PRIME IDEALS OF SUBRINGS OF MATRIX RINGS

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Abstract. In a ring $R_n(K, J)$ where $K$ is a commutative ring with identity and $J$ is an ideal of $K$, all prime ideals of $R_n(K, J)$ are of the form either $M_n(P)$ or $R_n(P, P \cap J)$. Therefore there is a one to one correspondence between prime ideals of $K$ not containing $J$ and prime ideals of $R_n(K, J)$.

1. Introduction

Let $NT_n(K)$ be the ring of all lower triangular $n \times n$ matrices over an associative ring $K$ with zeros on and above the main diagonal.

It is well known that all ideals (resp. prime ideals) of a matrix ring $M_n(R)$ where $R$ is an associative ring with identity are $M_n(I)$ (resp. $M_n(P)$) where $I$ (resp. $P$) is an ideal (resp. a prime ideal) of $R$. Dubish and Perlis [1, Thm.9] gave a uniform construction of all ideals of the ring $NT_n(K)$ over a field $K$. Levchuk [4, sec.2] similarly constructed ideals of the ring $NT_n(K)$ over a division ring $K$. It is impossible to give a similar description of the ideals of $NT_n(K)$ for the case $K = Z$ (see [4]).

Now we assume that $K$ is a commutative ring with identity and $J$ is an ideal of $K$ and denote the ring $NT_n(K) + M_n(J)$ by $R_n(K, J)$.

Kuzucuoglu and Levchuk [3] described all ideals of $R_n(K, J)$ when $J$ is a strongly maximal ideal of $K$. In this paper we will characterize all prime ideals of $R_n(K, J)$ when $J$ is an ideal (not necessarily strongly maximal ideal) of $K$. The results are followings;

(1) If $P \subsetneq J$ is a $J$-submodule of $K$. Then $P$ is a prime ideal of $K$ if and only if $M_n(P)$ is a prime ideal of $R_n(K, J)$.

(2) If $P$ is $J$-submodule of $K$ such that $P \not\subset J$ and $P \nsubseteq J$. Then $P$ is a prime ideal of $K$ if and only if $R_n(P, P \cap J)$ is a prime ideal of $R_n(K, J)$.

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For the convenience we denote following notations:
1. \( e_{ij} \); matrix unit on \( M_n(K) \).
2. \( \pi_{km} \); the canonical projection on \( M_n(K) \).
   That means \( \pi_{km}(e_{ij}) = 1 \) if \( (k, m) = (i, j) \) and \( \pi_{km}(e_{ij}) = 0 \) if \( (k, m) \neq (i, j) \).
3. \( (k, m) < (i, j) \) means that \( k \leq i, j \leq m \) and \( (k, m) \neq (i, j) \).

2. **Prime ideals of** \( R_n(K, J) \)

   If \( H \) is an ideal of \( R_n(K, J) \), \( \pi_{n1}(H) \) is a \( J \)-submodule (not necessarily an ideal) of \( K \). The following example shows that \( \pi_{n1}(H) \) is \( J \)-submodule but not an ideal of \( K \).

**Example 2.1.** For \( K = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \), \( J = 0 \oplus 0 \oplus 2\mathbb{Z} \) and \( \tilde{T} = \{(n, n, k)e_{21} + (0, 0, 2k)e_{12} \mid n, k \in \mathbb{Z}\} \), let

\[
H = Je_{11} + \tilde{T} + Je_{22} + M_2(J^2).
\]

Then \( H \) is an ideal of \( R_2(K, J) \) and \( H \neq \sum \pi_{ij}(H)e_{ij} \). Also, \( \pi_{21}(H) = \{(n, n, k) \mid n, k \in \mathbb{Z}\} \) is a \( J \)-submodule of \( K \) but not an ideal of \( K \).

But we can get the following lemma.

**Lemma 2.2.** Let \( H \) be a subset of \( R_n(K, J) \). If \( H \) is an ideal of \( R_n(K, J) \) Then the followings hold;
1. \( \pi_{ij}(H) \) is a \( J \)-submodule of \( K \).
2. For all \( (k, m) < (i, j) \), \( K\pi_{km}(H) \subseteq \pi_{ij}(H) \).
3. For all \( s > i \), \( J\pi_{sj}(H) \subseteq \pi_{ij}(H) \).
4. For all \( t < j \), \( J\pi_{it}(H) \subseteq \pi_{ij}(H) \).

