ON THE COMPACT RIEMANNIAN MANIFOLDS
WITH SOME GEODESICAL PROPERTIES

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ABSTRACT. In the paper, we study an n-dimensional compact Riemannian manifold \((M, g)\) with the property that the lengths of the images \(c(\mathbb{R})\) in \(M\) of any geodesic curves \(c : \mathbb{R} \mapsto M\) are finite.

1. Introduction

In the differential geometry, we know

**Theorem 1.1.** (H. Whitney) Any \(n\)-dimensional differentiable manifold can be embedded into \(\mathbb{R}^{2n+1}\) \([5]\).

So, in order to study any differentiable manifolds, it is sufficient to consider only the submanifolds of \(\mathbb{R}^m\) for each integer \(m \geq 1\).

We also see

**Theorem 1.2.** (J. Nash) Every Riemannian manifold \((M, g)\) can be isometrically embedded into some Euclidean space \(\mathbb{R}^n\) \([1], [3], [4]\).

Thus, by Theorem 1.2, in the paper we consider a Riemannian manifold \((M, g)\) as a submanifold of some Euclidean space \(\mathbb{R}^n\). Now, we can consider the following Question:

**Question 1.1.** Let \(M = S^1 \times S^2\). Is there a geodesic curve \(c : \mathbb{R} \mapsto M\) such that the length of its image \(c(\mathbb{R})\) in \(M\) is infinite?

More generally, we can also give the following Question:

**Question 1.2.** Let \((M, g)\) be an \(n\)-dimensional compact Riemannian manifold. Is there a geodesic curve \(c : \mathbb{R} \mapsto M\) such that the length of its image \(c(\mathbb{R})\) in \(M\) is infinite?

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For the answers of the above Questions, we have the following Main Theorems.

**Main Theorem 1.** Let \((M, g)\) be an \(n\)-dimensional compact Riemannian manifold with the property that for each geodesic curve \(c : \mathbb{R} \rightarrow M\), the length of its image \(c(\mathbb{R})\) in \(M\) is finite. Then we have

\[\text{either } \pi_1(M) = \mathbb{Z} \text{ or } \pi_1(M) = \mathbb{Z}_p \text{ for some } p \in \mathbb{N}.\]

**Main Theorem 2.** Let \((M, g)\) be an \(n\)-dimensional compact Riemannian manifold with the property that for each geodesic curve \(c : \mathbb{R} \rightarrow M\), the length of its image \(c(\mathbb{R})\) in \(M\) is finite. If \(n \geq 2\), then we obtain

\[\pi_1(M) = \mathbb{Z}_p \text{ for some } p \in \mathbb{N}.\]

But we know

**Theorem 1.3.** (Synge) Any compact oriented even-dimensional Riemannian manifold with positive sectional curvature is simply connected [2].

So, we can compare Main Theorem 2 with Theorem 1.3.

In the paper, we will use the following notations:

- \(\mathbb{R}\) = the set of real numbers
- \(\mathbb{R}^+\) = the set of positive real numbers
- \(\mathbb{Q}\) = the set of rational numbers
- \(\mathbb{N}\) = the set of natural numbers
- \(\mathbb{Z}\) = the set of integers
- \(\mathbb{Z}_p\) = the quotient group \(\mathbb{Z}/p\mathbb{Z}\)
- \(I = [0, 1]\)
- \(L(c)\) = the length of a curve \(c\)
- \(T_p M\) = the tangent space of \(M\) at \(p \in M\)
- \(\nabla\) = the Levi-Civita connection of a metric \(g\) in \(M\)
- \(|v| = \sqrt{g(v, v)}\) for \(v \in T_p M\)
- \(U_p M = \{v \in T_p M \mid |v| = 1\}\)
- \(\gamma_t(p, v)\) = the (closed) geodesic curve in \((M, g)\) with the initial conditions: \(\gamma_0(p, v) = p, \gamma_0'(p, v) = v\)
- \(S^n\) = the \(n\)-dimensional unit sphere
- \(\pi_1(M)\) = the fundamental group of \(M\)
2. Proofs of the main theorems

