VARIOUS INVERSE SHADOWING
IN LINEAR DYNAMICAL SYSTEMS

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ABSTRACT. In this paper, we give a characterization of hyperbolic linear dynamical systems via the notions of various inverse shadowing. More precisely it is proved that for a linear dynamical system \( f(x) = Ax \) of \( \mathbb{C}^n \), \( f \) has the \( T_h \)-inverse (\( T_h \)-orbital inverse or \( T_h \)-weak inverse) shadowing property if and only if the matrix \( A \) is hyperbolic.

1. Introduction

Consider a dynamical system generated by a homeomorphism \( f \) of a metric space \( X \) with a metric \( d \). For a point \( x \in X \), we denote by \( O(x, f) \) its orbit in the system \( f \); i.e., the set
\[
O(x, f) = \{ f^n(x) : n \in \mathbb{Z} \}.
\]

We say that a sequence \( \xi = \{ x_n \in X : n \in \mathbb{Z} \} \) is a \( \delta \)-pseudo orbit of \( f \) if the inequalities
\[
d(f(x_n), x_{n+1}) < \delta, \ n \in \mathbb{Z}
\]
hold. A \( \delta \)-pseudo orbit is a natural model of computer output in a process of numerical investigation of the system \( f \). In this case, the value \( \delta \) measures errors of the method, round-off errors, etc.

Recall that \( f \) has the shadowing property if given \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( \delta \)-pseudo orbit \( \xi = \{ x_n : n \in \mathbb{Z} \} \) we can find a point \( y \in X \) with the property
\[
d(f^n(y), x_n) < \varepsilon, \ n \in \mathbb{Z}.
\]

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Of course, if \( f \) has the shadowing property formulated above, then the results of its numerical study with a proper accuracy reflect its qualitative structure.

Let \( N(\varepsilon, A) \) be the \( \varepsilon \)-neighborhood of \( A \). It is said that \( f \) has the weak shadowing property [resp. orbital shadowing property] if given \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( \delta \)-pseudo orbit \( \xi = \{x_n\} \) of \( f \) we can find a point \( y \in X \) with the property

\[
\xi \subset N(\varepsilon, O(y, f)) \quad [\text{resp. } \xi \subset N(\varepsilon, O(y, f)) \text{ and } O(y, f) \subset N(\varepsilon, \xi)],
\]

where \( d_H \) denotes the Hausdorff distance on the set of compact subsets of \( X \). The weak shadowing property was introduced in [12] and the orbital shadowing property was introduced in [11].

Let \( X^Z \) be the space of all two sided sequences \( \xi = \{x_n : n \in \mathbb{Z}\} \) with elements \( x_n \in X \), endowed with the product topology. For \( \delta > 0 \), let \( \Phi_f(\delta) \) denote the set of all \( \delta \)-pseudo orbits of \( f \). A mapping \( \varphi : X \to \Phi_f(\delta) \subset X^Z \) is said to be a \( \delta \)-method for \( f \) if \( \varphi(x)_0 = x \), where \( \varphi(x)_0 \) denotes the 0th component of \( \varphi(x) \). Then each \( \varphi(x) \) is a \( \delta \)-pseudo orbit of \( f \) through \( x \). For convenience, write \( \varphi(x) \) for \( \{\varphi(x)_k\}_{k \in \mathbb{Z}} \). Say that \( \varphi \) is a continuous \( \delta \)-method for \( f \) if the map \( \varphi \) is continuous. The set of all \( \delta \)-methods [resp. continuous \( \delta \)-methods] for \( f \) will be denoted by \( T_0(f, \delta) \) [resp. \( T_c(f, \delta) \)]. If \( g : X \to X \) is a homeomorphism with \( d_\infty(f, g) < \delta \), where \( d_\infty(f, g) = \sup_{x \in X} \{d(f(x), g(x)), d(f^{-1}(x), g^{-1}(x))\} \), then \( g \) induces a continuous \( \delta \)-method \( \varphi_g \) for \( f \) by defining

\[
\varphi_g(x) = \{g^n(x) : n \in \mathbb{Z}\}.
\]

Let \( T_h(f, \delta) \) denote the set of all continuous \( \delta \)-methods \( \varphi_g \) for \( f \) which are induced by \( g \in Z(X) \) with \( d_\infty(f, g) < \delta \). We define \( T_\alpha(f) \) by

\[
T_\alpha(f) = \bigcup_{\delta > 0} T_\alpha(f, \delta),
\]

where \( \alpha = 0, c, h \). Clearly,

\[
T_h(f) \subset T_c(f) \subset T_0(f).
\]

The concept of inverse shadowing for homeomorphisms as a "dual" notion of shadowing property was established by Corless and Pilyugin [2], and Kloeden et al [4, 5] redefined this property using the concept of a method. We say that \( f \) has the \( T_\alpha \)-inverse shadowing property, for short \( IS_\alpha \), (\( \alpha = 0, c, h \)), if for any \( \varepsilon > 0 \) there is \( \delta > 0 \) such that for any \( \delta \)-method \( \varphi \) in \( T_\alpha(f, \delta) \) and any point \( x \in X \) there exists a point \( y \in X \) for which

\[
d(f^n(x), \varphi(y)_n) < \varepsilon, \ n \in \mathbb{Z}.
\]
Clearly we have the following relations among the various notions of inverse shadowing

\[ IS_0 \Rightarrow IS_c \Rightarrow IS_h. \]

When we study the inverse shadowing property in the qualitative theory of differentiable dynamical systems, an appropriate choice of the class of admissible pseudo orbits is crucial here ([2, 3, 5, 6, 10]). Moreover the inverse shadowing properties are not related to the shadowing property in general.

