ON AUTOMORPHISM GROUPS
OF AN \( \epsilon \)-FRAMED MANIFOLD


Abstract. Two examples of \( \epsilon \)-framed manifolds are constructed. It is proved that an \( \epsilon \)-framed structure on a manifold is not unique. Automorphism groups of \( \epsilon \)-framed manifolds are studied. Lastly we prove that a connected Lie group \( G \) admits a left invariant normal \( \epsilon \)-framed structure if and only if the Lie algebra of all left invariant vector fields on \( G \) is an \( \epsilon \)-framed Lie algebra.

1. Introduction

In 1963 K. Yano ([14]) introduced the notion of a \( f \)-structure on a manifold, which is defined by a \((1,1)\) tensor field \( f \) satisfying \( f^3 + f = 0 \). The concept of \( f \)-structure includes the notions of almost complex and almost contact structures and it is well known that it is really a more general structure. For instance, hypersurfaces of almost contact manifolds are not in general almost complex manifolds, but they have always \( f \)-structures associated to them.

Almost product structure is another type of structure widely studied by several authors (see [15], [5]). Analogously to the situation for almost complex and almost contact structures, almost paracontact structures ([16]) are closely related to almost product structures. Moreover, concept of structure defined by a \((1,1)\) tensor field \( f \) satisfying \( f^3 - f = 0 \) ([9], [10]) includes the notions of almost product and almost paracontact structures. The purpose of including in a general notion all the mentioned structures and others (\( \tau \)-contact, \( \tau \)-paracontact, etc.) leads to introduce the notion of \( f(3, \epsilon) \)-structure ([12]) which is defined by a \((1,1)\) tensor field \( f \) satisfying \( f^3 - \epsilon f = 0 \), \( (\epsilon = \pm 1) \). It turns out that

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f is of constant rank and there are two complementary distributions associated with the f(3, ε)-structure, as it happens with f-structures and some other known cases.

In [4], A. Morimoto introduced the notion of isomorphism and automorphism of almost contact structures. He also treated the left invariant normal almost contact structure on a Lie group and showed that the problem can be reduced to a purely algebraic one in Lie algebras.

Thus motivated sufficiently, in the present paper, we study automorphism groups of ε-framed structure manifolds. We begin with giving preliminary definitions and related concepts which we need to introduce the notion of ε-framed structure. This is a general structure which includes almost complex structures, almost product structures, almost contact structures, almost paracontact structures, etc. In section 3, two examples of ε-framed manifolds are constructed. It is proved that an ε-framed structure on a manifold is not unique. Automorphism groups of ε-framed manifolds are studied in section 4. In the last section, we show that the problem to find a left invariant normal ε-framed structure on a group manifold is equivalent to a purely algebraic problem in Lie algebra. In fact, we prove that a connected Lie group G admits a left invariant normal ε-framed structure if and only if the Lie algebra of all left invariant vector fields on G is an ε-framed Lie algebra.

2. Preliminaries

Let M be an n-dimensional differentiable manifold and let there be given a non-null (1, 1) tensor field \( \varphi \) satisfying

\[
\varphi^3 - \varepsilon \varphi = 0.
\]

We call such a structure an \( \varphi (3, \varepsilon) \)-structure ([7]). Following [11], we know that rank of \( \varphi \) is constant. Let \( \text{rank} (\varphi) = k \). If we put

\[
l = \varepsilon \varphi^2, \quad m = I - \varepsilon \varphi^2,
\]

then the tensors \( l \) and \( m \) acting in the tangent space at each point of \( M \) are complementary projection operators which define complementary distributions \( \mathcal{L} \) and \( \mathcal{M} \), respectively. Then the dimension of the distribution \( \mathcal{L} \) is \( k \) and the dimension of the distribution \( \mathcal{M} \) is \( (n - k) \). If \( \varepsilon = -1 \), \( \varphi (3, \varepsilon) \)-structure becomes a f-structure ([14]) and in this case \( \text{rank} (\varphi) \) is even.

