SHARP WEIGHTED ESTIMATE FOR
MULTILINEAR SINGULAR INTEGRAL OPERATOR

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ABSTRACT. In this paper, a sharp inequality for some multilinear
singular integral operators are obtained. As the applications, we get
the weighted $L^p (p > 1)$ norm inequalities and $L \log L$ type estimate
for the multilinear operators.

1. Preliminaries and results

As the development of Calderón-Zygmund operators and their com-
mutators, multilinear singular integral operators have been well studied.
In this paper, we will study some multilinear singular integral operators
as following.

Fix $\varepsilon > 0$. Let $T : S \to S'$ be a linear operator. Suppose that $T$
is bounded on $L^p (R^n)$ for $1 < p < \infty$ and weak $(L^1, L^1)$-bounded and
there exists a locally integrable function $K(x, y)$ on $R^n \times R^n \setminus \{(x, y) \in
R^n \times R^n : x = y\}$ such that

$$Tf(x) = \int_{R^n} K(x, y) f(y) dy$$

for every bounded and compactly supported function $f$, where $K$ satis-
fies:

$$|K(x, y)| \leq C|x - y|^{-n}$$

and

$$|K(y, x) - K(z, x)| + |K(x, y) - K(x, z)| \leq C|y - z|^\varepsilon |x - z|^{-n - \varepsilon}$$

when $2|y - z| \leq |x - z|$. Let $m_j$ be the positive integers ($j = 1, \ldots, l$),
$m_1 + \cdots + m_l = m$ and $A_j$ be the functions on $R^n$ ($j = 1, \ldots, l$). The

Received February 28, 2005.
2000 Mathematics Subject Classification: 42B20, 42B25.
Key words and phrases: multilinear operator, singular integral operator, sharp
estimate, BMO, $A_p$-weight.
multilinear operator related to $T$ is defined by

$$T^A(f)(x) = \int_{R^n} \prod_{j=1}^t R_{m_j+1}^j(A_j; x, y) \frac{K(x, y)f(y)dy}{|x - y|^m},$$

where

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha| \leq m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x - y)^\alpha.$$ 

Note that when $m = 0$, $T^A$ is just the multilinear commutator of $T$ and $A$ (see [8]). While when $m > 0$, $T^A$ is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been studied by many authors (see [1-5]). In [7], Hu and Yang proved a variant sharp estimate for the multilinear singular integral operators. In [11], Perez and Trujillo-Gonzalez prove a sharp estimate for some multilinear commutator when $A_j \in \mathcal{Osc}_{expH_j}$. The main purpose of this paper is to prove a sharp inequality for the multilinear singular integral operators. As the applications, we obtain the weighted $L^p(p > 1)$ norm inequalities and $L\log L$ type estimate for the multilinear operators.

First, let us introduce some notations. Throughout this paper, $Q$ will denote a cube of $R^n$ with sides parallel to the axes. For any locally integrable function $f$, the sharp function of $f$ is defined by

$$f^\#(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q|dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that(see[6])

$$f^\#(x) = \sup_{x \in Q} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c|dy.$$ 

We say that $f$ belongs to $BMO(R^n)$ if $f^\#$ belongs to $L^\infty(R^n)$ and $||f||_{BMO} = ||f^\#||_{L^\infty}$. For $0 < r < \infty$, we denote $f^\#_r$ by

