ON THE AXIOM OF CHOICE OF WEAK TOPOS $\mathcal{F}_{uz}$

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Abstract. Topos is a set-like category. In topos, the axiom of choice can be expressed as (AC1), (AC2) and (AC3). Category $\mathcal{F}_{uz}$ of fuzzy sets has a similar function to the topos $\text{Set}$ and it forms weak topos. But $\mathcal{F}_{uz}$ does not satisfy (AC1), (AC2) and (AC3). So we define (WAC1), (WAC2) and (WAC3) in weak topos $\mathcal{F}_{uz}$. And we show that they are equivalent in $\mathcal{F}_{uz}$.

1. Introduction

In a topos, the axiom of choice can be expressed as following.

(AC1) Every epimorphism is a retraction.

(AC2) For any noninitial object $A$ and $f : A \to B$, there exists a morphism $g : B \to A$ such that $f \circ g \circ f = f$.

(AC3) For any noninitial object $A$, there exists $\sigma : \Omega^A \to A$ such that for all $f : 1 \to \Omega^A$, we have $\sigma \circ f \in f'$ where $f' : A' \to A$ is a monomorphism, provided that $ev \circ (f \times i_A)$ is not the characteristic morphism of $0 \to A$.

They are not necessarily so related ([1], [4]). But in the topos $\text{Set}$ they are equivalent.

Category $\mathcal{F}_{uz}$ of fuzzy sets has a similar function to the topos $\text{Set}$. $\mathcal{F}_{uz}$ has finite products, middle object, equalizers, exponentials and weak subobject classifier. So $\mathcal{F}_{uz}$ forms a weak topos ([5], [6]). But $\mathcal{F}_{uz}$ does not satisfy (AC1), (AC2) and (AC3).

In this paper, we define (WAC1), (WAC2), (WAC3) as following.

(WAC1) Every epimorphism $f : (A, \alpha_A) \to (B, \alpha_B)$, where $\alpha_B(b) = \text{Max}\{\alpha_A[A'] | b = f[A'], A' \subseteq A\}$ and there exists an element $a' \in A$ such that $\alpha_A(a') \geq \alpha_B(b)$ for all $b \in B$, is a retraction.

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(WAC2) For any noninitial object \( A \) and \( f : A \to B \),
where \( \alpha_B(b) = \text{Max}\{\alpha_A(f[A']) | b = f[A'], A' \subseteq A\} \) and there exists an element \( a' \in A \) such that \( \alpha_A(a') \geq \alpha_B(b) \) for all \( b \in B \), there exists a morphism \( g : B \to A \) such that \( f \circ g \circ f = f \).

(WAC3) For any noninitial object \( A \), there exists \( \sigma : \Omega^A \to A \) such that for all \( f : A \to \Omega \), we have \( \alpha_A(\sigma(f)) = 1 \) where \( \alpha_A : A \to I \).

And we show that (WAC1), (WAC2), (WAC3) are equivalent in weak topos \( \mathcal{F}uz \).

2. Preliminaries

In this section, we state some definitions and properties which will serve as the basic tools for the arguments used to prove our results.

**Definition 2.1.** An elementary topos is a category \( \mathcal{E} \) such that

(T1) \( \mathcal{E} \) is finitely complete,

(T2) \( \mathcal{E} \) has exponentiation,

(T3) \( \mathcal{E} \) has a subobject classifier.

(T2) means that for every object \( A \) in \( \mathcal{E} \), endofunctor \( (-) \times A \) has its right adjoint \( (-)^A \). Hence for every object \( A \) in \( \mathcal{E} \), there exists an object \( B^A \), and a morphism \( ev_A : B^A \times A \to B \), called the evaluation map of \( A \), such that for any \( Y \) and \( f : Y \times A \to B \) in \( \mathcal{E} \), there exists a unique morphism \( g \) such that \( ev_A \circ (g \times i_A) = f \);

\[
\begin{array}{ccc}
Y \times A & \xrightarrow{f} & B \\
\downarrow_{g \times i_A} & & \downarrow_{i_B} \\
B^A \times A & \xrightarrow{ev_A} & B
\end{array}
\]