Moreover, If \( \sum \pi_{ij}(H)e_{ij} = H \), the converse is true.

**Proof.** (1) is clear.
(2) It is enough to show

\[
K\pi_{i-1,j}(H) \subseteq \pi_{ij}(H) .
\]

for \( i > 1 \) and

\[
K\pi_{i,j+1}(H) \subseteq \pi_{ij}(H)
\]

for \( j < n \).

Since \( H \) is an ideal, \( K e_{i,i-1}H \subseteq H \). So,

\[
K\pi_{i-1,j}(H) = \pi_{ij}(Ke_{i,i-1}H) \subseteq \pi_{ij}(H).
\]
On the other hand, since \( H \) is an ideal, \( HK_+ \subseteq H \). so,

\[
K \pi_{i,j+1}(H) = \pi_{ij}(HK_+) \subseteq \pi_{ij}(H).
\]

(3) For \( s > i \) the fact \( Je_{is} \subseteq R_n(K, J) \) implies

\[
J \pi_{sj}(H) = \pi_{ij}(Je_{is}H) \subseteq \pi_{ij}(H).
\]

(4) For \( t < j \) the fact \( Je_{tj} \subseteq R_n(K, J) \) implies

\[
J \pi_{it}(H) = \pi_{it}(H)J = \pi_{ij}(HJe_{tj}) \subseteq \pi_{ij}(H).
\]

Moreover, by hypothesis and by (1), \( H \) is a subgroup of \( R_n(K, J) \). So,

\[
\begin{align*}
\pi_{ij}(R_n(K, J)H) \\
= \sum_{\lambda=1}^{n} \pi_{i\lambda}(R_n(K, J))\pi_{\lambda j}(H) \\
= \sum_{\alpha=1}^{i-1} \pi_{i\alpha}(R_n(K, J))\pi_{\alpha j}(H) + \sum_{\beta=i}^{n} \pi_{i\beta}(R_n(K, J))\pi_{\beta j}(H) \\
= \sum_{\alpha=1}^{i-1} K\pi_{\alpha j}(H) + \sum_{\beta=i}^{n} J\pi_{\beta j}(H) \subseteq \pi_{ij}(H)
\end{align*}
\]

by (2) and (3).

Thus, \( R_n(K, J)H \subseteq H \). Similarly, by (2) and (4) \( HR_n(K, J) \subseteq H \). Therefore, \( H \) is an ideal of \( R_n(K, J) \). \( \square \)

**Remark.** If \( H \) is an ideal of \( R_n(K, J) \), \( \pi_{n1}(H) \supseteq \pi_{ij}(H) \) by Lemma 2.2 (2). This means that if \( H \) is a nonzero ideal, then \( \pi_{n1}(H) \) is a nonzero \( J \)-submodule of \( K \).

**Proposition 2.3.** If \( H \) is an ideal of \( R_n(K, J) \) such that \( \pi_{n1}(H) \equiv T \). Then \( H \supseteq M_n(TJ^2) \).

**Proof.** Since \( H \) is an ideal, \( \bar{H} \equiv (Je_{1n})H(Je_{1n}) \subseteq H \) and

\[
\pi_{ij}(\bar{H}) = \begin{cases} 
TJ^2, & \text{if } (i, j) = (1, n) \\
0, & \text{if } (i, j) \neq (1, n).
\end{cases}
\]
That is, $\bar{H} = TJ^2e_{1n}$. So, for $s \neq 1$

$$(Ke_{s1})\bar{H} = TJ^2e_{sn},$$

for $t \neq n$

$$\bar{H}(Ke_{nt}) = TJ^2e_{tt},$$

and for $s \neq 1$ and $t \neq n$

$$(Ke_{s1})\bar{H}(Ke_{nt}) = TJ^2e_{st}.$$

Therefore,

$$M_n(TJ^2) = \sum_{s=2}^{n}(Ke_{s1})\bar{H} + \sum_{t=1}^{n-1}\bar{H}(Ke_{nt}) + \sum_{s=2}^{n} \sum_{t=1}^{n-1}(Ke_{s1})\bar{H}(Ke_{nt}) + \bar{H} \subseteq H.$$

If $J$ is a zero ideal of $K$, then $R_n(K, J) = NT_n(K)$ is a nilpotent ring and there is no prime ideals of $NT_n(K)$. So, in this paper we assume $J$ is a nonzero ideal of $K$.

