TROTTER-KATO TYPE APPROXIMATIONS OF CONVOLUTED SOLUTION OPERATOR FAMILIES

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Abstract. Trotter-Kato type approximations of the convoluted solution operator families for the Volterra integral equations (VEₙ):

\[ v(t) = Aₙ \int_0^t v(t-s)dμₙ(s) + fₙ(t), \quad t \geq 0 \]

and the convergence of the solutions to the equations (VEₙ) are studied.

1. Introduction

Approximations of \( C₀ \)- and integrated- semigroups for the abstract Cauchy problems

\[ (ACPₙ) \quad u'(t) = Aₙu(t), \quad t \geq 0 ; \quad u(0) = xₙ \]

and those of the cosine families for the second order Cauchy problems

\[ (CPₙ) \quad u''(t) - Bₙu'(t) - Aₙu(t) = 0, \quad t \geq 0 ; \quad u(0) = xₙ, \quad u'(0) = yₙ \]

for \( n \in \mathbb{N} \) and their applications have been studied in the papers [1], [3], [6], [7], and etc.. Integrated and convoluted solution operator families which are general notion of the integrated and convoluted semigroups are suitable for studying the generalized well-posedness of the Volterra equation (VE) that follows (see [5]). The approximations of the integrated or convoluted solution operator families for the Volterra integral equations

\[ (VEₙ) \quad v(t) = Aₙ \int_0^t v(t-s)dμₙ(s) + fₙ(t), \quad t \geq 0 \]

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for \( n \in \mathbb{N} \) to an integrated or a convoluted solution operator family for the equation

\[
(VE) \quad v(t) = A \int_0^t v(t - s) d\mu(s) + f(t), \quad t \geq 0,
\]

also have been studied in [4] and [5], where it is assumed that \( A \) and \( A_n \), \( n \in \mathbb{N} \) are closed linear operators on a Banach space \( X \), that \( \mu \) and \( \mu_n \), \( n \in \mathbb{N} \) are scalar valued functions of local bounded variation on \([0, \infty)\) which are normalized, i.e., vanish at 0, and that \( f \) and \( f_n \), \( n \in \mathbb{N} \) are \( X \) valued Laplace transformable functions defined on \([0, \infty)\).

The objective of this paper is to improve a Trotter-Kato type approximation result of convoluted solution operator families (Theorem 4.2 in [5]) by showing that the result holds under weaker conditions on the scalar functions \( \mu \) and \( \mu_n \) and additionally, to formulate the approximation results (Theorems 4.1 and 4.2 in [5]) to the case that the spaces where the operators \( A_n \) are defined vary depending on \( n \in \mathbb{N} \). In practical examples however, only the norms might vary on a fixed set depending on \( n \). Since the former result with weaker conditions is immediately deduced from the latter ones with operators on varying spaces, we will prove the latter first. A convergence of the solutions of the equations \( (VE_n) \) to a solution of the equation \( (VE) \) is deduced from a convergence theorem of the functions in \( Lip_{\omega}([0, \infty); X) \) in terms of their Laplace-Stieltjes transforms and a solution characterization for \( (VE) \). Another convergence of the solutions of \( (VE_n) \) to a solution of \( (VE) \) is deduced from an integrated solution operator family result in [4] and the approximation result of convoluted solution operator families. For examples, those in [2] can be referred to.

2. Approximations of convoluted solution operator families and the convergence of the solutions of \((VE_n)\)

We set assumptions on the operator \( A \) and the function \( \mu \) for the equation \( (VE) \) in this section:

(A) Let \( A \) be a closed linear operator with domain \( D(A) \) and range in a Banach space \( X \) and for some constant \( \epsilon \geq 0 \), \( \mu \in BV_{\epsilon}([0, \infty); \mathbb{C}) \), i.e., \( \mu \) is a \( \mathbb{C} \)-valued, normalized function of local bounded variation on \([0, \infty)\) whose variation on \([0, t]\) does not exceed \( M e^{\epsilon t} \) for some constant \( M \geq 0 \) and for every \( t \geq 0 \).
Theorem 1 through Theorem 5 are preliminaries for the new results. See [1], [3], [4] or [5] for Theorems 1 and 2, Definition 3, Theorem 5, notation, and details. By \( \text{Lip}_\omega([0,\infty);X) \) for \( \omega \in \mathbb{R} \) we denote the space consisting of those functions \( F : [0,\infty) \to X \) with \( F(0) = 0 \) and for which \( \|F\|_{\text{Lip}_\omega} \) defined as

\[
\inf\{M \mid \|F(t+h) - F(t)\| \leq M \int_t^{t+h} e^{\omega r} dr \text{ for } t, h \geq 0\}
\]

is finite. It is clear that if \( F \in \text{Lip}_\omega([0,\infty);X) \) for some \( \omega \geq 0 \), the exponential bound \( \omega(F) \) of \( F \) is less than or equal to \( \omega \). If \( f \in L_{1,\text{loc}}^1([0,\infty);X) \) with \( \omega(f) < \infty \), \( f^{[1]} \in \text{Lip}_\omega([0,\infty);X) \) for any number \( \omega > \omega(f) \). By \( C^\infty_\omega((\omega,\infty);X) \) we denote the Widder space consisting of all those functions \( r \in C^\infty((\omega,\infty);X) \) for which

\[
\|r\|_{W,\omega} := \sup_{k \in \mathbb{N}_0, \lambda > \omega} \|(\lambda - \omega)^{k+1} \cdot \frac{1}{k!} r^{(k)}(\lambda)\| < \infty.
\]

It is well-known that the Laplace-Stieltjes transform \( F \mapsto \widehat{dF}(\lambda) := \int_0^\infty e^{-\lambda t} dF(t), \lambda > \omega \) is an isometric isomorphism from \( \text{Lip}_\omega([0,\infty);X) \) onto \( C^\infty_\omega((\omega,\infty);X) \) (see [3], [4], or [5] for example).