And subobject classifier in (T3) is an \( \mathcal{E} \)-object \( \Omega \), together with a morphism \( \top : 1 \to \Omega \) such that for any monomorphism \( h : D \to C \), there is unique morphism \( \chi_h : C \to \Omega \), called the character of \( h : D \to C \) that makes the following diagram a pull-back;

\[
\begin{array}{ccc}
D & \xrightarrow{1} & 1 \\
\downarrow_{h} & & \downarrow_{\top} \\
C & \xrightarrow{\chi_h} & \Omega
\end{array}
\]
Example 2.2. Category $\mathbf{Set}$ is a topos. The terminal object is the one-element sets $\{\ast\}$. The subobject classifier is $\top : \{\ast\} \to \Omega$ with $\Omega = \{0, 1\}$ defined by $\top(\ast) = 1$. If we define

$$
\begin{cases}
\chi_h = 1 & \text{if } c = h(d) \text{ for some } d \in D, \\
\chi_h = 0 & \text{otherwise},
\end{cases}
$$

then $\chi_h$ is a characteristic function of $D$.

Category $\mathbf{Fuz}$ of fuzzy sets is a category whose object is $(A, \alpha_A)$ where $A$ is an $\mathbf{Set}$-object and $\alpha_A : A \to I$ is a $\mathbf{Set}$-morphism with $I = (0, 1]$ and morphism from $(A, \alpha_A)$ to $(B, \alpha_B)$ is a $\mathbf{Set}$-morphism $f : A \to B$ such that $\alpha_A(a) \leq \alpha_B \circ f(a)$.

Definition 2.3. We say that an object $(I, \alpha_I)$ is a middle object of $\mathbf{Fuz}$ if there exists a unique morphism $f : A \to I$ such that $\alpha_A(a) = \alpha_I \circ f(a)$ for all $(A, \alpha_A)$ and $a \in A$.

Definition 2.4. We say that an object $(J, \alpha_J)$ is a weak subobject classifier of $\mathbf{Fuz}$ if there exists a unique morphism $\alpha_f : (A, \alpha_A) \to (J, \alpha_J)$ for all monomorphism $f : (B, \alpha_B) \to (A, \alpha_A)$ where $J = [0, 1]$ and $\alpha_J(j) = 1$ for all $j \in J$ such that $\alpha_f(a) \leq \alpha_A(a)$ and the following diagram is a pull-back.

$$
\begin{array}{ccc}
(B, \alpha_B) & \xrightarrow{\alpha_B} & (I, \alpha_I) \\
| f \downarrow & & \downarrow i \\
(A, \alpha_A) & \xrightarrow{\alpha_f} & (J, \alpha_J)
\end{array}
$$

Definition 2.5. A weak topos is a category $\mathcal{W}$ such that

- (WT1) $\mathcal{W}$ has equalizer, finite product and exponentiation,
- (WT2) $\mathcal{W}$ has a middle object,
- (WT3) $\mathcal{W}$ has a weak subobject classifier.

3. Main parts

Theorem 3.1. In a weak topos $\mathbf{Fuz}$ the following statements are equivalent:

(WAC1) Every epimorphism $f : (A, \alpha_A) \to (B, \alpha_B)$, where $\alpha_B(b) = \max\{\alpha_A[f(A')] \mid b = f(A'), A' \subseteq A\}$ and there exists an element $a' \in A$ such that $\alpha_A(a') \geq \alpha_B(b)$ for all $b \in B$, is a retraction.
(WAC2) For any noninitial object \( A \) and \( f : A \to B \), where \( \alpha_B(b) = \max \{ \alpha_A[f[A']] \mid b = f[A'], A' \subseteq A \} \) and there exists an element \( a' \in A \) such that \( \alpha_A(a') \geq \alpha_B(b) \) for all \( b \in B \), there exists a morphism \( g : B \to A \) such that \( f \circ g \circ f = f \).