Let \( n - k = r \). If \( M \) admits \( r \) linearly independent vector fields \( \xi_1, \ldots, \xi_r \) spanning the distribution \( \mathcal{M} \) at each point of \( M \) and if in
addition, there are $r$ 1-forms $\eta^1, \ldots, \eta^r$ such that

\begin{equation}
\varphi (\xi_\alpha) = 0,
\end{equation}

\begin{equation}
\varphi^2 = \epsilon (I - \eta^\alpha \otimes \xi_\alpha),
\end{equation}

then the structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha)$ is called an $\epsilon$-framed structure on $M$, and the pair $(M, \sum)$ or simply $M$ is called an $\epsilon$-framed manifold.

From the above two equations it follows that

\begin{equation}
\eta^\alpha \circ \varphi = 0,
\end{equation}

\begin{equation}
\eta^\alpha (\xi_\beta) = \delta^\alpha_\beta.
\end{equation}

The $\epsilon$-framed structure is a generalized structure which in special cases reduces to several known structures shown below which have been widely studied in the past.

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$r$</th>
<th>Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$f(2, \epsilon)$-structure ([7])</td>
</tr>
<tr>
<td>$-1$</td>
<td>1</td>
<td>framed structure ([15])</td>
</tr>
<tr>
<td>$-1$</td>
<td>1</td>
<td>almost r-contact structure ([13])</td>
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<tr>
<td>$-1$</td>
<td>1</td>
<td>almost contact structure ([2])</td>
</tr>
<tr>
<td>$-1$</td>
<td>0</td>
<td>almost complex structure ([15])</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>almost r-paracocontact structure ([3])</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>almost paracontact structure ([6])</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>almost product structure ([15])</td>
</tr>
</tbody>
</table>

If $(M, \sum)$ and $(\overline{M}, \overline{\sum}) = (\overline{M}, \varphi, \xi_\alpha, \overline{\eta}^\alpha)$ be two $\epsilon$-framed manifolds, then a $(1, 1)$ tensor field $F$ on the product manifold $M \times \overline{M}$ defined by [8]

\begin{equation}
F (X, \overline{X}) = (\varphi X + \epsilon \eta^\alpha (\overline{X}) \xi_\alpha, \varphi \overline{X} + \eta^\alpha (X) \xi_\alpha),
\end{equation}

where $X$ and $\overline{X}$ are any vector fields on $M$ and $\overline{M}$ respectively, satisfies

\[ F^2 = \epsilon I, \]

which is a $F(2, \epsilon)$-structure on $M \times \overline{M}$ ([7]).

An $\epsilon$-framed manifold $(M, \sum)$ is normal ([8]) if it satisfies

\begin{equation}
\overline{N} = [\varphi, \varphi] - \epsilon d\eta^\alpha \otimes \xi_\alpha = 0.
\end{equation}
3. Non-uniqueness of an \( \epsilon \)-framed structure

First, we construct two examples of \( \epsilon \)-framed manifolds.

**Example 3.1.** Let \( \theta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) be a smooth function. Define \( r \) functions \( \theta^\alpha : \mathbb{R}^{2n+r} \rightarrow \mathbb{R}, \; \alpha \in \{1, \ldots, r\} \), by

\[
\theta^\alpha (x^1, \ldots, x^{2n+r}) \equiv \theta (x^1, \ldots, x^{2n}) + x^{\alpha'}, \quad \alpha' = 2n + \alpha.
\]