**THEOREM 1.** Let \( \{F_n\}_n \) be a sequence of functions in \( \text{Lip}_\omega([0,\infty);X) \) for which there exists a constant \( M \geq 0 \) such that \( \|F_n\|_{\text{Lip}_\omega} \leq M \) for all \( n \in \mathbb{N} \). Then the following are equivalent.

(i) There exist constants \( a > \omega \) and \( b > 0 \) such that \( \lim_{n \to \infty} \widehat{dF_n}(\lambda_k) \) exists for all \( \lambda_k := a + kb, k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N} \).

(ii) There exists an \( F \in \text{Lip}_\omega([0,\infty);X) \) such that \( \|\widehat{dF}\|_{W,\omega} \leq M \) and \( \{\widehat{dF}_n(\cdot)\}_n \) converges to \( \widehat{dF}(\cdot) \) uniformly on compact subsets of \((\omega,\infty)\).

(iii) \( \lim_{n \to \infty} F_n(t) \) exists for every \( t \geq 0 \).

(iv) There exists an \( F \in \text{Lip}_\omega([0,\infty);X) \) with \( \|F\|_{\text{Lip}_\omega} \leq M \) such that \( \{F_n(\cdot)\}_n \) converges to \( F(\cdot) \) uniformly on compact subsets of \([0,\infty)\).

The following is a characterization of solutions of \((VE)\) in [4]. By \( \widehat{f}(\lambda) \) we denote the Laplace transform \( \int_0^\infty e^{-\lambda t} f(t)dt \) of \( f \).

**THEOREM 2.** Suppose that the assumptions in \((A)\) hold. Let \( f \in L_{1,\text{loc}}^1([0,\infty);X) \) be Laplace transformable. Let \( v \in C([0,\infty);X) \) with \( \omega(v) < \infty \) and let \( \omega \) be a number such that \( \omega \geq \max\{\epsilon, \text{abs}(f), \omega(v)\} \). Then the following are equivalent.
(i) \( v \) solves \((VE)\).

(ii) \( \tilde{v}(\lambda) \in D(A) \) and \((I - \hat{d}\mu(\lambda)A)\tilde{v}(\lambda) = \hat{f}(\lambda) \) if \( \lambda \in \mathbb{C}_\omega \).

(iii) \( \tilde{v}(\omega) \in D(A) \) and \((I - \hat{d}\mu(\omega)A)\tilde{v}(\omega) = \hat{f}(\omega) \) if \( \omega < k \in \mathbb{N} \).

The following definition of convoluted solution operator families is taken from [5].

**Definition 3.** Suppose that the assumptions for \( A \) and \( \mu \) in (A) hold. Let \( k \in L^1_{loc}([0, \infty); \mathbb{C}) \) be Laplace transformable. Let \( M > 0 \) and \( \omega \geq \max\{\varepsilon, \text{abs}(k)\} \) be some constants. Suppose that \((I - \hat{d}\mu(\lambda)A)^{-1} \in L(X) \) for all \( \lambda > \omega \). A strongly continuous mapping \( S : [0, \infty) \to L(X) \) is said to be a \( k \)-convoluted solution operator family (\( k \)-c.s.o.f. for short) of exponential type \((M; \omega)\) with generator \((A, \mu)\) if the following hold.

(i) \( ||S(t)|| \leq Me^{\omega t} \) for all \( t \geq 0 \).

(ii) \( \tilde{k}(\lambda)(I - \hat{d}\mu(\lambda)A)^{-1}x = \tilde{S}(\lambda)x = \int_0^\infty e^{-\lambda t} S(t)x \, dt \) for every \( \lambda > \omega \) and every \( x \in X \).

**Remark 4.** (i) If \( S \) is a \( k \)-c.s.o.f. of exponential type \((M; \omega)\) with generator \((A, \mu)\), then

\[
S(t)x = \int_0^t S(t - s)Ax\,ds(s) + k(t)x
\]

for every \( t \geq 0 \) and \( x \in D(A) \). (ii) If \( k(t) = \frac{tm}{m!} \), \( t \geq 0 \) for some \( m \in \mathbb{N}_0 \), a \( k \)-convoluted solution operator family with generator \((A, \mu)\) is an \( m \)-times integrated solution operator family with generator \((m \text{-i.s.o.f. for short}) (A, \mu)\).

One can refer to Corollary 3.1.12 in [4] for the following.

**Theorem 5.** Let \( S \) be an \( m \)-i.s.o.f. with generator \((A, \mu)\) for some \( m \in \mathbb{N}_0 \). Suppose that \( f \in C^{(m+1)}([0, \infty); X) \), i.e., \( f = g^{[m+1]} \) for some \( g \in C([0, \infty); X) \). Then the function \( v \) defined as \( v(t) := \int_0^t S(t - s)g(s)\,ds \), \( t \geq 0 \) is a solution of \((VE)\).