Let \((M,g)\) be an \(n\)-dimensional compact Riemannian manifold with the property that for each geodesic curve \(c : \mathbb{R} \to M\), the length of its image \(c(\mathbb{R})\) in \(M\) is finite. Then we have

**Proposition 2.1.** Every complete geodesic curve in \((M,g)\) is closed.

**Proof.** Suppose that there exists a nontrivial complete geodesic curve \(c : \mathbb{R} \to M\) such that \(c\) is not closed. Define a curve \(\tilde{c} : \mathbb{R} \to M\) by

\[
\tilde{c}(s) := c\left(\frac{s}{|c'(0)|}\right) \quad \text{for } s \in \mathbb{R}.
\]

Then we know that \(\tilde{c}\) is also geodesic in \((M,g)\) such that

\[
|\tilde{c}'(s)| = 1 \quad \text{for } s \in \mathbb{R}.
\]

By the definition of a geodesic curve, i.e., since every geodesic curve \(\tilde{c}(t)\) with \(\tilde{c}(0) = p\) and \(\tilde{c}'(0) = v\) in \((M,g)\) is the unique solution of the initial value problem:

\[
\nabla_{\tilde{c}'} \tilde{c}' = 0, \quad \tilde{c}(0) = p, \quad \tilde{c}'(0) = v,
\]

we obtain that the length of the image of the curve \(\tilde{c}(s)\) from 0 to \(s_0\) is equal to \(s_0\).

i.e., \(L(\tilde{c}([0,s_0])) = s_0\).

That is, if the curve \(\tilde{c}(s)\) has self-intersection points in the image \(\tilde{c}(\mathbb{R}) \subset M\), then they must intersect transversely.

Thus,

\[
L(\tilde{c}([0,\infty])) = \lim_{s_0 \to \infty} L(\tilde{c}([0,s_0])) = \lim_{s_0 \to \infty} s_0 = \infty.
\]

This contradicts the hypothesis. Therefore, the result follows. \(\square\)

**Proposition 2.2.** For each non-zero tangent vector \(v, v' \in T_pM\), \(p \in M\), there is a homotopy between the closed geodesic curve \(\gamma_t(p,v)\) and the closed geodesic curve \(\gamma_t(p,v')\).

**Proof.** Define a map \(\tilde{t} : U_pM \to \mathbb{R}\) by

\[
\tilde{t}(w) := \min\{t_0 \in \mathbb{R}^+ | \gamma_{t_0}(p,w) = p \text{ and } \gamma'_{t_0}(p,w) = w\} \quad \text{for } w \in U_pM.
\]

Then by Proposition 2.1, \(\tilde{t}\) is well-defined. For each non-zero tangent vector \(v, v' \in T_pM\), define a map \(F : I \times I \to M\) by

\[
F(t,s) := \gamma_{\tilde{t}}(p,\tilde{v}),
\]
where \( \hat{t} = \frac{\vec{v}}{|\vec{v}|} \cdot t \) and \( \vec{v} = (1 - s)v + sv' \) for \( s, t \in I \). Then clearly, \( F \) is a homotopy between the closed curve \( \gamma_t(p, v) \) and the closed curve \( \gamma_t(p, v') \).

**Proof of the Main Theorem 1.** Since each class in \( \pi_1(M) \) can be represented by a closed geodesic curve \([2], \) by Proposition 2, the result follows. 

**Example 2.1.** a) For the unit circle \( S^1 \), we know that the lengths of the images \( c(\mathbb{R}) \) of any non-trivial geodesic curves \( c : \mathbb{R} \mapsto S^1 \) are equal to \( 2\pi \), and so finite. But \( \pi_1(S^1) = \mathbb{Z} \).

b) For the unit sphere \( S^2 \), let \( c : \mathbb{R} \mapsto S^2 \) be a non-trivial geodesic curve in \( S^2 \). Then the length of the image \( c(\mathbb{R}) \subset S^2 \) is equal to \( 2\pi \), and so finite. But \( \pi_1(S^2) = 0 \).