We define \( r \) 1-forms \( \eta^\alpha \), \( r \) vector fields \( \xi_\alpha \), and a \((1,1)\) tensor field \( \varphi \) on \( \mathbb{R}^{2n+r} \) as follows:

\[
\eta^\alpha \equiv \text{grad} (\theta^\alpha),
\]

\[
\xi_\alpha \equiv \frac{\partial}{\partial x^{\alpha'}},
\]

\[
\varphi X \equiv \varphi \left( X^a \frac{\partial}{\partial x^a} + X^{a'} \frac{\partial}{\partial x^{a'}} + X^{\alpha'} \frac{\partial}{\partial x^{\alpha'}} \right),
\]

\[
\equiv X^a \frac{\partial}{\partial x^a} + \epsilon X^a \frac{\partial}{\partial x^{a'}} - \left( \theta_a X^{a'} + \epsilon \theta_{a'} X^a \right) \sum_{\alpha'} \frac{\partial}{\partial x^{\alpha'}},
\]

where \( \epsilon = \pm 1 \), \( a \in \{1, \ldots, n\}, \; a' = n + a, \; \theta_i = \frac{\partial \theta}{\partial x^i}, \; i \in \{1, \ldots, 2n\} \), and

\[
P^a Q_{a'} \equiv P_1 Q_{1'} + \cdots + P^n Q_{n'}, \quad P^a Q_{a'} \equiv P_1 Q_1 + \cdots + P^n Q_n.
\]

Then \((\varphi, \xi_\alpha, \eta^\alpha)\) is an \( \epsilon \)-framed structure on \( \mathbb{R}^{2n+r} \).

**Example 3.2.** We construct another example of an \( \epsilon \)-framed structure in the Euclidean space \( \mathbb{R}^6 \). We define \((\varphi, \xi_1, \xi_2, \eta^1, \eta^2)\) in \( \mathbb{R}^6 \) by their matrices as follows:

\[
\varphi = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\epsilon & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \epsilon & 0 & 0 & 0 \\
\end{bmatrix}, \quad \xi_1 = \begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}, \quad \xi_2 = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix},
\]

\[
\eta^1 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \epsilon & 0 & 0 & 0 \\
\end{bmatrix}, \quad \eta^2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \epsilon & 0 & 0 \\
\end{bmatrix}.
\]

The above set provides the required structure on \( \mathbb{R}^6 \).

In view of relations (2)--(5), we are able to state the following theorem.

**Theorem 3.3.** Let \((\varphi, \xi_\alpha, \eta^\alpha)\) and \((\varphi, \xi_\alpha, \tilde{\eta}^\alpha)\) (resp. \((\varphi, \xi_\alpha, \eta^\alpha)\)) be two \( \epsilon \)-framed structures on a manifold \( M \), then we have \( \eta^\alpha = \tilde{\eta}^\alpha \) (resp. \( \xi_\alpha = \tilde{\xi}_\alpha \)).
Thus we see that two $\epsilon$-framed structures having same $\varphi$ and same $\xi_\alpha$ (resp. $\eta^\alpha$) on a manifold are always identical. However, an $\epsilon$-framed structure on a manifold $M$ always induces another $\epsilon$-framed structure on $M$. This is stated in the following theorem.

**Theorem 3.4.** An $\epsilon$-framed structure on a manifold is not unique.

**Proof.** Let $(\varphi, \xi_\alpha, \eta^\alpha)$ be an $\epsilon$-framed structure on a manifold $M$. Let $\psi$ be a non-singular $(1,1)$ tensor field on $M$. Defining

$$\tilde{\varphi} = \psi^{-1} \varphi \psi, \quad \tilde{\xi}_\alpha = \psi^{-1} \xi_\alpha, \quad \tilde{\eta}^\alpha = \eta^\alpha \circ \psi,$$

one can show that $(\tilde{\varphi}, \tilde{\xi}_\alpha, \tilde{\eta}^\alpha)$ is also an $\epsilon$-framed structure on a manifold $M$. \qed

4. Automorphism groups of $\epsilon$-framed manifolds

We begin this section with the definition of the $\epsilon$-framed structure isomorphisms and automorphisms of $\epsilon$-framed manifolds, analogous to that of A. Morimoto ([4]) as follows:

