$L_p$ SPACES STRUCTURE OF THE 
BANACH ENVELOPE OF $W_{EAK}L^1$

JEONGHEUNG KANG

ABSTRACT. The Banach envelope of $W_{Eak}L_1$ contains a complemented Banach sublattice that is isometrically isomorphic to $L_p(\mu)$ space.

1. Introduction

In this paper we will show that the Banach envelope of $W_{Eak}L_1$ (denoted $wL_1$) contains a complemented Banach sublattice that is isometrically isomorphic to $L_p(\mu)$ space where $\mu$ is a separable probability measure. For $1 < p < \infty$, we can find a lattice isometry $T : L_p(\mu) \to wL_1$ such that the range of $T$ is a complemented subspace of the Banach envelope of $W_{Eak}L_1$. In [7, Theorem 3.7], J. Kupka and T. Peck proved that there exists a lattice isometry $T$ from $L_1$ into $wL_1$ such that the range of $T$ is a complemented subspace of $wL_1$. In [11, Theorem 1], T. Peck and M. Talagrand proved that if $E$ is a separable Banach lattice with order continuous norm, then there is a lattice isometry of $E$ into $wL_1$. Since $L_p$ ($1 < p < \infty$) is also a sublattice of $wL_1$, naturally we can ask that $L_p$ is also a complemented sublattice of $wL_1$. We will give answer for this question. The main result of this paper is the extension of J. Kupka and T. Peck’s theorem 3.9 in [7].

The space $W_{Eak}L_1$, as a Lorentz space $L(1,\infty)$, was introduced in analysis when key operators of harmonic analysis did not map $L_1$ into $L_1$. As examples of such operators, one can give the Hardy-Littlewood maximal function and the Hilbert transform. It became natural to investigate $W_{Eak}L_1$, the space of measurable functions $f$ satisfying $\mu(\{x \in \Omega : |f(x)| > y\}) \leq \frac{c}{y}$, from these important operators in analysis.
It is known that (except for some trivial measure space), $WeakL_1$ is not normable (see [1]). The question therefore arose as to whether any nontrivial continuous linear functionals on $WeakL_1$ exists. In [1, Theorem 6], the answer for this question was observed. This implies $WeakL_1$ has a nontrivial dual space. In [7], J. Kupka and T. Peck studied the structure of $WeakL_1$. They showed that the space $L_\infty$ is dense in the dual of $WeakL_1$ with $weak^*$-topology and showed lattice embeddings of $L_1$, $l_1[0,1]$, $l_\infty$ and $c_0[0,1]$ into $wL_1$ where $wL_1$ is the Banach envelope of $WeakL_1$. Later on, T. Peck and M. Talagrand proved that every separable order continuous Banach lattice is lattice isometric to a sublattice of $wL_1$ in [11, Theorem 1]. Finally, H. Lotz and T. Peck removed the hypothesis of order continuity in the separable case, in [10, Theorem 2].

As a Lorentz space, we’ll study the space $L(1, \infty)$ which is called $WeakL_1$ (denote $wL_1$).

\begin{equation}
(1.1) \quad wL_1 = \{ f \in L_0 : \mu(\{ x \in \Omega : |f(x)| > y \}) < \frac{c}{y} \},
\end{equation}

where $c > 0$ is independent of $y > 0$. As we mentioned, $wL_1$ is not normable, but we can find nontrivial linear functionals on $wL_1$. This was first observed by M. Cwikel and Y. Sagher in [1, Theorem 6].

In [3], if $\mu$ is nonatomic, then we can get an equivalent integral-like seminorm

\begin{equation}
(1.2) \quad \| f \|_{wL_1} = \lim_{n \to \infty} \sup_{\frac{1}{2} \geq \frac{1}{p} \geq n} \frac{1}{\ln \frac{q}{p}} \int_{\frac{1}{p} \leq |f| \leq q} |f| d\mu.
\end{equation}