**Theorem 2.4.** $K$ is a prime ring if and only if $R_n(K, J)$ is a prime ring.

**Proof.** ($\Rightarrow$) Let $H_1$ and $H_2$ be nonzero ideals of $R_n(K, J)$. Then $\pi_{n1}(H_1) \equiv T_1 \neq 0$ and $\pi_{n1}(H_2) \equiv T_2 \neq 0$. Then $H_1 \supseteq M_n(T_1J^2) \neq 0$ and $H_2 \supseteq M_n(T_2J^2) \neq 0$ by Proposition 2.3 and primeness of $K$. So, since $T_1T_2J^4 \neq 0$, $H_1H_2 \supseteq M_n(T_1J^2)M_n(T_2J^2) \supseteq M_n(T_1T_2J^4) \neq 0$.

Therefore $R_n(K, J)$ is a prime ring.

($\Leftarrow$) Let $I_1$ and $I_2$ be nonzero ideals of $K$. Then $R_n(I_1, I_1 \cap J)$ and $R_n(I_2, I_2 \cap J)$ are nonzero ideals of $R_n(K, J)$. Since $R_n(K, J)$ is a prime ring, $R_n(I_1, I_1 \cap J)R_n(I_2, I_2 \cap J) \neq 0$. So, $0 \neq \pi_{n1}\{R_n(I_1, I_1 \cap J)R_n(I_2, I_2 \cap J)\} \subseteq I_1I_2$. This implies $I_1I_2 \neq 0$.

Therefore, $K$ is a prime ring. □

**Lemma 2.5.** Let $H$ be an ideal of $R_n(K, J)$ such that $\pi_{n1}(H) \equiv T$.

1. If $T \supseteq J$, then $H$ is not a prime ideal.
2. If $T \subseteq J$ and $H \subsetneq M_n(T)$, then $H$ is not a prime ideal.
3. If $T \subsetneq J$, $T \not\supset J$ and $H \subsetneq R_n(T, T \cap J)$, then $H$ is not a prime ideal.
PROOF. (1) Since \( \{R_n(K, J)\}^n \subseteq M_n(J) \) and by Proposition 2.3,
\[
\{R_n(K, J)\}^{3n} \subseteq \{M_n(J)\}^3 \subseteq M_n(TJ^2) \subseteq H.
\]
So, \( H \) is not a prime ideal.

(2) Suppose \( H \) is a prime ideal and let \( L \equiv M_n(KT) \). Then \( L \) is an ideal of \( R_n(K, J) \) such that \( L \nsubseteq H \). Since \( \{R_n(K, J)\}^n \subseteq M_n(J) \)
\[
L \{R_n(K, J)\}^{2n} \subseteq M_n(KTJ^2) = M_n(TJ^2) \subseteq H
\]
by Proposition 2.3. Since \( L \nsubseteq H \), \( \{R_n(K, J)\}^{2n} \subset H \). So, \( R_n(K, J) \subseteq H \).

This is a contradiction.

(3) Suppose \( H \) is a prime ideal and let \( L \equiv R_n(KT, KT \cap J) \). Then \( L \) is an ideal of \( R_n(K, J) \) such that \( L \nsubseteq H \). Since \( \{R_n(K, J)\}^n \subseteq M_n(J) \)
\[
L \{R_n(K, J)\}^{2n} \subseteq M_n(KTJ^2) = M_n(TJ^2) \subseteq H
\]
by Proposition 2.3. Since \( L \nsubseteq H \), \( \{R_n(K, J)\}^{2n} \subset H \). So, \( R_n(K, J) \subseteq H \).

This is a contradiction. \( \square \)

REMARK. By Lemma 2.5, the prime ideals of \( R_n(K, J) \) are of the form \( M_n(T) \) or \( R_n(T, T \cap J) \) for some \( J \)-submodule \( T \) of \( K \). Next theorem shows that \( T \) is actually an ideal of \( K \).