**Definition 4.1.** Let $(M, \Sigma)$ and $(\bar{M}, \bar{\Sigma})$ be two $\epsilon$-framed manifolds. A diffeomorphism $f$ of $M$ onto $\bar{M}$ is called an $\epsilon$-framed structure isomorphism of $M$ onto $\bar{M}$ if the following conditions are satisfied:

$$f_\ast \circ \varphi = \tilde{\varphi} \circ f_\ast,$$

$$f_\ast (\xi_\alpha) = \tilde{\xi}_\alpha,$$

$$f_\ast (\eta^\alpha) = \tilde{\eta}^\alpha,$$

where $f_\ast$ is the differential of $f$. In particular, if $(M, \Sigma) = (\bar{M}, \bar{\Sigma})$, then $f$ is called an $\epsilon$-framed structure automorphism of $M$. The set of all $\epsilon$-framed structure automorphisms of $(M, \Sigma)$ forms a group of transformations of $M$. This group is denoted by $A\Sigma(M)$.

**Lemma 4.2.** A diffeomorphism $f$ of $(M, \Sigma)$ onto $(\bar{M}, \bar{\Sigma})$ is an $\epsilon$-framed structure isomorphism if and only if $f_\ast \circ \varphi = \tilde{\varphi} \circ f_\ast$ and $f_\ast (\xi_\alpha) = \tilde{\xi}_\alpha$ (or $f_\ast (\eta^\alpha) = \tilde{\eta}^\alpha$).

**Lemma 4.3.** A diffeomorphism $f$ of $(M, \Sigma)$ onto itself is an $\epsilon$-framed structure automorphism if and only if $f_\ast \circ \varphi = \varphi \circ f_\ast$ and $f_\ast (\xi_\alpha) = \xi_\alpha$ (or $f_\ast (\eta^\alpha) = \eta^\alpha$).

The proofs of above two Lemmas are straightforward and hence omitted.
DEFINITION 4.4. Let $\widetilde{M}$ be an $F(2, \epsilon)$-structure manifold equipped with an $F(2, \epsilon)$-structure given by $F^2 = \epsilon I$, $\epsilon = \pm 1$. We define an $F(2, \epsilon)$-structure automorphism $f$ of $M$ by a diffeomorphism $f$ of $\widetilde{M}$ onto itself which leaves invariant the $F(2, \epsilon)$-structure of $\widetilde{M}$, that is, $f_\ast \circ F = F \circ f_\ast$. We denote by $A(\widetilde{M})$ the group of all $F(2, \epsilon)$-structure automorphisms of $\widetilde{M}$.

DEFINITION 4.5. Let $D(M)$ denote the group of all diffeomorphisms of a differentiable manifold $M$ onto itself. For any two differentiable manifolds $M$ and $\widetilde{M}$, a homomorphism $H$ of $D(M) \times D(\widetilde{M})$ into $D(M \times \widetilde{M})$ is defined by

$$H(f, g) = f \times g$$

for $f \in D(M)$ and $g \in D(\widetilde{M})$, where

$$(f \times g)(p, q) = (f(p), g(q)), \quad (p, q) \in M \times \widetilde{M}.$$ 

THEOREM 4.6. Let $(M, \Sigma)$ and $(\widetilde{M}, \Sigma)$ be two $\epsilon$-framed manifolds. Then

$$H\left(A_{\Sigma}(M) \times A_{\Sigma}(\widetilde{M})\right) \subset A(M \times \widetilde{M}),$$

where $M \times \widetilde{M}$ is considered as an $F(2, \epsilon)$-structure manifold with the induced $F(2, \epsilon)$-structure by $\Sigma$ and $\Sigma$ defined by equation (6).