Later on, in [4] actually the Banach envelope seminorm on $wL_1$ was calculated to be exactly as above. Note that the seminorm on $wL_1$ defined in (1.2) is a lattice seminorm. This is not quite obvious, but using integration by parts one can readily show that the seminorm $\| \cdot \|_{wL_1}$ is exactly same as (see [7, 1.5])

\begin{equation}
(1.3) \quad \lim_{n \to \infty} \sup_{\frac{1}{2} \geq \frac{1}{p} \geq n} \frac{1}{\ln \frac{q}{p}} \int_{\frac{1}{p}}^{q} \mu(\{ x \in \mu : |f(x)| > t \}) dt.
\end{equation}

Even though $wL_1$ is complete with respect to the quasinorm $q(f) = \sup_{a > 0} \mu(\{ x \in \Omega : |f(x)| > a \})$, it is not complete with respect to the seminorm $\| \cdot \|_{wL_1}$. This is due to M. Cwikel and C. Fefferman in [3] and also we can see this in [7, 1.4]. Let $\mathcal{N} = \{ f \in wL_1 : \| f \|_{wL_1} = 0 \}$. Then we obtain the quotient space $wL_1/\mathcal{N}$. We define $wL_1$ as the normed envelope (and its completion as the Banach envelope) of $wL_1$. 

2. \( L_p \) space structure in \( wL_1 \)

To study this subject, we need some basic facts about the dual of \( wL_1 \). We would like to change nonlinear limit superior expression (1.4) for \( \| \cdot \|_{wL_1} \) into a linear expression by directing the number \( I_n^b(f) = \frac{1}{\text{ln} n} \int_{[a \leq |f| \leq b]} |f| \, d\mu \) in some fashion. By [4, Section 1], we can define (1.4) as

\[
(2.1) \quad \|f\|_{wL_1} = \lim_{n \to \infty} (\sup \{ I_n^b(f) : b/a \geq n \}).
\]

For this, we introduce an ultrafilter \( \mathcal{U} \) so that the limit of the \( I_n^b \) along \( \mathcal{U} \) will determine a canonical integral-like linear functional \( I_{\mathcal{U}} \in wL_1^* \).

We now construct an ultrafilter \( \mathcal{U} \) (see [7, Section 2.1]). For \( n = 1, 2, \ldots \), let \( F_n = \{(a, b) : 1 \leq a < b, b/a \geq n \} \) and then define \( \mathcal{F} = \{ F_n : n \geq 1 \} \). Treating \( \mathcal{F} \) as a filter of subsets of the set \( S = [1, \infty) \times [1, \infty) \), we obtain from Zorn's lemma an ultrafilter \( \mathcal{U} \) of subsets of \( S \) such that \( \mathcal{F} \subset \mathcal{U} \). From now, we'll fix the ultrafilter \( \mathcal{F} \subset \mathcal{U} \). The significance of the ultrafilter property lies in the fact that for every function \( f \in wL_1 \), and for every integer \( n \) sufficiently large \( \{ I_n^b(f) : (a, b) \in F_n \} \) is bounded, so that the limit \( l = \lim_{\mathcal{U}} I_n^b(f) \) always exists (for every \( \epsilon > 0 \), there is a set \( \mathcal{U} \in \mathcal{U} \) such that \( |I_n^b(f) - l| < \epsilon \) whenever \( (a, b) \in \mathcal{U} \)).

Define the "ersatz integral" \( I_{\mathcal{U}} \) for every nonnegative function \( f \in wL_1 \) by \( I_{\mathcal{U}}(f) = \lim_{\mathcal{U}} I_n^b(f) \). For more properties of \( I_{\mathcal{U}}(f) \), refer to [7, 2.3 key lemma]. We define for an arbitrary function \( f \in wL_1 \) by \( I_{\mathcal{U}}(f) = I_{\mathcal{U}}(f^+) - I_{\mathcal{U}}(f^-) \). Then we have \( |I_{\mathcal{U}}(f)| \leq \|f\|_{wL_1} \). Define \( \|f\|_{\mathcal{U}} = I_{\mathcal{U}}(|f|) \). Note that (see [7, 2.12])

\[
(2.2) \quad \|f\|_{\mathcal{U}} \leq \|f\|_{wL_1}.
\]

For the dual of \( wL_1 \) (or \( wL_1 \)), we state the theorem which is due to J. Kupka and T. Peck in [7, 2.8].