Theorems 1 and 2 imply a convergence of the solutions of \((VE_n)\) to a solution of \((VE)\).

**Theorem 6.** Suppose that the assumptions in (A) hold. Let \( A_n, n \in \mathbb{N} \) be closed linear operators on the Banach space \( X \) and \( \mu_n \in BV(X) \) for every \( n \in \mathbb{N} \). Let \( f, f_n \in L^1_{loc}([0, \infty); X) \) be Laplace transformable functions for which there exists a constant \( a \geq 0 \) such that \( \text{abs}(f), \text{abs}(f_n) \leq a \) for all \( n \in \mathbb{N} \). Let \( v_n \in \text{Lip}_\omega([0, \infty); X) \) be
a solution of \((VE_n)\) for every \(n \in \mathbb{N}\) for which there exist constants \(M > 0\) and \(\omega \geq \max\{\epsilon, a\}\) such that \(\|v_n\|_{Lip_\omega} \leq M\) for all \(n \in \mathbb{N}\). Suppose that \((I - \widehat{d\mu}(\lambda)A)^{-1}\) and \((I - \widehat{d\mu_n}(\lambda)A_n)^{-1}\) exist in \(L(X)\) for all \(n \in \mathbb{N}\) and that \(\lim_{n \to \infty} (I - \widehat{d\mu_n}(\lambda)A_n)^{-1}f_n(\lambda) = (I - \widehat{d\mu}(\lambda)A)^{-1}f(\lambda)\) for every \(\lambda > \omega\). Then \(v_n(t)\) converges to a solution of \((VE)\) uniformly on compact subsets of \([0, \infty)\).

**Proof.** Let \(\lambda > \omega\). Since \(v_n\) is an exponentially bounded solution of \((VE_n)\) with \(\omega(v_n) \leq a \leq \omega\), by Theorem 2, \(\lambda(I - \widehat{d\mu_n}(\lambda)A_n)^{-1}f_n(\lambda) = \lambda \widehat{w}_n(\lambda) = \widehat{d\delta v}(\lambda)\) for every \(n \in \mathbb{N}\). Since \(\|v_n\|_{Lip_\omega} \leq M\) for all \(n \in \mathbb{N}\) and since by the hypothesis, \(\lim_{n \to \infty} \widehat{d\delta v}(\lambda) = \lambda(I - \widehat{d\mu}(\lambda)A)^{-1}\widehat{f}(\lambda)\) for every \(\lambda > \omega\), it follows from Theorem 1 that the solution \(v_n(t)\) of \((VE_n)\) converges to a function \(v(t)\) in \(Lip_\omega([0, \infty); X)\) uniformly on compact subsets of \([0, \infty)\). By Theorem 2 and the uniqueness of a limit, \(\lambda \widehat{\delta v}(\lambda) = \widehat{d\delta v}(\lambda) = \lambda(I - \widehat{d\mu}(\lambda)A)^{-1}\widehat{f}(\lambda)\) for every \(\lambda > \omega\). Thus, \((I - \widehat{d\mu}(\lambda)A)\widehat{\delta v}(\lambda) = \widehat{f}(\lambda)\) for every \(\lambda > \omega\). By Theorem 2, \(v(t)\) is a solution of the equation \((VE)\). \(\square\)

Let \(X\) be a Banach space and let \(M > 0\) and \(\omega \geq 0\) be some constants. A sequence \(\{S_n\}_n\) of operator valued functions \(S_n : [0, \infty) \to L(X)\) is said to be \((M; \omega)\)-stable (or simply stable) if \(\|S_n(t)\| \leq Me^{\omega t}\) for all \(t \geq 0\) and \(n \in \mathbb{N}\). The Trotter-Kato type approximations of convoluted solution operator families, Theorems 4.1 and 4.2 in [5] can be formulated to the case that the spaces on which the operators \(A_n\) are defined vary depending on \(n\). For the results we set assumptions for \((VE)\) and \((VE)_n\):

(B) Let \((X, \|\cdot\|)\) and \((X_n, \|\cdot\|_n)\), \(n \in \mathbb{N}\) be Banach spaces. Let \(A\) and \(A_n\), \(n \in \mathbb{N}\) be closed linear operators on \(X\) and \(X_n\), \(n \in \mathbb{N}\) respectively. For some \(\epsilon \geq 0\), let \(\mu, \mu_n \in BV_{\epsilon}([0, \infty); \mathbb{C})\) for all \(n \in \mathbb{N}\).

In addition we assume that \(f \in L^1_{loc}([0, \infty); X)\) and \(f_n \in L^1_{loc}([0, \infty); X_n)\), \(n \in \mathbb{N}\) are Laplace transformable. We prove the new results analogously to Theorems 4.1 and 4.2 in [5] but under weaker conditions on the scalar functions in Theorem 9 that follows.