PROOF. For $(f, g) \in A_{\Sigma}(M) \times A_{\Sigma}(\widetilde{M})$, we put $H = H(f, g)$. Let $H_\ast, f_\ast, g_\ast$ denote the differentials of $H$, $f$, $g$ respectively. Then for any tangent vectors $X_p \in T_p(M)$ and $\overline{X}_q \in T_q(\widetilde{M})$, we have

$$FH_\ast(X_p, \overline{X}_q)$$

$$= F_{f(p), g(q)}(f_\ast(X_p), g_\ast(\overline{X}_q))$$

$$= \left(\varphi(f_\ast(X_p)) + \epsilon \eta^\alpha(g_\ast(\overline{X}_q))\xi_{\alpha(f(p))}, \xi_{\alpha(g(q))}\right)$$

$$+ \eta^\alpha(f_\ast(X_p))\xi_{\alpha(g(q))}$$

$$= \left(f_\ast \varphi(X_p) + \epsilon g^\alpha(\overline{X}_q)f_\ast(\xi_{\alpha(p)}), g_\ast\varphi(\overline{X}_q) + f^\alpha(\overline{X}_q)g_\ast(\xi_{\alpha(q)})\right)$$

$$= \left(f_\ast(\varphi(X_p) + \epsilon \eta^\alpha(\overline{X}_q)f_\ast(\xi_{\alpha(p)})), g_\ast(\varphi(\overline{X}_q) + \eta^\alpha(\overline{X}_q)f_\ast(\xi_{\alpha(q)}))\right)$$

$$= H_\ast F(X_p, \overline{X}_q).$$

Hence, $FH_\ast = H_\ast F$, which proves that $H \in A(M \times \widetilde{M})$. 

Let $(M, \Sigma)$ and $(\widetilde{M}, \Sigma)$ be two $\epsilon$-framed manifolds, and let $M \times \widetilde{M}$ admit an $F(2, \epsilon)$-structure defined by equation (6). We denote by
\( A(M \times \overline{M}) \) the Lie algebra of all infinitesimal \( F(2, \varepsilon) \)-structure automorphisms of \( M \times \overline{M} \). The homomorphism \( H \) induces a homomorphism of \( \mathcal{X}(M) \times \mathcal{X}(\overline{M}) \) into \( \mathcal{X}(M \times \overline{M}) \), where \( \mathcal{X}(M) \) is the Lie algebra of all vector fields on \( M \), such that

\[
H(X, \overline{X}) = X + \overline{X}
\]

for \( X \in \mathcal{X}(M) \) and \( \overline{X} \in \mathcal{X}(\overline{M}) \).

In the following theorem we determine the inverse image of \( A(M \times \overline{M}) \) by the homomorphism \( H \).

**Theorem 4.7.** Let \( A(M \times \overline{M}) \) be the Lie algebra of all infinitesimal \( F(2, \varepsilon) \)-structure automorphisms of \( M \times \overline{M} \). Then \( X + \overline{X} \in A(M \times \overline{M}) \) if and only if the following equations are satisfied:

\[
\mathcal{L}_X \varphi = 0,
\]

\[
\mathcal{L}_{\overline{X}} \varphi = 0,
\]

and

\[
(\mathcal{L}_X \eta^\alpha) \xi_\beta = (\mathcal{L}_{\overline{X}} \eta^\alpha) \overline{\xi}_\beta,
\]

where \( \mathcal{L} \) is the operator of Lie derivative.

**Proof.** Let \( X + \overline{X} \in A(M \times \overline{M}) \). Since \( X + \overline{X} \) is an infinitesimal transformation, we have \( \mathcal{L}_{(X+\overline{X})} F = 0 \), that is, the Lie bracket satisfies

\[
[X + \overline{X}, F(Y + \overline{Y})] = F[X, Y] + \overline{X}, Y + \overline{Y}
\]

for all \( Y \in \mathcal{X}(M) \) and \( \overline{Y} \in \mathcal{X}(\overline{M}) \). Using equation (6), we get the left hand side of (15) equal to

\[
[X, \varphi Y] + \varepsilon [\eta^\alpha(\overline{Y})] X, \xi_\alpha + X (\eta^\alpha(Y)) \overline{\xi}_\alpha
\]

\[
+ [\overline{X}, \varphi \overline{Y}] + \varepsilon \overline{X} (\eta^\alpha(\overline{Y})) \xi_\alpha + \eta^\alpha(Y) [\overline{X}, \overline{\xi}_\alpha].
\]