$$f^\#_r(x) = \left( ||f^\#(x) ||^r \right)^{1/r}.$$ 

Let $M$ be the Hardy-Littlewood maximal operator, that is, that $M(f)(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y)|dy$. For $k \in N$, we denote by $M^k$ the operator $M$ iterated $k$ times, i.e., $M^1(f)(x) = M(f)(x)$ and $M^k(f)(x) = M(M^{k-1}(f))(x)$ for $k \geq 2$. 
Let $\Phi$ be a Young function and $\check{\Phi}$ be the complementary associated to $\Phi$, we define the $\Phi$-average of a function of $f$ over a cube $Q$ by
\[
\|f\|_{\Phi, Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi \left( \frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\}
\]
and the maximal function associated to $\Phi$ by
\[
M_{\Phi}(f)(x) = \sup_{x \in Q} \|f\|_{\Phi, Q};
\]
The Young functions to be using in this paper are $\Phi(t) = \exp(t^r) - 1$ and $\Psi(t) = t\log^r(t + e)$, the corresponding $\Phi$-average and maximal functions denoted by $\|\cdot\|_{\exp L^r, Q}, M_{\exp L^r}$ and $\|\cdot\|_{L(\log L)^r, Q}, M_{L(\log L)^r}$. We have the following inequality, for any $r > 0$ and $m \in N$(see[11])
\[
M(f) \leq M_{L(\log L)^r}(f), \quad M_{L(\log L)^m}(f) \approx M^{m+1}(f);
\]
For $r \geq 1$, we denote that
\[
\|b\|_{\text{osc}_{\exp L^r}} = \sup_Q ||b - b_Q||_{\exp L^r, Q},
\]
the space $O_{\text{osc}_{\exp L^r}}$ is defined by
\[
O_{\text{osc}_{\exp L^r}} = \{ b \in L^1_{\log}(\mathbb{R}^n) : \|b\|_{\text{osc}_{\exp L^r}} < \infty \}.
\]
It has been known that(see[11])
\[
\|b - b_{2Q}\|_{\exp L^r, 2^kQ} \leq Ck\|b\|_{O_{\text{osc}_{\exp L^r}}},
\]
It is obvious that $O_{\text{osc}_{\exp L^r}}$ coincides with the $BMO$ space if $r = 1$. And $O_{\text{osc}_{\exp L^r}} \subset BMO$ if $r > 1$. We denote the Muckenhoupt weights by $A_p$ for $1 \leq p < \infty$(see[6]).

Now we state our main results as following.

**Theorem 1.** Let $r_j \geq 1$ and $D^\alpha A_j \in O_{\text{osc}_{\exp L^r}},$ for all $\alpha$ with $|\alpha| = m_j$ and $j = 1, \ldots, l$. Denote that $1/r = 1/r_1 + \cdots + 1/r_l$. Then for any $0 < p < 1$, there exists a constant $C > 0$ such that for any $f \in C^\infty_0(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,
\[
(T^A(f))_p^#(x) \leq C \prod_{j=1}^l \left( \sum_{|\alpha_j| = m_j} \|D^\alpha A_j\|_{O_{\text{osc}_{\exp L^r}}} \right) M_{L(\log L)^{1/r}}(f)(x).
\]

**Theorem 2.** Let $r_j \geq 1$ and $D^\alpha A_j \in O_{\text{osc}_{\exp L^r}},$ for all $\alpha$ with $|\alpha| = m_j$ and $j = 1, \ldots, l$. 
(1) If $1 < p < \infty$ and $w \in A_p$, then

$$
\|T^A(f)\|_{L^p(w)} \leq C \prod_{j=1}^l \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{O_{sc \exp L^q_j}} \right) \|f\|_{L^p(w)};
$$

(2) If $w \in A_1$. Denote that $1/r = 1/r_1 + \cdots + 1/r_l$ and $\Phi(t) = t \log^{1/r}(t + e)$. Then there exists a constant $C > 0$ such that for all $\lambda > 0$,

$$
w(\{x \in R^n : |T^A(f)(x)| > \lambda\}) \leq C \int_{R^n} \Phi \left( \lambda^{-1} \prod_{j=1}^l \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{O_{sc \exp L^q_j}} \right) |f(x)| \right) w(x) dx.
$$

2. Some lemmas

We give some preliminaries lemmas.

**Lemma 1.** ([3]) Let $A$ be a function on $R^n$ and $D^\alpha A \in L^q(R^n)$ for all $\alpha$ with $|\alpha| = m$ and some $q > n$. Then

$$
|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\alpha|=m} \left( \frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x, y)} |D^\alpha A(z)|^q dz \right)^{1/q},
$$

where $\tilde{Q}$ is the cube centered at $x$ and having side length $5\sqrt{n}|x - y|$.

**Lemma 2.** ([6, p.485]) Let $0 < p < q < \infty$ and for any function $f \geq 0$. We define that, for $1/r = 1/p - 1/q$

$$
\|f\|_{WL^q} = \sup_{\lambda > 0} \lambda^{1/q} \{|x \in R^n : f(x) > \lambda\}|^{1/q},
$$

$$
N_{p,q}(f) = \sup_E \|f \chi_E\|_{L^p}/\|\chi_E\|_{L^r},
$$

where the sup is taken for all measurable sets $E$ with $0 < |E| < \infty$. Then

$$
\|f\|_{WL^q} \leq N_{p,q}(f) \leq (q/(q - p))^{1/p} \|f\|_{WL^q}.
$$
LEMMA 3. ([11]) Let \( r_j \geq 1 \) for \( j = 1, \ldots, m \), we denote that \( 1/r = 1/r_1 + \cdots + 1/r_m \). Then
\[
\frac{1}{|Q|} \int_Q |f_1(x) \cdots f_m(x)g(x)|dx \\
\leq \|f\|_{\exp L^{r_1},Q} \cdots \|f\|_{\exp L^{r_m},Q} \|g\|_{L(\log L)^{1/r},Q}.
\]

3. Proof of Theorem

It is only to prove Theorem 1.