**Theorem 7.** Suppose that the assumptions in (B) hold. Let \(k\) which is in \(L^1_{loc}([0, \infty); \mathbb{C})\) be Laplace transformable. Let \(M > 0, L > 0, \) and \(\omega \geq \epsilon\) be some constants. Let \(\{S_n\}_n\) be an \((M; \omega)\)-stable sequence of \(k\)-convoluted solution operator families \(S_n\) with generators \((A_n, \mu_n)\) for \(n \in \mathbb{N}\). Let \(P_n \in L(X; X_n)\) and \(Q_n \in L(X_n; X)\) with \(\|P_n\|, \|Q_n\| \leq L\) for all \(n \in \mathbb{N}\). Suppose that \((I - \widehat{d\mu}(\lambda)A)^{-1}\) exists as an operator in \(L(X)\) for
every $\lambda > \omega$ and that $\lim_{n \to \infty} Q_n(I - \hat{d}\mu_n(\lambda))A_n^{-1}P_n x = (I - \hat{d}\mu(\lambda))A^{-1}x$ for every $\lambda > \omega$ and $x \in X$. Then $(A, \mu)$ generates a $k^{[1]}$-c.s.o.f. $T \in Lip_\omega([0,\infty); L(X))$ with $\|T\|_{Lip_\omega} \leq L^2M$. Moreover, for every $x \in X$, the sequence $\{Q_nS_n^{[1]}(t)P_n x\}$ converges to $T(t)x$ uniformly on compact subsets of $[0,\infty)$. If in addition, $A$ is densely defined and $\mu$ is absolutely continuous on $[0,\infty)$, then there exists a $k$-c.s.o.f. $S$ of exponential type $(M; \omega)$ with generator $(A, \mu)$. In fact, $S(t)x = \frac{dT(t)x}{dt}$ for every $t \geq 0$ and $x \in X$.

**Proof.** Define $T_n(t)x := Q_nS_n^{[1]}(t)P_n x := Q_n \int_0^t S_n(s)P_n x ds$ for every $n \in \mathbb{N}$, $t \geq 0$, and $x \in X$. Then clearly, $T_n \in Lip_\omega([0,\infty); L(X))$ with $\|T_n\|_{Lip_\omega} \leq L^2M$ for all $n \in \mathbb{N}$ and so $\|T_n(\cdot)x\|_{Lip_\omega} \leq L^2M\|x\|$ for all $n \in \mathbb{N}$ and all $x \in X$. Then it follows from $\lim_{n \to \infty} Q_n(I - \hat{d}\mu_n(\lambda))A_n^{-1}P_n x = (I - \hat{d}\mu(\lambda))A^{-1}x$ that $\hat{d}T_n(\lambda)x = Q_nS_n(\lambda)P_n x = \hat{k}(\lambda)Q_n(I - \hat{d}\mu_n(\lambda))A_n^{-1}P_n x$ converges to $\hat{k}(\lambda)(I - \hat{d}\mu(\lambda))A^{-1}x$ for every $\lambda > \omega$ and $x \in X$. Thus, by Theorem 1, it holds that for every $x \in X$, there exists $T_x \in Lip_\omega([0,\infty); X)$ with $\|T_x\|_{Lip_\omega} \leq L^2M\|x\|$ such that $\{T_n(\cdot)x\}$ converges to $T_x(\cdot)$ uniformly on compact subsets of $[0,\infty)$. Define $T(t)x := T_x(t)$ for every $t \geq 0$ and $x \in X$. Then by the uniqueness of a limit, $T(t) : X \to X$ is linear for every $t \geq 0$. Moreover, $T \in Lip_\omega([0,\infty); L(X))$ with $\|T\|_{Lip_\omega} \leq L^2M$. Thus, it follows from Theorem 1 that for every $x \in X$, $\{\hat{d}T_n(\lambda)x\}$ converges to $\hat{d}T(\lambda)x$ uniformly on compact subsets of $(\omega,\infty)$. By the uniqueness of a limit, $\hat{k}^{[1]}(\lambda)(I - \hat{d}\mu(\lambda))A^{-1}x = \frac{\hat{k}(\lambda)}{\hat{\chi}}(I - \hat{d}\mu(\lambda))A^{-1}x = \frac{\hat{d}T(\lambda)}{\hat{\chi}}x = \hat{T}(\lambda)x$ for every $\lambda > \omega$ and $x \in X$. Thus, $T$ is a $k^{[1]}$-c.s.o.f. with generator $(A, \mu)$. Assuming that $A$ is densely defined and $\mu$ is absolutely continuous on $[0,\infty)$, it follows from the second half of the proof of Theorem 3.4 in [5] that $S(t)x := \frac{dT(t)x}{dt}$ exists for all $t \geq 0$ and $x \in X$, that $S(t) \in L(X)$ for every $t \geq 0$, and that $S(t)x$ is continuous on $[0,\infty)$ for every $x \in X$, and finally that $S$ is a $k$-c.s.o.f. with generator $(A, \mu)$. □

If $P_nQ_n$ is the identity operator on $X_n$ and $\lim_{n \to \infty} Q_nP_n x = x$ for all $x \in X$ in Theorem 7, it is deduced under additional assumptions on $\mu$ and $\mu_n$ that for every $x \in X$, the sequence $\{Q_nS_n(t)P_n x\}$ converges to $S(t)x$ uniformly on compact subsets of $[0,\infty)$. We use the following elementary fact for the result.