**Theorem 2.1.** Define a linear operator \( T_{\mathcal{U}} : L_\infty(\mu) \to wL_1^* \) by \( T_{\mathcal{U}}(m)(f) = I_{\mathcal{U}}(mf) \) for all \( m \in L_\infty \), and for all \( f \in wL_1 \). Then \( T_{\mathcal{U}} \) constitutes an isometric order isomorphism of \( L_\infty(\mu) \) into \( wL_1^* \). Moreover, the linear span of the subspaces \( T_{\mathcal{U}}(L_\infty(\mu)) \), as \( \mathcal{U} \) ranges over the collection of ultrafilter (of subset of \( S \)) which contain \( \mathcal{F} \) constitutes a norming, and hence a weak* dense, subspace of \( wL_1^* \).

The operator \( T_{\mathcal{U}} \) of Theorem 2.1 determines an isometric order isomorphic embedding of \( L_\infty(\mu) \) into \( wL_1(\mathcal{U})^* \) where \( wL_1(\mathcal{U}) = wL_1(\mathcal{U})/ \)
$N_{\mathcal{U}}$ and $N_{\mathcal{U}} = \{ f \in wL_1 : \|f\|_{\mathcal{U}} = 0 \}$. Moreover, the range of this embedding is norming, and hence weak* dense in $wL_1(\mathcal{U})^*$.
Let $L(\mathcal{U}) = \{ f \in wL_1 : \|f\|_{wL_1} = \|f\|_{\mathcal{U}} \}$. Then $L(\mathcal{U})$ is a closed subset of $wL_1$ (see [6]) and if $f$ is a $\frac{1}{q}$-like function, then $\|f\|_{wL_1} = \|f\|_{\mathcal{U}} = I_{\mathcal{U}}(f)$.

**Lemma 2.2.** If $\phi \neq 0$ is a linear functional on $wL_1(\mathcal{U})$, then $\phi$ is a linear functional on $wL_1$ with $\|\phi\| \neq 0$.

**Proof.** Let $\phi \neq 0$ be a linear functional on $wL_1(\mathcal{U})$. Then for any $f \in wL_1$ with $\|f\|_{\mathcal{U}} > 0$ (since $f \in wL_1$ is also regarded as $f \in wL_1(\mathcal{U})$).

$$0 < |\phi(f)| \leq \|\phi\||f||_{\mathcal{U}}$$

$$\leq \|\phi\||f||_{wL_1} \quad \text{by (2.2)}.$$

Hence, $\|\phi\| \neq 0$ on $wL_1$. This implies $\phi \neq 0$ is a linear functional on $wL_1$. \qed

We now give a lemma about linear functionals on $wL_1$ which is actually due to J. Kupka and T. Peck (see [7, 2.20]).

**Lemma 2.3.** For a ultrafilter $\mathcal{U}$ defined as above, let $f \in wL_1$ be a nonnegative function with $\|f\|_{\mathcal{U}} = 1$. Then for any $g \in wL_1$, disjointly supported from $f$, we can find a positive $\phi \in wL_1^*$ such that $\|\phi\| = 1$, $\phi(f) = 1$ and $\phi(g) = 0$.

Let $(f_n)_{n=1}^\infty$ be a sequence of nonnegative elements in $wL_1$ with $\|f_n\|_{\mathcal{U}} = 1$ for all $n = 1, 2, 3, \cdots$ and such that the $f_n$ have pairwise disjoint supports. Applying the inductive argument to Lemma 2.3, for each $f_n$, we can find a linear functional $\phi_n$ on $wL_1$ such that $\phi_n(f_n) = 1$, $\|\phi_n\| = 1$ and $\phi_n(f_m) = 0$ if $n \neq m$.

**Lemma 2.4.** Let $(f_n)_{n=1}^\infty$ be a sequence of nonnegative elements in $wL_1$ such that the $f_n$ have pairwise disjoint supports with $\|f_n\|_{\mathcal{U}} = 1$, for all $n = 1, 2, \cdots$ and let $(\phi_n)_{n=1}^\infty$ be a sequence of positive linear functionals on $wL_1$ selected as above. Then for any $f \in wL_1$, we have $\sum_{n=1}^\infty |\phi_n(f)| \leq \|f\|_{wL_1}$.