PROOF OF THEOREM 1. It suffices to prove for \( f \in C_0^\infty (R^n) \) and some constant \( C_0 \), the following inequality holds:
\[
\left( \frac{1}{|Q|} \int_Q |T^A(f)(x) - C_0|^p dx \right)^{1/p} \\
\leq C \prod_{j=1}^l \left( \sum_{|\alpha_j|=m_j} \|D^\alpha_j A_j\|_{\text{osc}_{\exp L^{r_j}}} \right) M_{L(\log L)^{1/r}}(f)(x).
\]

Without loss of generality, we may assume \( l = 2 \). Fix a cube \( Q = Q(x_0,d) \) and \( \bar{x} \in Q \). Let
\[
\tilde{Q} = 5\sqrt{n}Q
\]
and
\[
\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha| = m_j} \frac{1}{\alpha!} (D^\alpha A_j) \bar{x}^\alpha.
\]

Then \( R_{m_j}(A_j; x, y) = R_{m_j}(\tilde{A}_j; x, y) \) and \( D^\alpha \tilde{A}_j = D^\alpha A_j - (D^\alpha A_j) \bar{x} \) for \( |\alpha| = m_j \). We write, for \( f_1 = f\chi_Q \) and \( f_2 = f\chi_{R^n \setminus \tilde{Q}} \),
\[
T^A(f)(x) \\
= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x-y|^m} K(x, y) f(y)dy \\
= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x-y|^m} K(x, y) f_2(y)dy \\
+ \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m} K(x, y) f_1(y)dy \\
- \sum_{|\alpha_1| = m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y) (x-y)^{\alpha_1}}{|x-y|^m} D^\alpha \tilde{A}_1(y) K(x, y) f_1(y)dy
\]
\[- \sum_{|\alpha_2| = m_2} \frac{1}{\alpha_2!} \int_{\mathbb{R}^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{A}_2(y) K(x, y) f_1(y) dy \]

\[+ \sum_{|\alpha_1| = m_1, \ |\alpha_2| = m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{\mathbb{R}^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{A}_1(y) D^{\alpha_2} \tilde{A}_2(y)}{|x-y|^m} \times K(x, y) f_1(y) dy, \]

then

\[|T^\tilde{A}(f)(x) - T^\tilde{A}(f_2)(x_0)| \leq \left| \int_{\mathbb{R}^n} \prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y) \frac{K(x, y) f_1(y) dy}{|x-y|^m} \right| \]

\[+ \sum_{|\alpha_1| = m_1} \frac{1}{\alpha_1!} \int_{\mathbb{R}^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \tilde{A}_1(y) K(x, y) f_1(y) dy \]

\[+ \sum_{|\alpha_2| = m_2} \frac{1}{\alpha_2!} \int_{\mathbb{R}^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{A}_2(y) K(x, y) f_1(y) dy \]

\[+ \sum_{|\alpha_1| = m_1, \ |\alpha_2| = m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{\mathbb{R}^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{A}_1(y) D^{\alpha_2} \tilde{A}_2(y)}{|x-y|^m} \times K(x, y) f_1(y) dy \]

\[+ |T^\tilde{A}(f_2)(x) - T^\tilde{A}(f_2)(x_0)| \]

\[:= I_1(x) + I_2(x) + I_3(x) + I_4(x) + I_5(x), \]

thus,

\[\left( \frac{1}{|Q|} \int_Q \left| T^\tilde{A}(f)(x) - T^\tilde{A}(f_2)(x_0) \right|^p dx \right)^{1/p} \leq \left( \frac{C}{|Q|} \int_Q I_1(x)^p dx \right)^{1/p} + \left( \frac{C}{|Q|} \int_Q I_2(x)^p dx \right)^{1/p} \]

\[+ \left( \frac{C}{|Q|} \int_Q I_3(x)^p dx \right) \quad \left( \frac{C}{|Q|} \int_Q I_4(x)^p dx \right)^{1/p} \]