**Proof.** (WAC1) \( \Rightarrow \) (WAC2) Since \( f : (A, \alpha_A) \to (B, \alpha_B) \) is factored by \( (f[A], \alpha_{f[A]}) \), we get \( f = m \circ e \) where \( e \) is an epimorphism and \( m \) is a monomorphism. By hypothesis there exists a morphism \( s : f[A] \to A \) such that \( e \circ s = i_{f[A]} \). Since \( A \) is a disjoint union of \( f[A] \) and \( B - f[A] \), we can construct a morphism \( h : (B - f[A]) \to A \) defined by \( h(b') = a' \) for all \( b' \in B - f[A] \), where \( \alpha_A(a') \geq \alpha_B(b) \) for all \( b \in B \). So we have that \( \alpha_A \circ h \leq \alpha_{B - f[A]} \). By the property of coproduct, there exists a morphism \( g : B \to A \) such that \( g \circ m = s \). That is, the following diagram commute.

\[
\begin{array}{ccc}
A & \xleftarrow{h} & B - f(A) \\
\downarrow{i_A} & & \downarrow{} \\
A & \xleftarrow{g} & B \\
\downarrow{} & & \downarrow{m} \\
f[A] & \xleftarrow{s} & f[A]
\end{array}
\]

We only claim that \( f \circ g \circ f = f \).

\[
f \circ g \circ f = (m \circ e) \circ g \circ (m \circ e) \\
= (m \circ e) \circ (g \circ m) \circ e \\
= (m \circ e) \circ s \circ e \\
= m \circ (e \circ s) \circ e \\
= m \circ e = f.
\]

(WAC2) \( \Rightarrow \) (WAC1) Let \( f : (A, \alpha_A) \to (B, \alpha_B) \) be an epimorphism such that \( \alpha_A \leq \alpha_B \circ f \). By hypothesis there exists a morphism \( g : (B, \alpha_B) \to (A, \alpha_A) \) such that \( \alpha_B \leq \alpha_A \circ g \) and \( e \circ s \circ e = e \). Since \( e \) is an epimorphism, we have \( f \circ g = i_B \). Hence \( f \) is a retraction. \( \square \)

**Theorem 3.2.** In a weak topos \( \mathcal{F}uz \) which is normal, the following statements are equivalent:

(WAC1) Every epimorphism \( f : (A, \alpha_A) \to (B, \alpha_B) \), where \( \alpha_B(b) = \max \{ \alpha_A[f[A']] \mid b = f[A'], A' \subseteq A \} \) and there exists an element \( a' \in A \) such that \( \alpha_A(a') \geq \alpha_B(b) \) for all \( b \in B \), is a retraction.
(WAC3) For any noninitial object $A$, there exists $\sigma : \Omega^A \to A$ such that for all $f : A \to \Omega$, we have $\alpha_A(\sigma(f)) = 1$ where $\alpha_A : A \to I$.

Proof. (WAC1) $\Rightarrow$ (WAC3) Consider a morphism $ev : A \times \Omega^A \to \Omega$ defined by $ev(a, s) = s(a)$. By the property of product, for any two morphisms $ev : A \times \Omega^A \to \Omega$ and $p_2 : A \times \Omega^A \to \Omega^A$ there exists a morphism $(ev, p_2) : A \times \Omega^A \to \Omega \times \Omega^A$ such that $p_2 \circ (ev, p_2) = p_2$ and $p_1 \circ (ev, p_2) = ev$ where $p_1 : \Omega \times \Omega^A \to \Omega$ and $p_2' : \Omega \times \Omega^A \to \Omega^A$. That is, the following diagram commute.

\[
\begin{array}{ccc}
A \times \Omega^A & \xrightarrow{ev} & \Omega \\
\downarrow i & & \downarrow p_1' \\
A \times \Omega^A & \xrightarrow{(ev, p_2)} & \Omega \times \Omega^A \\
\downarrow i & & \downarrow p_2' \\
A \times \Omega^A & \xrightarrow{p_2} & \Omega^A
\end{array}
\]