**Proof.** For an arbitrary element $f \in wL_1$, the number $\phi_n(f)$ is the limit of a subnet of the sequence $\{ I_{\mathcal{U}}(x_{E_{n,k}} \cdot f) \}$ where $(E_{n,k})_{k=1}^\infty$ is a decreasing sequence of subsets of $E_n = \text{supp}(f_n)$, and $f_n$ is bounded on $E_{n,k}$ for all $k$ (see [7, 2.20]). Fix $n \neq m$, let $(E_{n,k})_{k=1}^\infty$ be the decreasing sequence of measurable sets for $f_n$ and $(E_{m,k})_{k=1}^\infty$ the corresponding
sequence for $f_m$. Let $r = sgn I_{U}(\chi_{E_{n,k}} \cdot f)$, $s = sgn I_{U}(\chi_{E_{m,k}} \cdot f)$. Put $m = r\chi_{E_{n,k}} + s\chi_{E_{m,k}}$ so that $\|m\|_{\infty} = 1$. By Theorem 2.1 and Lemma 2.3, we can identify $T_{U}(m) = \hat{m}$ as a linear functional on $wL_{1}$. Then we have

$$
\hat{m}(f) = |I_{U}(\chi_{E_{n,k}} \cdot f)| + |I_{U}(\chi_{E_{m,k}} \cdot f)| \\
= I_{U}(m \cdot f) \\
\leq \|m\|_{\infty}\|f\|_{U} \quad \text{since } \|m\|_{\infty} = 1 \\
= \|f\|_{U} \quad \text{by (2.2)} \\
\leq \|f\|_{wL_{1}}.
$$

By the additive rule for nets [5, Lemma 6, p.28], we can say that in the limit

$$
|\phi_{n}(f)| + |\phi_{m}(f)| \leq \|f\|_{U} \quad \text{by (2.2)} \\
\leq \|f\|_{wL_{1}}.
$$

To show $\sum_{n=1}^{\infty} |\phi_{n}(f)| \leq \|f\|_{wL_{1}}$, it suffices to show that for any $N \in \mathbb{N}$, $\sum_{n=1}^{N} |\phi_{n}(f)| \leq \|f\|_{wL_{1}}$. For $n = 1, 2, \ldots$, let $(E_{n,k})_{k=1}^{\infty}$ be the decreasing sequence of measurable sets for $f_{n}$ and $E_{n} = \text{supp}(f_{n})$. Let $r_{n} = sgn(\chi_{E_{n,k}} \cdot f)$. Put $m = \sum_{n=1}^{N} r_{n}\chi_{E_{n,k}}$. Then we have $\|m\|_{\infty} = 1$. By the same argument as above, one can get

$$
\hat{m}(f) = \sum_{n=1}^{N} |I_{U}(\chi_{E_{n,k}} \cdot f)| \\
= I_{U}(m \cdot f) \\
\leq \|m\|_{\infty}\|f\|_{U} \quad \text{since } \|m\|_{\infty} = 1 \text{ and by (2.2)} \\
\leq \|f\|_{wL_{1}}.
$$

By the additive rule for nets [5, Lemma 6, p.28], we can say that in the limit

$$
\sum_{n=1}^{N} |\phi_{n}(f)| \leq \|f\|_{U} \quad \text{since } \|m\|_{\infty} = 1 \\
\leq \|f\|_{wL_{1}}.
$$

We can therefore say that $\sum_{n=1}^{\infty} |\phi_{n}(f)| \leq \|f\|_{wL_{1}}$. This proves the lemma. \qed
We now need to recall the T. Peck and M. Talagrand’s theorem. In [11, Theorem 1], one can see the following theorem; Let $\Omega$ be a set and $\Omega_{i,n}, n \geq 1, 1 \leq i \leq 2^n$ be a set of $\Omega$ such that $\Omega_{1,0} = \Omega, \Omega_{i,n} \cap \Omega_{j,n} = \emptyset$, if $i \neq j$ and $\Omega_{i,n} = \Omega_{2i-1,n+1} \cup \Omega_{2i,n+1}$. Let $\chi_{i,n}$ be the characteristic function of $\Omega_{i,n}, n > 0, 1 \leq i \leq 2^n$ and let $Y$ be the linear span of the functions $\chi_{i,n}, n > 0, 1 \leq i \leq 2^n$.