\[+ \left( \frac{C}{|Q|} \int_Q I_5(x)^p dx \right)^{1/p} \]

\[:= I_1 + I_2 + I_3 + I_4 + I_5. \]
Now, let us estimate $I_1$, $I_2$, $I_3$, $I_4$ and $I_5$, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, by Lemma 1, we get

$$R_{m_j}(\tilde{A}_j; x, y) \leq C|x - y|^m \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{L^r_{exp}'_j},$$

thus, by Lemma 2 and the weak type $(1,1)$ of $T$, we obtain

$$I_1 \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{L^r_{exp}'_j} \right) \left( \frac{1}{|Q|} \int_Q |T(f_1)(x)|^p dx \right)^{1/p}$$

$$= C \prod_{j=1}^{2} \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{L^r_{exp}'_j} \right) |Q|^{-1} \frac{||T(f_1)|_{L^p}|_{L^1}}{|Q|^{1/p - 1}}$$

$$\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{L^r_{exp}'_j} \right) |Q|^{-1} ||T(f_1)||_{L^1}$$

$$\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{L^r_{exp}'_j} \right) M(f)(\tilde{x})$$

$$\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{L^r_{exp}'_j} \right) M_L(log L)^{1/r}(f)(\tilde{x});$$

For $I_2$, note that $||\chi_Q||_{L^2} \leq C$, similar to the proof of $I_1$ and by using Lemma 3, we get

$$I_2 \leq C \sum_{|\alpha_2| = m_2} ||D^{\alpha_2} A_2||_{L^r'_{exp}}$$

$$\times \sum_{|\alpha_1| = m_1} \left( \frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} \tilde{A}_1 f_1)(x)|^p dx \right)^{1/p}$$

$$\leq C \sum_{|\alpha_2| = m_2} ||D^{\alpha_2} A_2||_{L^r'_{exp}}$$
\[
\times \sum_{|\alpha_1|=m_1} |Q|^{-1} ||T(D^{\alpha_1} \tilde{A}_1 f_1)(x)\chi_Q||_{W^{1,1}}
\leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} A_2||_{O_{sc_{expL^2}}} \\
\times \sum_{|\alpha_1|=m_1} \frac{1}{|Q|} \int_{R^n} |D^{\alpha_1} \tilde{A}_1(x) f_1(x)|\,dx
\leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} A_2||_{O_{sc_{expL^2}}}, Q ||\chi_Q||_{expL^2}, Q \\
\times \sum_{|\alpha_1|=m_1} ||D^{\alpha_1} A_1 - (D^{\alpha_1} A_1)_\tilde{Q}||_{expL^2}, Q ||f||_{L(logL)^{1/r}, \tilde{Q}}
\leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{O_{sc_{expL^2}}^j} \right) M_{L(logL)^{1/r}} (f)(\bar{x});
\]

For $I_3$, similar to the proof of $I_2$, we get

\[
I_3 \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{O_{sc_{expL^2}}^j} \right) M_{L(logL)^{1/r}} (f)(\bar{x});
\]

Similarly, for $I_4$, by using Lemma 3, we get

\[
I_4 \leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left( \frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 f_1)(x)|^p\,dx \right)^{1/p}
\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1} ||T(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 f_1)\chi_Q||_{W^{1,1}}
\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|} \int_{R^n} |D^{\alpha_1} \tilde{A}_1(x) D^{\alpha_2} \tilde{A}_2(x) f_1(x)|\,dx
\leq C \sum_{|\alpha_1|=m_1} ||D^{\alpha_1} A_1 - (D^{\alpha_1} A_1)_\tilde{Q}||_{expL^1}, \tilde{Q} \\
\times \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} A_2 - (D^{\alpha_2} A_2)_\tilde{Q}||_{expL^2}, \tilde{Q} ||f||_{L(logL)^{1/r}, \tilde{Q}}
\leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{O_{sc_{expL^2}}^j} \right) M_{L(logL)^{1/r}} (f)(\bar{x});
\]
For $I_5$, we write

$$T^\mathcal{A}(f_2)(x) - T^\mathcal{A}(f_2)(x_0)$$

$$= \int_{\mathbb{R}^n} \left( \frac{K(x, y)}{|x-y|^m} - \frac{K(x_0, y)}{|x_0-y|^m} \right) \prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y) f_2(y) dy$$