**Lemma 8.** Let $\{T_n\}_n$ be an $(M; \omega)$-stable sequence of $k$-convoluted solution operator families $T_n : [0,\infty) \to L(X)$ for every $n \in \mathbb{N}$ and for
some $\epsilon \geq 0$, let $\mu_n \in BV_c((0, \infty); \mathbb{C})$ for all $n \in \mathbb{N}$. If $\lim_{n \to \infty} \mu_n(t) = 0$, then $\lim_{n \to \infty} \int_0^t T_n(s)xd\mu_n(s) = 0$ for every $t \geq 0$ and every $x \in X$.

**Proof.** Case 1. Suppose that $\mu_n$ is an increasing function in the space $BV_c([0, \infty); \mathbb{R})$ for every $n \in \mathbb{N}$. Let $t \geq 0$ and $x \in X$. Since $\| \int_0^t T_n(s)xd\mu_n(s)\| \leq \int_0^t \|T_n(s)x\|d\mu_n(s) \leq Me^{\omega t}\|x\|\mu_n(t)$ and since the last term converges to 0 as $n \to \infty$, it holds that $\lim_{n \to \infty} \int_0^t T_n(s)xd\mu_n(s) = 0$.

Case 2. Suppose that $\mu_n \in BV_c([0, \infty); \mathbb{C})$ for every $n \in \mathbb{N}$. Then $\mu_n = \alpha_n - \beta_n$ for some increasing functions $\alpha_n$ and $\beta_n$ in $BV_c([0, \infty); \mathbb{R})$ and so $\int_0^t T_n(s)xd\mu_n(s) = \int_0^t T_n(s)x\alpha_n(s) - \int_0^t T_n(s)x\beta_n(s)$ for every $n \in \mathbb{N}$. Thus, it follows from Case 1 that $\lim_{n \to \infty} \int_0^t T_n(s)xd\mu_n(s) = 0$ for every $t \geq 0$ and $x \in X$.

Case 3. Suppose that $\mu_n \in BV_c([0, \infty); \mathbb{R})$ for every $n \in \mathbb{N}$. Then $\mu_n = \gamma_n + i\delta_n$ for some $\gamma_n, \delta_n \in BV_c([0, \infty); \mathbb{R})$ for every $n \in \mathbb{N}$ implies that $\lim_{n \to \infty} \int_0^t T_n(s)xd\mu_n(s) = 0$ for every $t \geq 0$ and $x \in X$. $\square$

**Theorem 9.** Suppose that the assumptions in (B) hold. Additionally suppose that $A$ and $A_n$, $n \in \mathbb{N}$ are densely defined and that $\mu$ and $\mu_n$, $n \in \mathbb{N}$ are absolutely continuous. Suppose that there exist constants $R, a > 0$ such that $\mu_n' \in Lip_a([0, \infty); \mathbb{C})$ with $\|\mu_n'\|_{Lip_a} \leq R$ for all $n \in \mathbb{N}$ and that the sequence $\{\mu_n(t)\}_n$ converges to $\mu(t)$ uniformly on compact subsets of $[0, \infty)$. Let $k \in L^1_{\text{loc}}([0, \infty); \mathbb{C})$ be a Laplace transformable function which is bounded on compact subsets of $[0, \infty)$. Let $M > 0$ and $\omega \geq \max\{\epsilon, a, \text{abs}(k)\}$ be some constants. Let $\{S_n\}_n$ be an $(M; \omega)$-stable sequence of $k$-convoluted solution operator families $S_n$ with generators $(A_n, \mu_n)$ for $n \in \mathbb{N}$. Suppose that $\hat{\mu} \neq 0$ on $(\omega, \infty)$ and that $\hat{(I - d\mu(\lambda)A)^{-1}}$ exists in $L(X)$ for every $\lambda > \omega$. Let $P_n \in L(X; X_n)$ and $Q_n \in L(X_n; X)$ such that $P_nQ_n = I_n$, the identity operator on $X_n$ and $\lim_{n \to \infty} Q_nP_nx = x$ for all $x \in X$. Suppose that there exists a constant $L \geq 0$ such that $\|P_n\|, \|Q_n\| \leq L$ for all $n \in \mathbb{N}$ and that $\lim_{n \to \infty} Q_n(I - d\mu(\lambda)A_n)^{-1}P_nx = (I - d\mu(\lambda)A)^{-1}x$ for every $\lambda > \omega$ and $x \in X$. Then there exists a $k$-c.s.o.f. $S = \{S(t)\}_{t \geq 0}$ of exponential type $(L^2M; \omega)$ with generator $(A, \mu)$ such that for every $x \in X$, the sequence $\{Q_nS_n(t)P_nx\}_n$ converges to $S(t)x$ uniformly on compact subsets of $[0, \infty)$.

