A NEW NON-MEASURABLE
SET AS A VECTOR SPACE

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ABSTRACT. We use Cauchy’s functional equation to construct a
new non-measurable set which is a (vector) subspace of \( \mathbb{R} \) and is
of a codimension 1, considering \( \mathbb{R} \), the set of real numbers, as a
vector space over a field \( \mathbb{Q} \) of rational numbers. Moreover, we show
that \( \mathbb{R} \) can be partitioned into a countable family of disjoint non-
measurable subsets.

In every book on classical measure theory it is not hard to find a
statement given by G. Vitali that there exists a subset of \( \mathbb{R} \) which is
not Lebesgue measurable. We call the subset a non-measurable set for
simplicity.

In general the existence of non-measurable set is guaranteed by the
Axiom of Choice, or equivalent theorems such as Hausdorff maximality
principle, Zorn’s lemma, and so on. Conversely, it is well known that
the axiom of choice is essential for the existence.

Here we construct a new non-measurable set which is a (vector) sub-
space of \( \mathbb{R} \), the set all real numbers, and has a codimension 1 when we
consider \( \mathbb{R} \) as a vector space over a field \( \mathbb{Q} \) of rational numbers. We
construct it via the Cauchy functional equation using a method which
seems to be simpler, more heuristic and less logical, in author’s opinion,
than we have done with the classical one.

Moreover, it will be shown that \( \mathbb{R} \) can be expressed as a countable
union of disjoint family of the non-measurable subspace and its transla-
tions.

First, consider the following famous Lemma whose proof is seen, for
example, in [1]:

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Lemma 1. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a (Lebesgue) measurable function satisfying

(1) Cauchy's Equation: \( f(x + y) = f(x) + f(y) \), \( x, y \in \mathbb{R} \). Then \( f(x) = ax \), \( x \in \mathbb{R} \), for a constant \( a = f(1) \).

Throughout this \( \mathbb{R} \) is considered as a vector space over a field \( \mathbb{Q} \). First of all, with the help of Hausdorff maximality principle we take a maximal linearly independent subset \( \Lambda \) containing the number 1 as an element. We write the set as

\[
\Lambda = \{1\} \cup \{v_\alpha | \alpha \in I\}
\]

for a some index set \( I \). Note here that \( I \) is uncountable and each \( v_\alpha \) is an irrational number. The subset \( \Lambda \) is usually called a Hamel basis for \( \mathbb{R} \), in a sense that every real number \( x \) can be uniquely expressed as

(2) \( x = r + \sum_{\alpha \in I} a_\alpha v_\alpha \),

where \( r \) and \( a_\alpha \) are rational numbers which are zero except only a finite number of them.

We define a linear map \( \varphi \) on the vector space \( \mathbb{R} \) into itself by its values on the basis \( \Lambda \) as follows:

(3) \( \varphi(1) = 1 \), \( \varphi(v_\alpha) = 0 \), \( \alpha \in I \).

Then using the expression as (2), for every \( x = r + \sum_{\alpha \in I} a_\alpha v_\alpha \) and \( y = s + \sum_{\alpha \in I} b_\alpha v_\alpha \) we have

\[
\varphi(x + y) = \varphi((r + s) + \sum_{\alpha \in I} (a_\alpha + b_\alpha) v_\alpha) = r + s = \varphi(x) + \varphi(y).
\]

For each \( v_\alpha \) in \( \Lambda \) and a sequence \( (r_j) \) in \( \mathbb{Q} \) converging to \( v_\alpha \) we see that \( \varphi(r_j) = r_j \) converges to \( v_\alpha \) as \( j \) goes to \( \infty \), but \( \varphi(v_\alpha) = 0 \). This implies the discontinuity of \( \varphi \) at each point \( x = v_\alpha \). In view of the above Lemma 1 we conclude that \( \varphi \) is eventually not measurable.

Remark. The function defined above is a variant of an example seen in the book [2].

Now we denote by \( \mathbb{Q}_\Lambda \) the set of all real numbers whose expressions with respect to the basis \( \Lambda \) are of the form \( \sum_{\alpha \in I} a_\alpha v_\alpha \). In fact, \( \mathbb{Q}_\Lambda \) is a subspace generated by the set \( \Lambda \setminus \{1\} \). In other words, a real number \( x \) belongs to \( \mathbb{Q}_\Lambda \) if and only if there exist indices \( \alpha_1, \alpha_2, \ldots, \alpha_k \in I \) such that

\[
x = a_{\alpha_1} v_{\alpha_1} + a_{\alpha_2} v_{\alpha_2} + \cdots + a_{\alpha_k} v_{\alpha_k},
\]

for some rational numbers \( a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_k} \).
THEOREM 2. The set $Q_A$ is a subspace of codimension 1 and non-measurable as a subset of $\mathbb{R}$.

PROOF. It is easy to see that $Q_A$ is a subspace and has a codimension 1, since $\mathbb{R} \cong Q \oplus Q_A$.

To prove the second statement let $(r_j)$ be an enumeration of $Q$ and $r + Q_A$ be a translation of $Q_A$ by a number $r$. Then we have

\begin{equation}
\mathbb{R} = \bigcup_{j=1}^{\infty} (r_j + Q_A).
\end{equation}

Here, it is not hard to see that $(r_i + Q_A) \cap (r_j + Q_A) = \emptyset$ for $i \neq j$, using the unique expression of real numbers with respect to the Hamel basis and $\varphi(x) = r_j$ for all $x \in r_j + Q_A$ from its values defined by (3).

Now we recall that the function $\varphi$ defined above is a non-measurable function. Therefore, we find a real number $\gamma$ for which the set $A = \{x|\varphi(x) > \gamma\}$ is non-measurable. Hence, we can write

\begin{equation}
A = \bigcup_{r_j > \gamma} (r_j + Q_A).
\end{equation}

Therefore, there exists a $j_0$ with $r_{j_0} > \gamma$ such that $r_{j_0} + Q_A$ is non-measurable. Otherwise, the set $A$ would be measurable, which leads a contradiction. But, since the Lebesgue measure is translation-invariant, the set $Q_A$ is also non-measurable. \hfill \Box

In fact, since the subspace $Q_A$ is non-measurable, so is its every translation $r + Q_A$. Therefore, in view of (4) we can say as follows:

COROLLARY 3. $\mathbb{R}$ can be partitioned into a countable family of disjoint non-measurable subset, i.e.,

\begin{equation}
\mathbb{R} = \bigcup_{j=1}^{\infty} (r_j + Q_A),
\end{equation}

where $(r_j)$ be an enumeration of $Q$.

References


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