c) For the generalized flat torus \( T^2 := \mathbb{R}^2/s\mathbb{Z} \oplus t\mathbb{Z} \) with \( s, t \in \mathbb{R}^+ \), we know that \( \pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z} \). But let \( \pi : \mathbb{R}^2 \mapsto T^2 \) be the natural projection. Then for any straight lines \( l : y = ax + b \) with \( a, b \in \mathbb{R} \) in \( \mathbb{R}^2 \), we have

\[
L(c) = \begin{cases} 
\sqrt{(sn)^2 + (tm)^2} & \text{if } a = \frac{m}{n} \in \mathbb{Q} - \{0\} \text{ and } m, n \text{ integers}, \\
\infty & \text{if } a \in \mathbb{R} - \mathbb{Q}, \\
s & \text{if } a = 0, \\
t & \text{if } l : x = p, \ p \in \mathbb{R},
\end{cases}
\]

where \( c \) is the image \( \pi(l) \) in \( T^2 \). So, the torus \( T^2 \) has a geodesic curve whose image is of infinite length in \( T^2 \).

d) Let \( (M, g) \) be an \( n \)-dimensional compact connected Riemannian manifold such that neither \( \pi_1(M) = \mathbb{Z} \) nor \( \pi_1(M) = \mathbb{Z}_p \) for some \( p \in \mathbb{N} \). Then there exists a geodesic curve in \( (M, g) \) such that the length of its image in \( M \) is infinite.

**Proof of the Main Theorem 2.** Consider the map \( \widetilde{\gamma} : U_pM \mapsto \mathbb{R} \), defined by

\[
\widetilde{\gamma}(v) := \min\{t_0 \in \mathbb{R}^+ \mid \gamma_{t_0}(p, v) = p \text{ and } \gamma'_{t_0}(p, v) = v\} \text{ for } v \in U_pM.
\]

Let \( \vec{\partial} \) be the north pole of the unit sphere \( S^n \) and \( T : T_{\vec{\partial}}S^n \mapsto T_pM \) an isometry. For each closed geodesic curve \( \gamma_t(p, v) : [0, \bar{\tau}(v)] \mapsto M \) with \( v \in U_pM \), let \( w := T^{-1}(v) \in U_{\vec{\partial}}S^n \). Then define a map \( f_v : S^n \mapsto M \) by

\[
f_v(\vec{\partial}) := p.
\]
and for any
\[ \bar{w} \in U_\bar{o}S^n, \quad f_v(c_t(\bar{o}, \bar{w})) := \gamma_\bar{t}(p, T(\bar{w})) \] for
\[ \begin{cases} 
\bar{t} \in [0, \pi] & \text{if } \bar{w} \neq w, \\
\bar{t} \in [0, \pi] & \text{if } \bar{w} = w,
\end{cases} \]
where \( \bar{t} = \frac{\bar{t}(T(\bar{w}))}{2\pi} \cdot t \) and \( c_t(\bar{o}, \bar{w}) \) is the geodesic curve in \( S^n \) such that \( c_0(\bar{o}, \bar{w}) = \bar{o} \) and \( c_0(\bar{o}, \bar{w}) = \bar{w} \). Then by the definition of \( f_v \), obviously, \( f_v \) is a well-defined map. Let \( \bar{o} \) be the south pole of \( S^n \). Then we can also show that
\[ f_v \mid_{S^n - \{\bar{o}\}}: S^n - \{\bar{o}\} \rightarrow M \text{ is continuous,} \]
where \( f_v \mid_{S^n - \{\bar{o}\}} \) is the restriction map of \( f_v \) to the subset \( S^n - \{\bar{o}\} \).

For each \( v \in U_pM \), we get \( w = T^{-1}(v) \in U_\bar{o}S^n \). Then conveniently, we may assume \( w = (-1, 0, \ldots, 0) \in \mathbb{R}^{n+1} \). Consider the map \( F : I \times I \hookrightarrow S^n \), given by
\[ F(t, s) := (-s \sin 2\pi t, \sqrt{2s(1 - s)(1 - \sin 2\pi t)}, 0, \ldots, 0, 1 - s + s \cos 2\pi t) \]
for \( (t, s) \in I \times I \) and the inclusion map \( i : I \times [0, 1) \hookrightarrow I \times I \).