Similarly, the right hand side of (15) is equal to

\[
\varphi [X, Y] + \varepsilon [\eta^\alpha([X, \overline{Y}])] \xi_\alpha + \varphi [\overline{X}, \overline{Y}] + \eta^\alpha([X, Y]) \overline{\xi}_\alpha.
\]

Comparing (16) and (17), we get the following four conditions, equivalent to (15):

\[
\varphi [X, Y] = [X, \varphi Y],
\]

\[
\eta^\alpha([X, Y]) \xi_\alpha = \eta^\alpha(Y) [\overline{X}, \overline{\xi}_\alpha] + X (\eta^\alpha(Y)) \overline{\xi}_\alpha,
\]

\[
\varphi [\overline{X}, \overline{Y}] = [\overline{X}, \varphi \overline{Y}],
\]
and
\begin{equation}
\eta^\alpha ([X, Y]) \xi_\alpha = \eta^\alpha (Y) [X, \xi_\alpha] + \left[ \eta^\alpha (Y) \right] \xi_\alpha.
\end{equation}

From (18) and (20), we get (12) and (13), respectively. Putting $Y = \xi_\beta$ in (21), we have
\begin{equation}
\eta^\alpha ([X, \xi_\beta]) \xi_\alpha = [X, \xi_\beta].
\end{equation}

Operating by $\eta^\gamma (\gamma = 1, \ldots, r)$ to above equation, we get (14).

Conversely, let relations (12)–(14) be true. Since (18) and (20) follow from (12) and (13), respectively, putting $Y = \xi_\alpha$ in (18), we have $\varphi [X, \xi_\alpha] = 0$, and hence $\varphi^2 [X, \xi_\alpha] = 0$, that is,
\begin{equation}
[X, \xi_\alpha] = \eta^\beta ([X, \xi_\alpha]) \xi_\beta = \eta^\beta ([X, \xi_\alpha]) \xi_\beta,
\end{equation}

where (3) and (14) are used. Therefore to prove (21), it is sufficient to prove
\begin{equation}
\eta^\alpha ([X, Y]) = \bar{X} (\eta^\alpha (Y)) + \bar{Y} (\eta^\alpha (Y)) \eta^\beta ([X, \xi_\beta]).
\end{equation}

Now, (24) is obvious for $Y = \xi_\alpha$. Taking $Y \in \mathcal{X} (\mathcal{M})$ such that $\eta^\alpha (Y) = 0$, we get $\varphi^2 Y = Y$, and then using (20) and (4) we get
\begin{equation}
\eta^\alpha ([X, Y]) = \bar{Y} (\eta^\alpha (Y)) = \epsilon \eta^\alpha (\varphi [X, \bar{Y}]) = 0.
\end{equation}

Therefore, (24) holds for $Y \in \mathcal{X} (\mathcal{M})$ such that $\eta^\alpha (Y) = 0$. For an arbitrary vector field $Y \in \mathcal{X} (\mathcal{M})$, $Y$ can be written as $Y = Y_1 + \eta^\alpha (Y) \xi_\alpha$, where $Y_1 = Y - \eta^\alpha (Y) \xi_\alpha$ and $\eta^\beta (Y_1) = 0$. This shows that (24) is also true for all $Y \in \mathcal{X} (\mathcal{M})$. The condition (19) is verified in the similar manner. Thus $X + \bar{X} \in \mathcal{A} (\mathcal{M} \times \mathcal{M})$ which proves the theorem.

5. $\epsilon$-framed Lie algebra

Let $G$ be a connected Lie group. For any element $g \in G$, the left translation $L_g$ of $G$ is defined by
\begin{equation}
L_g (x) = gx, \quad x \in G.
\end{equation}

An $\epsilon$-framed structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha)$ on $G$ will be called left invariant if $L_g \in \mathcal{A}_\sum (G)$ for all $g \in G$. In this section we shall show that the problem to find a left invariant normal $\epsilon$-framed structure on a group manifold is reduced to a purely algebraic problem in Lie algebra.