**Proof.** Since the hypotheses include the assumptions for Theorem 7, $(A, \mu)$ generates a $k$-c.s.o.f. $S = \{S(t)\}_{t \geq 0}$ of exponential type
For the uniform convergence of \( \{S_n(t)\}_n \) to \( S(t) \) on compact subsets of \([0, \infty)\), we first show that for every \( y \in D(A) \), the sequence \( \{Q_nS_n(t)P_ny\}_n \) converges to \( S(t)y \) uniformly on compact subsets of \([0, \infty)\). Let \( y \in D(A) \) and \( y_n \in D(A_n) \) for \( n \in \mathbb{N} \). Since \( S \) and \( S_n \), \( n \in \mathbb{N} \) are \( k \)-convoluted solution operator families with generators \((A, \mu)\) and \((A_n, \mu_n)\), respectively and \( Q_n \) are bounded linear operators, it follows from Remark 4 following Definition 3 that

\[
(1) \quad S(t)y = \int_0^t S(t-s)Ay \mu(s) + k(t)y
\]

and

\[
(2) \quad Q_nS_n(t)y_n = \int_0^t Q_nS_n(t-s)A_ny_n \mu(s) + k(t)Q_ny_n
\]

hold for every \( t \geq 0 \). Let \( h(\lambda) := (I - \widetilde{d \mu}(\lambda)A)^{-1} \) and let \( h_n(\lambda) := (I - \widetilde{d \mu_n}(\lambda)A_n)^{-1} \) for \( \lambda > \omega \) and \( n \in \mathbb{N} \). Then from the hypothesis,

\[
\lim_{n \to \infty} Q_nh_n(\lambda)P_n x = h(\lambda)x \text{ for every } \lambda > \omega \text{ and } x \in X.
\]

Let \( \lambda_0 > \omega \) such that \( \widetilde{d \mu}(\lambda_0) \neq 0 \) and let \( z := (I - \widetilde{d \mu}(\lambda_0)A)y \). Then \( y = h(\lambda_0)z \).

\[
||Q_nS_n(t)P_n y - S(t)y|| \leq ||Q_nS_n(t)P_n \left(h(\lambda_0)z - Q_nh_n(\lambda_0)P_n z\right)||
\]

\[
+ ||Q_nS_n(t)h_n(\lambda_0)P_n z - S(t)h(\lambda_0)z||.
\]

Since \( Q_nS_n(t)P_n \) are uniformly bounded on compact subsets of \([0, \infty)\) and since \( \lim_{n \to \infty} Q_nh_n(\lambda_0)P_n z = h(\lambda_0) \), it suffices to estimate the convergence of the second term in (3). It is deduced from the condition \( \mu_n \in Lip_\alpha([0, \infty); \mathbb{C}) \) with \( |\mu_n|_{Lip_\alpha} \leq R \) for all \( n \in \mathbb{N} \) that \( \mu_n \in Lip_\alpha([0, \infty); \mathbb{C}) \) and \( ||\mu_n||_{Lip_\alpha} \leq R \) for all \( n \in \mathbb{N} \) (see [4] or [5] for example). Since \( \lim_{n \to \infty} \mu_n(t) = \mu(t) \) for every \( t \geq 0 \) with \( ||\mu_n||_{Lip_\alpha} \leq R \) for all \( n \in \mathbb{N} \), by Theorem 1, \( \lim_{n \to \infty} \widetilde{d \mu_n}(\lambda_0) = \widetilde{d \mu}(\lambda_0) \). Since \( \lim_{n \to \infty} \widetilde{d \mu_n}(\lambda_0) = \widetilde{d \mu}(\lambda_0) \neq 0 \), to estimate the convergence of the second term

\[
||Q_nS_n(t)h_n(\lambda_0)P_n z - S(t)h(\lambda_0)z||
\]

in (3) is equivalent to estimate that of the sequence

\[
\{||\widetilde{d \mu}(\lambda_0)\widetilde{d \mu_n}(\lambda_0)(Q_nS_n(t)h_n(\lambda_0)P_n z - S(t)h(\lambda_0)z)||\}_n.
\]
By (1) and (2),

\[
\| \widehat{d\mu}(\lambda_0) \widehat{d\mu_n}(\lambda_0) \left( Q_n S_n(t) h_n(\lambda_0) P_n z - S(t) h(\lambda_0) z \right) \| \\
\leq \| \widehat{d\mu}(\lambda_0) \widehat{d\mu_n}(\lambda_0) \left( \int_0^t Q_n S_n(t-s) A_n h_n(\lambda_0) P_n z \, d\mu_n(s) \\
- \int_0^t S(t-s) A h(\lambda_0) z \, d\mu(s) \right) \| \\
+ \| \widehat{d\mu}(\lambda_0) \widehat{d\mu_n}(\lambda_0) k(t) \left( Q_n h_n(\lambda_0) P_n z - h(\lambda_0) z \right) \|.
\]

(4)

Since the second term converges to 0 uniformly on compact subsets of \([0, \infty)\), it suffices to estimate the convergence of the first term in (4).

\[
\| \widehat{d\mu}(\lambda_0) \widehat{d\mu_n}(\lambda_0) \left( \int_0^t Q_n S_n(t-s) A_n h_n(\lambda_0) P_n z d\mu_n(s) \\
- \int_0^t S(t-s) A h(\lambda_0) z d\mu(s) \right) \| \\
= \| \widehat{d\mu}(\lambda_0) \int_0^t Q_n S_n(t-s)(h_n(\lambda_0) - I) P_n z d\mu_n(s) \\
- \widehat{d\mu_n}(\lambda_0) \int_0^t S(t-s)(h(\lambda_0) - I) z d\mu(s) \| \\
\leq \| \left( \widehat{d\mu}(\lambda_0) - \widehat{d\mu_n}(\lambda_0) \right) \int_0^t Q_n S_n(t-s)(h_n(\lambda_0) - I) P_n z d\mu_n(s) \| \\
+ |\widehat{d\mu_n}(\lambda_0)| \left\| \int_0^t Q_n S_n(t-s)(h_n(\lambda_0) - I) P_n z d\mu_n(s) \\
- \int_0^t S(t-s)(h(\lambda_0) - I) z d\mu(s) \right\|.
\]