Let \( \tilde{F} := f_v \circ F \circ i : I \times [0, 1) \rightarrow M \). Then define the map \( \tilde{F} : I \times I \rightarrow M \) by
\[ \begin{cases} 
\tilde{F}(t, s) := \tilde{F}(t, s), & (t, s) \in I \times [0, 1) \\
\tilde{F}(t, 1) := \lim_{s \to 1} \tilde{F}(t, s), & (t, 1) \in I \times \{1\}.
\end{cases} \]

It is easy to show that \( \tilde{F} \) is continuous. By Main Theorem 1, there is an element \( \alpha \) in \( \pi_1(M) \) such that \( \alpha \) generates the group \( \pi_1(M) \). Then we have
\[ \alpha = [\gamma_\bar{t}(p, v_0)] \text{ for some } v_0 \in U_pM, \]
where \([\gamma_\bar{t}(p, v_0)]\) denotes the class represented by the closed geodesic curve \( \gamma_\bar{t}(p, v_0) \). Consider the map \( f_{v_0} \) and the map \( \tilde{F} \) obtained by using \( f_{v_0} \) instead of \( f_v \) as above. Then
\[ 0 = [\tilde{F}(t, 0)] = [\tilde{F}(t, 1)] = [\gamma_\bar{t}(p, v_0) + \bar{C}] = [\gamma_\bar{t}(p, v_0)] + [\bar{C}], \]
where \( \bar{C} = (\tilde{F}(t, 1) - \gamma_\bar{t}(p, v_0)) \cup \{\gamma_{\tilde{t}(v_0)}(p, v_0)\} \) and to obtain the last equality, with the abuse of the notations we handle those classes in the last equation with the new base point \( \gamma_{\tilde{t}(v_0)}(p, v_0) \) instead of the old original base point \( p \). Since the closed curves \( \gamma_\bar{t}(p, v_0) \) and \( \bar{C} \) have the same orientation in some sense, by Main Theorem 1, there exists a non-negative integer \( n \) such that \([\bar{C}] = n\alpha\). Thus we have
\[ (n + 1)\alpha = 0. \]
Therefore, we conclude
\[ \pi_1(M) = \mathbb{Z}_p \quad \text{for some } p \in \mathbb{N}. \]

Remark 2.1. a) Let \( \mathbb{P}^n \) be the \( n \)-dimensional real projective space with \( n \geq 2 \). i.e., let \( A : S^n \to S^n \) be the antipodal map, defined by \( A(q) = -q \) for \( q \in S^n \). Then we have \( \mathbb{P}^n := S^n/\{id, A\} \). We know \( \pi_1(\mathbb{P}^n) = \mathbb{Z}_2 \). But with some computations, we have
\[ 0 = \left[ \widehat{F}(t, 1) \right] = 2\alpha. \]

Let \( c : \mathbb{R} \to \mathbb{P}^n \) be a geodesic curve in \( \mathbb{P}^n \). Then the length of the image \( c(\mathbb{R}) \subset \mathbb{P}^n \) is equal to or less than \( \pi_1 \), and so finite.

b) Let \( M = S^1 \times S^2 \). Since \( \pi_1(M) = \mathbb{Z} \) and \( \dim M \geq 2 \), we know that there exists a geodesic curve in \( M \) such that the length of its image in \( M \) is infinite.

c) If \( \pi_1(M) \supset \mathbb{Z} \) with \( \dim M \geq 2 \), then for each point \( p \in M \) there is a tangent vector \( v \in U_pM \) such that the complete geodesic curve \( \gamma_t(p, v) \) with \( \gamma_0(p, v) = p \) and \( \gamma'_0(p, v) = v \) has the length of its image in \( M \) to be infinite.

d) Let \( N \) be a \( k \)-dimensional compact Riemannian manifold with \( k \geq 1 \). Let \( M = S^1 \times N \). Since \( \pi_1(M) \supset \mathbb{Z} \), we must have a geodesic curve in \( M \) whose image is of infinite length.

References


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