Since $(ev, p_2)$ is an epimorphism, there exists a morphism $h : \Omega \times \Omega^A \to A \times \Omega^A$ such that $(ev, p_2) \circ h = i_{\Omega \times \Omega^A}$. Also for a morphism $g : \Omega^A \to \Omega$ where $g(s) = 1$ for all $s \in \Omega^A$ and a morphism $i_{\Omega^A} : \Omega^A \to \Omega^A$ there exists a morphism $(g, i_{\Omega^A}) : \Omega^A \to \Omega \times \Omega^A$ such that $p_1 \circ (g, i_{\Omega^A}) = g$, $p_2 \circ (g, i_{\Omega^A}) = i_{\Omega^A}$. That is, the following diagram commute.

\[
\begin{array}{ccc}
\Omega & \xleftarrow{g} & \Omega^A \\
\downarrow p_1' & & \downarrow i \\
\Omega \times \Omega^A & \xleftarrow{(g, i_{\Omega^A})} & \Omega^A \\
\downarrow p_2' & & \downarrow i \\
\Omega^A & \xleftarrow{i_{\Omega^A}} & \Omega^A
\end{array}
\]

We get $h \circ (g, i_{\Omega^A})(s) = h(1, s) = (a, u)$ for some $a \in A$ and $u \in \Omega^A$. Also $u = p_2(a, u) = p_2 \circ h(1, s) = p_2' \circ (ev, p_2) \circ h(1, s) = p_2'(1, s) = s$. That is, $h \circ (g, i_{\Omega^A})(s) = (a, s)$. By $(ev, p_2) \circ h \circ (g, i_{\Omega^A})(s) = (ev, p_2)(a, s)$ and $g(s) = 1$ for all $s \in \Omega^A$, we get $s(a) = 1$. So $p_1 \circ h \circ (g, i_{\Omega^A})(s) = p_1(a, s) = a$. Let $\sigma = p_1 \circ h \circ (g, i_{\Omega^A})$, then $\sigma(s) = a$ and $s(a) = 1$. 
(WAC3) ⇒ (WAC1) For an epimorphism \( f : A \to B \), we construct a morphism \( \Omega^f : \Omega^B \to \Omega^A \) defined by \( \Omega^f(s) = s \circ f \) where \( s : B \to \Omega \). We only claim that \( f \circ \sigma \circ \Omega^f \circ j = i_B \) where \( j : B \to \Omega^B \) defined by \( j_B(b) = 1 \) and fixed for otherwise.

\[
\begin{array}{ccc}
\Omega^A & \overset{\Omega}{\longrightarrow} & \Omega^B \\
\sigma \downarrow & & \uparrow j \\
A & \overset{f}{\longrightarrow} & B
\end{array}
\]

Then \( f \circ \sigma \circ \Omega^f \circ j(b) = f \circ \sigma(j_B \circ f) \). Let \( \sigma(j_B \circ f) = a \) then, by definition of \( \sigma \), we get that \( j_B(f(a)) = 1 \) and \( b = f(a) \). Therefore \( f \circ \sigma \circ \Omega^f \circ j(b) = b \). \( \square \)

**Corollary 3.3.** In a weak topos \( \mathcal{F}uz \) which is normal, the following statements are equivalent:

(WAC1) Every epimorphism \( f : (A, \alpha_A) \to (B, \alpha_B) \), where \( \alpha_B(b) = \text{Max}\{\alpha_A(f[A']) \mid b = f[A'], A' \subseteq A\} \) and there exists an element \( a' \in A \) such that \( \alpha_A(a') \geq \alpha_B(b) \) for all \( b \in B \), is a retraction.

(WAC2) For any noninitial object \( A \) and \( f : A \to B \), where \( \alpha_B(b) = \text{Max}\{\alpha_A(f[A']) \mid b = f[A'], A' \subseteq A\} \) and there exists an element \( a' \in A \) such that \( \alpha_A(a') \geq \alpha_B(b) \) for all \( b \in B \), there exists a morphism \( g : B \to A \) such that \( f \circ g \circ f = f \).

(WAC3) For any noninitial object \( A \), there exists \( \sigma : \Omega^A \to A \) such that for all \( f : A \to \Omega \), we have \( \alpha_A(\sigma(f)) = 1 \) where \( \alpha_A : A \to I \).

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