(5)

Since \( \| \int_0^t Q_n S_n(t-s)(h_n(\lambda_0) - I) P_n z \, d\mu_n(s) \| \) are uniformly bounded on compact subsets of \([0, \infty)\) and since \( \lim_{n \to \infty} \widehat{d\mu_n}(\lambda_0) = \widehat{d\mu}(\lambda_0) \) in the first term of (5), it suffices to estimate the convergence of the term

\[
\| \int_0^t Q_n S_n(t-s)(h_n(\lambda_0) - I) P_n z d\mu_n(s) - \int_0^t S(t-s)(h(\lambda_0) - I) z d\mu(s) \|.
\]
in the second term of (5).

\[
\| \int_0^t Q_n S_n(t - s)(h_n(\lambda_0) - I)P_n z d\mu_n(s) \hspace{1cm} - \int_0^t S(t - s)(h(\lambda_0) - I)z d\mu(s) \| \\
= \| \int_0^t Q_n S_n(s)(h_n(\lambda_0) - I)P_n z d\mu_n(t - s) \hspace{1cm} - \int_0^t S(s)(h(\lambda_0) - I)z d\mu(t - s) \| \\
= \| \int_0^t Q_n S_n(s)P_n (Q_n h_n(\lambda_0)P_n z - Q_n P_n z) d\mu_n(t - s) \hspace{1cm} - \int_0^t S(s)(h(\lambda_0) - I)z d\mu(t - s) \| \\
\leq \| \int_0^t Q_n S_n(s)P_n (Q_n h_n(\lambda_0)P_n z - h(\lambda_0)z) d\mu_n(t - s) \| \\
\hspace{1cm} + \| \int_0^t (Q_n S_n(s)P_n - S(s))(h(\lambda_0) - Q_n P_n)z d\mu_n(t - s) \| \\
\hspace{1cm} + \| \int_0^t S(s)(I - Q_n P_n)z d\mu_n(t - s) \| \\
\hspace{1cm} + \| \int_0^t S(s)(h(\lambda_0) - I)z d\mu_n(t - s) \| \\
= \| \int_0^t Q_n S_n(s)P_n (Q_n h_n(\lambda_0)P_n z - h(\lambda_0)z) \mu_n'(t - s) ds \|
\hspace{1cm} + \| \int_0^t (Q_n S_n(s)P_n - S(s))(h(\lambda_0) - Q_n P_n)z \mu_n'(t - s) ds \| \\
\hspace{1cm} + \| \int_0^t S(s)(z - Q_n P_n z) \mu_n'(t - s) ds \| \\
\hspace{1cm} + \| \int_0^t S(t - s)(h(\lambda_0) - I)z d\left(\mu_n(s) - \mu(s)\right) \|.
\]

Since \(\|Q_n S_n(s)P_n\|\ |\mu_n'(t - s)|\) are uniformly bounded on compact subsets of \([0, \infty)\) and since \(\lim_{n \to \infty} Q_n h_n(\lambda_0)P_n z = h(\lambda_0)z\), the first term in (6) converges to 0 uniformly on compact subsets of \([0, \infty)\). Since \(\|S(s)\|\ |\mu_n'(t - s)|\) are uniformly bounded on compact subsets of \([0, \infty)\)
and \( \lim_{n \to \infty} Q_n P_n z = z \), the third term in (6) converges to 0 uniformly on compact subsets of \([0, \infty)\). Lemma 8 implies that the fourth term in (6) converges uniformly on compact subsets of \([0, \infty)\). Thus, it suffices to estimate the second term in (6). By the integration by parts,

\[
\left\| \int_0^t \left( Q_n S_n(s) P_n - S(s) \right) (h(\lambda_0) - Q_n P_n) z \mu_n'(t-s) ds \right\|
\leq \left\| \int_0^t \left( Q_n S_n^{[1]}(s) P_n - S^{[1]}(s) \right) (h(\lambda_0) z - Q_n P_n z) \mu''_n(t-s) ds \right\|
\leq \sup_{n \in \mathbb{N}} \| h(\lambda_0) z - Q_n P_n z \| \cdot \sup_{n \in \mathbb{N}} \sup_{s \in [0, t]} | \mu''_n(s) | \cdot \int_0^t \| Q_n S_n^{[1]}(s) P_n - S^{[1]}(s) \| ds.
\]

Since \( \sup_{n \in \mathbb{N}} | \mu'_n |_{Lip_a} \leq R \), \( \sup_{n \in \mathbb{N}} \sup_{s \in [0, t]} | \mu''_n(s) | < \infty \) for every \( t \geq 0 \).

Since \( \lim_{n \to \infty} Q_n P_n x = x \) for all \( x \in X \), \( \sup_{n \in \mathbb{N}} \| h(\lambda_0) z - Q_n P_n z \| \) is finite.