$$+ \int_{\mathbb{R}^n} \left( R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; x_0, y) \right)$$

$$\times \frac{R_{m_2}(\tilde{A}_2; x, y)}{|x_0-y|^m} K(x_0, y) f_2(y) dy$$

$$+ \int_{\mathbb{R}^n} \left( R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y) \right)$$

$$\times \frac{R_{m_1}(\tilde{A}_1; x_0, y)}{|x_0-y|^m} K(x_0, y) f_2(y) dy$$

$$- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{\mathbb{R}^n} \left[ \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} K(x, y) \right.$$

$$- \frac{R_{m_2}(\tilde{A}_2; x_0, y)(x_0-y)^{\alpha_1}}{|x_0-y|^m} K(x_0, y) \left] \right. \times D^{\alpha_1} \tilde{A}_1(y) f_2(y) dy$$

$$- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{\mathbb{R}^n} \left[ \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} K(x, y) \right.$$

$$\times D^{\alpha_2} \tilde{A}_2(y) f_2(y) dy$$

$$\left. + \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{\mathbb{R}^n} \left[ \frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} K(x, y) \right.$$

$$\times D^{\alpha_1} \tilde{A}_1(y) D^{\alpha_2} \tilde{A}_2(y) f_2(y) dy$$

$$= I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)};$$

By Lemma 1, we know that, for $x \in Q$ and $y \in 2^{k+1}\bar{Q} \setminus 2^k\bar{Q}$,

$$|R_{m_j}(\tilde{A}_j; x, y)| \leq C|x-y|^{m_j} \sum_{|\alpha_j|=m_j} (||D^{\alpha_j} A||_{\text{osc}_x L^{r_j}}$$
\[ + |(D^{\alpha_j} A)_{\tilde{Q}}(x,y) - (D^{\alpha_j} A)_{\tilde{Q}}| \]
\[ \leq C k |x - y|^m_j \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A||_{Osc_{expL^r_j}}. \]

Note that \( |x - y| \sim |x_0 - y| \) for \( x \in Q \) and \( y \in R^n \setminus \tilde{Q} \), we obtain, by the conditions on \( K \),

\[ |I_5^{(1)}| \leq C \int_{R^n} \left( \frac{|x - x_0|}{|x_0 - y|^{m+n+1}} + \frac{|x - x_0|}{|x_0 - y|^{m+n+\varepsilon}} \right) \]
\[ \times \prod_{j=1}^2 |R_{m_j}(\tilde{A}_j; x, y)| |f_2(y)| dy \]
\[ \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^r_j}} \right) \]
\[ \times \sum_{k=0}^{\infty} \int_{2k+1 \tilde{Q} \setminus 2^k \tilde{Q}} k^2 \left( \frac{|x - x_0|}{|x_0 - y|^{n+1}} + \frac{|x - x_0|}{|x_0 - y|^{n+\varepsilon}} \right) |f(y)| dy \]
\[ \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^r_j}} \right) \]
\[ \times \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-\varepsilon k}) \frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |f(y)| dy \]
\[ \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^r_j}} \right) M(f)(\bar{x}); \]

For \( I_5^{(2)} \), by the formula (see [3]):

\[ R_{m_j}(\tilde{A}; x, y) - R_{m_j}(\tilde{A}; x_0, y) = \sum_{|\beta| < m_j} \frac{1}{\beta!} R_{m_j - |\beta|} (D^\beta \tilde{A}; x, x_0) (x - y)^\beta \]

and Lemma 1, we have

\[ |R_{m_j}(\tilde{A}; x, y) - R_{m_j}(\tilde{A}; x_0, y)| \]
\[ \leq C \sum_{|\beta| < m_j} \sum_{|\alpha| = m_j} |x - x_0|^{m_j - |\beta|} |x - y|^{|\beta|} ||D^\alpha A||_{Osc_{expL^r_j}}, \]
thus

\[ |I_5^{(2)}| \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{exp} L^r_j} \right) \times \sum_{k=0}^{\infty} \int_{2^{k+1}\hat{Q}\setminus 2^k\hat{Q}} k \frac{|x-x_0|}{|x_0-y|^{n+1}} |f(y)|dy \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{exp} L^r_j} \right) M(f)(\tilde{x}); \]

Similarly,

\[ |I_5^{(3)}| \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{exp} L^r_j} \right) M(f)(\tilde{x}); \]