**Theorem 2.5** [11, T. Peck and M. Talagrand]. Let $X$ be the completion of $Y$ under some lattice norm on $Y$ where $Y$ is given the usual pointwise order. Then there is a lattice isometry of $X$ into $wL_1$.

T. Peck and M. Talagrand constructed for $n > 0, 1 \leq i \leq 2^n$ under lattice isometry $T, T\chi_{i,n} = f_{i,n}$, where $f_{i,n} = \sum_{m \geq n, 2^m - n} \sum_{j=1}^{2^m - n} e_{2^m - n (i-1) + j, m}$ and each $e_{i,n}(x) = \frac{b_{i,n}}{x - u_{i,n}}, x \in [v_{i,n}, w_{i,n}]$ is a $\frac{1}{x}$-like function. Note that $f_{i,n}$ are all nonnegative and pairwise disjointly supported in $wL_1$ and $f_{i,n} = f_{2i,n+1} + f_{2i+1,n+1},$ for all $n,$ and $1 \leq i \leq 2^n$ (see [11, proof of Theorem 1]).

**Theorem 2.6.** For $1 < p < \infty$, the Banach envelope of $\text{Weak}L_1$ contains a complemented sublattice that is isometrically isomorphic to $L_p(\Omega, \Sigma, \mu)$ where $\mu$ is a separable probability measure.

**Proof.** As an immediate corollary of Theorem 2.5 (see [11, Corollary 2]), we can see that if $E$ is a separable order continuous Banach lattice then there is a lattice isometry of $E$ into $wL_1$. Since for $1 < p < \infty$, $L_p(\mu)$ space is also an order continuous Banach lattice, we can find a lattice isometry $T$ of $L_p$ into $wL_1$ where $\mu$ is a separable probability measure. Also for $1 < p < \infty$, $L_p$ space is a reflexive Banach lattice, $T(L_p)$ is a reflexive sublattice $wL_1$. This implies that the unit ball $B_{TL_p}$ is weakly compact. Since every separable reflexive Banach lattice has an order continuous norm, $L_p$ does not have an order continuous norm. Hence we can apply the construction of $T$ in Theorem 2.5. Let $\chi_{i,n}\xi_{i,n} = 1$ be the subset of $L_p$ defined in Theorem 2.5. Without loss of generality, one can assume $\|\chi_{i,n}\| = 1$ for all $1 \leq i \leq 2^n$. Then we have $\overline{\text{span}}(\chi_{i,n})^{2^n} \subset L_p$.

Define $T\chi_{i,n} = f_{i,n}$, then $\overline{\text{span}}(f_{i,n}) \simeq \overline{\text{span}}(\chi_{i,n})$. Since $\{\chi_{i,n}\}$ form a dense subset of $L_p$, $\{f_{i,n}\}$ form a dense subset of $TL_p$. Moreover, for fixed $n$, the $f_{i,n}$ are pairwise disjointly supported nonnegative elements in $TL_p$ with $\|f_{i,n}\| = 1$. Hence by Lemma 2.3, we can find linear functionals $\phi_{i,n}$ on $wL_1$ such that $\phi_{i,n}(f_{j,n}) = \delta_{i,j}$ and $\|\phi_{i,n}\| = 1$, for all $i = 1, 2, \ldots$. For each $n$, let $B_n = \{f_{i,n}\}^{2^n}_{i=1}$ and define $P_{B_n} : wL_1 \longrightarrow \text{span} B_n$. 

\[ P_{B_n} : wL_1 \longrightarrow \text{span} B_n. \]
$\overline{\text{span}}(f_i, n)_{i=1}^{2^n} \subset TL_p$ by

$$ (2.3) \quad P_{B_n}(f) = \sum_{i=1}^{2^n} \phi_{i,n}(f) f_i,n. $$

Since, for all $f \in wL_1$

$$ \|P_{B_n}(f)\|_{wL_1} = \| \sum_{i=1}^{2^n} \phi_{i,n}(f) f_i,n \|_{wL_1} $$

$$ \leq \sum_{i=1}^{2^n} |\phi_{i,n}(f)| \|f_i,n\|_{wL_1} $$

$$ \leq \sum_{i=1}^{2^n} |\phi_{i,n}(f)| \quad \text{by Lemma 2.4 and } \|f_i,n\|_{wL_1} = 1 $$

$$ \leq \|f\|_{wL_1}. $$

(2.4)