Since for every \( x \in X \), \( \{ Q_n S_n^{[1]}(s) P_n x \}_n \) converges to \( S^{[1]}(s)x \) uniformly on compact subsets of \([0, \infty)\), (7) converges to 0 uniformly on compact subsets of \([0, \infty)\). Thus, \( Q_n S_n(t) P_n y \) converges to \( S(t)y \) uniformly on compact subsets of \([0, \infty)\) for every \( y \in D(A) \). Since \( \overline{D(A)} = X \) and since \( Q_n S_n(t) P_n y \) are uniformly bounded on compact subsets of \([0, \infty)\), \( Q_n S_n(t) P_n x \) converges to \( S(t)x \) uniformly on compact subsets of \([0, \infty)\) for every \( x \in X \).

If \( X_n = X \) and \( Q_n = P_n = I \), the identity operator on \( X \) for all \( n \in \mathbb{N} \) Theorem 7 becomes Theorem 4.1 in [5] and Theorem 9 does almost Theorem 4.2 in [5] but with weaker conditions on the scalar functions \( \mu_n \) and \( \mu \) as follows.

**Corollary 10.** (a) Let \( A \) and \( A_n \), \( n \in \mathbb{N} \) be densely defined closed linear operators on a Banach space \( X \) and let \( \mu \) and \( \mu_n \), \( n \in \mathbb{N} \) be absolutely continuous functions in \( BV_e([0, \infty); \mathbb{C}) \) for some \( \epsilon \geq 0 \) for all \( n \in \mathbb{N} \). Additionally suppose that \( D(A) \cap \bigcap_{n \in \mathbb{N}} D(A_n) \) contains a dense subset \( D \) of \( X \), that \( \mu_n(t) \) converges to \( \mu(t) \) uniformly on compact subsets of \([0, \infty), \) and that there exist constants \( L, a > 0 \) such that \( \mu'_n \in Lip_a([0, \infty); \mathbb{C}) \) and \( | \mu'_n |_{Lip_a} \leq L \) for all \( n \in \mathbb{N} \).

(b) Let \( k \in L^1_{loc}([0, \infty); \mathbb{C}) \) be a Laplace transformable function which is bounded on compact subsets of \([0, \infty)\). Let \( \omega \geq \max\{ \epsilon, a, \text{abs}(k) \} \)
and \( M > 0 \) be some constants. Let \( \{S_n\}_n \) be an \((M; \omega)\)-stable sequence of \( k\)-convoluted solution operator families \( S_n \) with generators \((A_n, \mu_n)\) for \( n \in \mathbb{N} \).

(c) Suppose that \( \hat{d} \mu \neq 0 \) on \((\omega, \infty)\) and that \((I - \hat{d} \mu(\lambda)A)^{-1} \) exists in \( L(X) \) for every \( \lambda > \omega \) and \( \lim_{n \to \infty} (I - \hat{d} \mu_n(\lambda)A_n)^{-1}x = (I - \hat{d} \mu(\lambda)A)^{-1}x \) for every \( \lambda > \omega \) and \( x \in X \).

Then there exists a \( k\)-c.s.o.f. \( S \) of exponential type \((M; \omega)\) with generator \((A, \mu)\) for which for every \( x \in X \), the sequence \( \{S_n(\cdot)x\}_n \) converges to \( S(\cdot)x \) uniformly on compact subsets of \([0, \infty)\).

Suppose that for some \( x \in X \), \( f(t) = f_n(t) = \frac{t^{m+1}}{(m+1)!}x \) for all \( t \geq 0 \) and \( n \in \mathbb{N} \) in \((VE_n)\) and \((VE)\) so that the equations \((VE_n)\) and \((VE)\) become

\[
(VE_n') \quad v(t) = A_n \int_0^t v(t-s)d\mu_n(s) + \frac{t^{m+1}}{(m+1)!}x, \quad t \geq 0
\]

\[
(VE') \quad v(t) = A \int_0^t v(t-s)d\mu(s) + \frac{t^{m+1}}{(m+1)!}x, \quad t \geq 0,
\]

respectively. If \( k(t) = \frac{t^m}{m!} \) for some \( m \in \mathbb{N}_0 \) in Corollary 10 so that \((A_n, \mu_n)\), \( n \in \mathbb{N} \) generate \( m \)-times integrated solution operator families, a solution of the equation \((VE')\) is obtained as a limit of the solutions of the equations \((VE_n')\) as follows.

Assume \((a')\) through \((c')\):

(a') the assumption \((a)\) in Corollary 10.

(b') Let \( m \in \mathbb{N}_0 \). For some constants \( M > 0 \) and \( \omega \geq \max\{\epsilon, a\} \), suppose that \( \{S_n\}_n \) is an \((M; \omega)\)-stable sequence of \( m \)-times integrated solution operator families \( S_n \) with generators \((A_n, \mu_n)\) for \( n \in \mathbb{N} \).

(c') the assumption \((c)\) in Corollary 10.

Then by Corollary 10, \((A, \mu)\) generates an \( m \)-i.s.o.f. \( S \) of exponential type \((M; \omega)\) for which for every \( x \in X \), \( \{S_n(\cdot)x\}_n \) converges to \( S(\cdot)x \) uniformly on compact subsets of \([0, \infty)\). Thus, \( v_n(t) := \int_0^t S_n(s)x\,ds \) converges to \( \int_0^t S(s)x\,ds \) uniformly on compact subsets of \([0, \infty)\). Note that by Theorem 5, \( v(t) := \int_0^t S(s)x\,ds \) is the unique solution of \((VE')\).

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