, gy) = 1
< \delta(\cup_{h \in H_{gf}}hO(x, y, f, g)) = 4
\end{align*}
\]
for \( fx \neq gy \). Thus the conditions of the above Corollary 2.1 and Theorem 3.1 are satisfied but Theorems 4.2 and 4.4 of Jungck [6] are not applicable since \( fg5 = 1 \neq 2 = gf5 \).

**Remark 3.2.** Example 4.4 of Jungck [6] shows that not both \( f \) and \( g \) of the above Theorem 3.1 need have a fixed point and that the fixed point may not be unique.

The proof of the following result goes in a similar fashion as that of Theorem 3.1, so we omit the proof.
THEOREM 3.2. Let $f$ and $g$ be self maps of a compact metric space $(X,d)$ satisfying $gf$ is continuous. Assume that there exist $S,T \in A_{gf}$ such that $S$ and $T$ are continuous and

$$d(Sx,Ty) > \inf\{d(uSux), d(uTy), d(Vux), d(uTy), d(x,y) \mid u \in H_{gf}\}$$

for $Sx \neq Ty$. Then at least one of $S$ or $T$ has a fixed point.

References


Guo-Jing Jiang
Dalian Management Cadre’s College
Dalian, Liaoning 116031
People’s Republic of China

Shin Min Kang
Department of Mathematics
Gyeongsang National University
Chinju 660-701, Korea
E-mail: smkang@mongae.gsu.ac.kr