THEOREM 2.6. If \( M_n(T) \) or \( R_n(T, T \cap J) \) is a prime ideal of \( R_n(K, J) \) where \( T \) is a \( J \)-submodule of \( K \). Then \( T \) is an ideal of \( K \).

PROOF. Suppose \( T \nsubseteq J, T \nsubseteq J \) and \( R_n(T, T \cap J) \) is a prime ideal. Then
\[
R_n(KT, KT \cap J)M_n(J) \subseteq M_n(KTJ) = M_n(TJ) \subseteq R_n(T, T \cap J).
\]
Since \( M_n(J) \nsubseteq R_n(T, T \cap J) \), \( R_n(KT, KT \cap J) \subseteq R_n(T, T \cap J) \). So, \( KT = T \), that is, \( T \) is an ideal of \( K \).
Similarly, we have that \( T \) is an ideal of \( K \) if \( M_n(T) \) is a prime ideal of \( R_n(K, J) \). \( \square \)

THEOREM 2.7. If \( P \subseteq J \) is a \( J \)-submodule of \( K \). Then \( P \) is a prime ideal of \( K \) if and only if \( M_n(P) \) is a prime ideal of \( R_n(K, J) \).
PROOF. By first isomorphism theorem, we can easily show that

\[ R_n(K, J)/M_n(P) \simeq R_n(K/P, J/P). \]

By Theorem 2.4, \( P \) is a prime ideal of \( K \) if and only if \( R_n(K/P, J/P) \) is a prime ring if and only if \( M_n(P) \) is a prime ideal of \( R_n(K, J) \).

\[ \square \]

THEOREM 2.8. If \( P \) is a \( J \)-submodule of \( K \) such that \( P \nsubseteq J \) and \( P \nsubseteq J \). Then \( P \) is a prime ideal of \( K \) if and only if \( R_n(P, P \cap J) \) is a prime ideal of \( R_n(K, J) \).

PROOF. (\( \Rightarrow \)) Suppose \( R_n(P, P \cap J) \) is not prime. Then there exist ideals \( H_1, H_2 (\nsubseteq R_n(P, P \cap J)) \) of \( R_n(K, J) \) such that \( H_1 H_2 \subseteq R_n(P, P \cap J) \). Let \( \pi_n(H_1) = T_1 \) and \( \pi_n(H_2) = T_2 \). We may assume \( T_1 \) and \( T_2 \) are ideals. Thus,

\[ P \supseteq \pi_n(H_1 H_2) \supseteq T_1 J^2 T_2 J^2 = T_1 T_2 J^4. \]

Now \( H_1 \nsubseteq R_n(P, P \cap J) \). So, for some \( i_1 > j_1, \pi_{i_1 j_1}(H_1) \supseteq P \) or for some \( i_2 \leq j_2, \pi_{i_2 j_2}(H_1) \supseteq P \cap J \). In both cases \( \pi_n(H_1) = T_1 \supseteq P \).

Similarly, \( \pi_n(H_2) = T_2 \supseteq P \). That is, \( T_1, T_2, J \nsubseteq P \).

Therefore, \( P \) is not prime.

(\( \Leftarrow \)) Suppose \( P \) is not prime. Then there exist ideals \( A, B (\supseteq P) \) of \( K \) such that \( AB \subseteq P \).

\[ \begin{align*}
(A \cap J)(B \cap J) &\subseteq AB \cap J \subseteq P \cap J, \\
R_n(A, A \cap J)R_n(B, B \cap J) &\subseteq R_n(P, P \cap J).
\end{align*} \]

But \( R_n(A, A \cap J), R_n(B, B \cap J) \) are ideals of \( R_n(K, J) \) such that \( R_n(A, A \cap J), R_n(B, B \cap J) \supseteq R_n(P, P \cap J) \).

Therefore, \( R_n(P, P \cap J) \) is not prime. \[ \square \]

COROLLARY 2.9. There is a one-to-one correspondence between the set of all prime ideals of \( K \) not containing \( J \) and the set of all prime ideals of \( R_n(K, J) \).

COROLLARY 2.10. A prime ideal of \( M_n(K) \) is \( M_n(P) \) where \( P \) is a prime ideal of \( K \).

PROOF. Since \( R_n(K, K) = M_n(K) \), we can easily prove by Theorem 2.7. \[ \square \]
References


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