This implies $\|P_{B_n}\| \leq 1$, and $P_{B_n}$ is a well defined linear map. Moreover, $f_{j,n} \in TL_p \subset wL_1$,

$$ P_{B_n}(f_{j,n}) = \sum_{i=1}^{2^n} \phi_{i,n}(f_{j,n}) f_i,n $$

$$ = \phi_{j,n}(f_{j,n}) f_{j,n} $$

$$ = f_{j,n}. $$

(2.5)

Hence $\|P_{B_n}(f_{j,n})\|_{wL_1} = \|f_{j,n}\|_{wL_1} = 1$, and $P_{B_n}^2 = P_{B_n}$. Hence $P_{B_n}$ is a projection $wL_1$ onto $\overline{\text{span}}(f_i, n)_{i=1}^{2^n} \subset TL_p$. From this, we want to find a projection $P$ from $wL_1$ onto $TL_p$. We define a partial order on $\{B_n\}_{n=1}^{\infty}$ by $B_n \prec B_m$ if $\overline{\text{span}}(f_i, n) \subset \overline{\text{span}}(f_i, m)$. Then for each $B_n$, we have $\|P_{B_n}(f)\|_{wL_1} \leq \|f\|_{wL_1}$, for all $f \in wL_1$ by (2.4). Hence the vector $P_{B_n}(f)$ belongs to $\{g \in TL_p : \|g\|_{wL_1} \leq \|f\|_{wL_1}\}$ which is a weakly compact subset in $TL_p$. Now consider the following product space;

$$ (2.6) \quad \prod_{f \in wL_1} \{g \in TL_p : \|g\|_{wL_1} \leq \|f\|_{wL_1}\}. $$

Note that by Tychnoff’s theorem, $\prod_{f \in wL_1} \{g \in TL_p : \|g\|_{wL_1} \leq \|f\|_{wL_1}\}$ is compact for the weak topology. Hence the net $\{P_{B_n}\}$ of projections
from $wL_1$ to $TL_p$ has a subnet which converges to some limit point $P$, in the topology of pointwise convergence on $wL_1$, taking the weak topology on $TL_p$. Let $\{P_{B_{n}}\}$ be a subnet of $\{P_{B_n}\}$ which converges to $P$. Then we have the weak limit $P(f) = \lim_{\alpha} P_{B_{n_{\alpha}}}(f)$, for all $f \in wL_1$. Since each $P_{B_{n}}$ is contractive, positive, and norm one, $P$ is contractive, positive, and norm one.

Finally, we need to show that for all $f \in TL_p$, $P(f) = f$. Since $(f_{i,n})$ are dense, given $\epsilon > 0$ one can find $B_n = \{f_{i,n}\}$ such that $\|\sum_{i=1}^{2^n} \alpha_i f_{i,n} - f\|_{wL_1} < \epsilon/2$ for some $(\alpha_i)_{i=1}^{2^n}$. Let $g = \sum_{i=1}^{2^n} \alpha_i f_{i,n}$. Then

$$\begin{align*}
\|P(f) - f\|_{wL_1} &\leq \|P(f) - P(g)\|_{wL_1} + \|P(g) - g\|_{wL_1} + \|g - f\|_{wL_1} \\
&\leq \|P(f - g)\|_{wL_1} + \|g - f\|_{wL_1}
\end{align*}$$

since $\|P(g) - g\|_{wL_1} = 0$

$$\leq \|f - g\|_{wL_1} + \|g - f\|_{wL_1} < \epsilon.$$

Hence $P(f) = f$ for all $f \in TL_p$. Therefore $P$ is a positive norm one projection from $wL_1$ onto $TL_p$. This proves the theorem. \qed

References

$L_p$ spaces structure of the $WeakL_1$ space

Department of Mathematics  
Korea Military Academy  
Seoul 139-799 Korea  
$E-mail$: jkang@kma.ac.kr