First, we need the following definition.
**Definition 5.1.** Let \( \mathcal{G} \) be a real Lie algebra, \( \varphi \) a linear map of \( \mathcal{G} \) into itself, \( \xi_\alpha^r \) elements of \( \mathcal{G} \) and \( \eta^a \) linear functions on \( \mathcal{G} \). Then \( \mathcal{G} \) is called an \( \epsilon \)-framed Lie algebra if
\[
(26) \quad \varphi^2 = \epsilon (I - \eta^a \otimes \xi_\alpha), \quad \epsilon = \pm 1,
\]
\[
(27) \quad \varphi(\xi_\alpha) = 0,
\]
\[
(28) \quad \epsilon [X, Y] + [\varphi X, \varphi Y] - \varphi [X, \varphi Y] - \varphi [\varphi X, Y] = 0.
\]

From (26) and (27) it follows that
\[
(29) \quad \eta^a \circ \varphi = 0,
\]
\[
(30) \quad \eta^a(\xi_3) = \delta^a_3.
\]

Now, we present the main result.

**Theorem 5.2.** An \( n \)-dimensional connected Lie group \( G \) admits a left invariant normal \( \epsilon \)-framed structure if and only if the Lie algebra \( \mathcal{G} \) of all left invariant vector fields on \( G \) is an \( \epsilon \)-framed Lie algebra.

**Proof.** Let \( G \) admit a left invariant normal \( \epsilon \)-framed structure \( \sum = (\varphi, \xi_\alpha, \eta^a) \). Then for any \( X \in G \), we have \( \varphi X \in G \), because
\[
(31) \quad (L_g)_* \circ \varphi X = \varphi \circ (L_g)_* X = \varphi X
\]
for all \( g \in G \). Hence the restriction \( \tilde{\varphi} \) of \( \varphi \) to \( \mathcal{G} \) maps \( \mathcal{G} \) into itself. Taking \( X \in \mathcal{G} \), since \( \eta^a \) and \( X \) are left invariant, we have \( \eta^a(X) \) is equal to a constant on \( \mathcal{G} \). Hence the restriction \( \tilde{\eta}^a \) of \( \eta^a \) to \( \mathcal{G} \) are linear functions on \( \mathcal{G} \). On the other hand, it is clear that \( \xi_\alpha \in \mathcal{G} \). Hence, by putting \( \xi_\alpha = \xi_\alpha \), the structure \( \sum = (\tilde{\varphi}, \tilde{\xi}_\alpha, \tilde{\eta}^a) \) satisfies (26) and (27). Since \( \eta^a(X) \) is constant for \( X \in \mathcal{G} \), we see that (7) implies (28), which proves that \( \mathcal{G} \) is an \( \epsilon \)-framed Lie algebra.

Conversely, suppose that \( G \) admits an \( \epsilon \)-framed structure \( \sum = (\tilde{\varphi}, \tilde{\xi}_\alpha, \tilde{\eta}^a) \) satisfying (26)–(28). Then for any \( X \in X(G) \) we can find \( n \) functions \( f^i \) on \( G \) such that \( X \) can be written uniquely as
\[
(32) \quad X = f^i X_i.
\]
We now define \( \sum = (\varphi, \xi_\alpha, \eta^a) \) as follows:
\[
(33) \quad \varphi X = f^i (\tilde{\varphi} X_i),
\]
\[
(34) \quad \eta^a(X) = f^i (\tilde{\eta}^a(X_i))
\]
\[
(35) \quad \xi_\alpha = \tilde{\xi}_\alpha.
\]
Then clearly $\sum$ satisfies (2) and (3), hence $\sum$ is an $\varepsilon$-framed structure on $G$. On the other hand, in view of (28) we have

$$\frac{1}{N}(X_i, X_j) = 0, \quad i, j \in \{1, \ldots, r\}.$$ 

Since $\frac{1}{N}$ is a tensor field on $G$, $\frac{1}{N}$ vanishes identically, which proves that $\sum$ is normal. Thus Theorem 5.2 is proved. $\square$

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