For \( I_5^{(4)} \), similar to the proof of \( I_5^{(1)} \), \( I_5^{(2)} \), and \( I_2 \), we get

\[ |I_5^{(4)}| \leq C \sum_{|\alpha_1|=m_1} \int_{R^n} \left| \frac{(x-y)^{\alpha_1} K(x,y)}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1} K(x_0,y)}{|x_0-y|^m} \right| \times |R_{m_2}(\tilde{A}_2;x,y)||D^{\alpha_1} \tilde{A}_1(y)||f_2(y)|dy \]

\[ + C \sum_{|\alpha_1|=m_1} \int_{R^n} |R_{m_2}(\tilde{A}_2;x,y) - R_{m_2}(\tilde{A}_2;x_0,y)| \times \left| \frac{(x_0-y)^{\alpha_1} K(x_0,y)}{|x_0-y|^m} \right| |D^{\alpha_1} \tilde{A}_1(y)||f_2(y)|dy \leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} A_2||_{Osc_{exp} L^r_2} \sum_{|\alpha_1|=m_1} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-\varepsilon k}) \times \frac{1}{|2^k\hat{Q}|} \int_{2^k\hat{Q}} |D^{\alpha_1} \tilde{A}_1(y)||f(y)|dy \]

\[ \leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} A_2||_{Osc_{exp} L^r_2} \sum_{|\alpha_1|=m_1} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-\varepsilon k}) \times ||D^{\alpha_1} A_1 - (D^{\alpha_1} A_1)_{\hat{Q}}||_{exp L^{r_1,2^k\hat{Q}}L_{L(\log L)^{1/r}} L_2} ||f||_{L(\log L)^{1/r,2^k\hat{Q}}} \]

\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{exp} L^r_j} \right) M_{L(\log L)^{1/r}}(f)(\tilde{x}); \]
Similarly,

\[
|I^{(5)}_5| \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^{r_j}}} \right) M_{L(logL)^{1/r}}(f)(\tilde{x});
\]

For \(I^{(6)}_5\), by using Lemma 3, we obtain

\[
|I^{(6)}_5| \leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \sum_{k=1}^{\infty} (2^{-k} + 2^{-\varepsilon k}) \times \frac{1}{2^{kQ}} \int_{2^kQ} |D^{\alpha_1} A_1(y)||D^{\alpha_2} A_2(y)||f(y)|dy
\]

\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^{r_j}}} \right) M_{L(logL)^{1/r}}(f)(\tilde{x});
\]

Thus

\[
|I_5| \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^{r_j}}} \right) M_{L(logL)^{1/r}}(f)(\tilde{x}).
\]

This completes the proof of Theorem 1.

By Theorem 1 and the \(L^p\)-boundedness of \(M_{L(logL)^{1/r}}\), we may obtain the conclusions (1)(2) of Theorem 2.

4. Example

In this section we shall apply Theorem 1 and 2 of the paper to the Calderón-Zygmund singular integral operator.

Let \(T\) be the Calderón-Zygmund operator (see [4], [6], [12]), the multilinear operator related to \(T\) is defined by

\[
T^A(f)(x) = \int_{R^n} \prod_{j=1}^{l} R_{m_j+1}(A_j; x, y) \frac{K(x, y)f(y)dy}{|x - y|^m}.
\]

In particular, the multilinear commutator related to \(T\) is(see[11])

\[
T^A(f)(x) = \int_{R^n} \left[ \prod_{j=1}^{l} (A_j(x) - A_j(y)) \right] K(x, y)f(y)dy.
\]
Then

\[(T^A(f))^\#_p(x) \leq C \prod_{j=1}^l \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\text{osc}_{L_{r_j}}} \right) M_{L\log L}^{1/r}(f)(x)\]

for any \(f \in C_0^\infty(R^n)\) and \(0 < p < 1\);

\[(2) \quad ||T^A(f)||_{L^p(w)} \leq C \prod_{j=1}^l \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\text{osc}_{L_{r_j}}} \right) ||f||_{L^p(w)}\]

for any \(w \in A_p\) and \(1 < p < \infty\);

\[w(\{x \in R^n : |T^A(f)(x)| > \lambda\})\]

\[(3) \quad \leq C \int_{R^n} \Phi \left( \lambda^{-1} \prod_{j=1}^l \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\text{osc}_{L_{r_j}}} \right) |f(x)| \right) w(x)dx\]

for any \(w \in A_1\) and all \(\lambda > 0\).

References


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