**Example 1.1.** [7] Consider the dynamical system \( f \) on the unit circle \( S^1 \) with coordinate \( x \in [0, 1) \), given by

\[ f(x) = x + \frac{1}{2\pi} \sin(2\pi x). \]

Then it has the shadowing property. Therefore it has the \( T_c \) inverse shadowing property. But it does not have the \( T_0 \) inverse shadowing property.

**Example 1.2.** [8] Pseudo-Anosov maps on a compact surface have the \( T_h \) inverse shadowing property but it does not have the shadowing property.

**Example 1.3.** [4] Let \( \{0, 1\}^\mathbb{Z} \) be the space of all two sided sequences \( x = \{x_i; n \in \mathbb{Z}\} \) with elements \( x_i \in \{0, 1\} \), endowed with a metric \( D \) defined by

\[ D(x, y) = \sup_{i \in \mathbb{Z}} \left\{ \frac{|x_i - y_i|}{2^{|i|}} \right\}, \]

where \( x, y \in \{0, 1\}^\mathbb{Z} \). We also write this space as \( \sum_2 \) to shorten the notation. Define a shift map \( \sigma : \sum_2 \to \sum_2 \) by

\[ \sigma(x)_i = x_{i+1} \quad (i \in \mathbb{Z}), \]

where \( x \in \sum_2 \). Then the shift homeomorphism \( \sigma \) is an expansive homeomorphism with the shadowing property, but it does not have the \( T_h \) inverse shadowing property.

Now we introduce the notion of weak [resp. orbital] inverse shadowing which is a "dual" notion of weak [resp. orbital] shadowing.

**Definition 1.4.** We say that \( f \) has the \( T_\alpha \)-weak [resp. \( T_\alpha \)-orbital] inverse shadowing property, for short WIS\( \alpha \) [resp. OIS\( \alpha \)], (\( \alpha = 0, c, h \)), if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( \delta \)-method \( \varphi \in T_\alpha(f, \delta) \) and any point \( x \in M \) there is a point \( y \in M \) for which

\[ \varphi(y) \subset N(\varepsilon, O(x, f)) \quad \text{[resp. } \xi \subset N(\varepsilon, O(y, f)) \text{ and } O(y, f) \subset N(\varepsilon, \xi) \text{].} \]
Clearly we have the following relations
\[ \text{WIS}_0 \Rightarrow \text{WIS}_c \Rightarrow \text{WIS}_h, \quad \text{OIS}_0 \Rightarrow \text{OIS}_c \Rightarrow \text{OIS}_h, \]
and
\[ \text{ISP}_\alpha \Rightarrow \text{OIS}_\alpha \Rightarrow \text{WIS}_\alpha \quad (\alpha = 0, c, h). \]

**Remark 1.5.** Suppose that \( \mathcal{T}_a(f) \subset \mathcal{T}_b(f) \) for \( a, b \in \{0, c, h\} \). If \( f \) has the \( \mathcal{T}_b \)-weak [resp. \( \mathcal{T}_b \)-orbital] inverse shadowing property then it has the \( \mathcal{T}_a \)-weak [resp. \( \mathcal{T}_a \)-orbital] inverse shadowing property. We can easily show that every irrational rotation \( f \) on the unit circle \( S^1 \) has the \( \mathcal{T}_c \)-weak (or \( \mathcal{T}_h \)-inverse) inverse shadowing property, but it does not have the \( \mathcal{T}_c \)-inverse (or \( \mathcal{T}_h \)-inverse) shadowing property. Furthermore we can show that every rational rotation on the unit circle has the \( \mathcal{T}_c \)-orbital inverse shadowing property, but it does not have the \( \mathcal{T}_c \)-weak inverse shadowing property. It can be checked that every shift homeomorphism does not have the \( \mathcal{T}_c \)-weak inverse shadowing property. Moreover Choi et al. [1] showed that the \( \mathcal{T}_h \)-weak inverse shadowing property is generic in the space of homeomorphisms on a compact metric space with the \( C^0 \) topology.

### 2. Main theorem

Let \( A \) be a nonsingular matrix on \( \mathbb{C}^n \). We consider the dynamical system \( f(x) = Ax \) of \( \mathbb{C}^n \). We say that the matrix \( A \) is called **hyperbolic** if the spectrum does not intersect the circle \( \{ \lambda : |\lambda| = 1 \} \).

**Lemma 2.1.** Let \((X,d)\) be a metric space. Assume that for two dynamical systems \( f \) and \( g \) on \( X \) there exists a homeomorphism \( h \) on \( X \) such that \( h \) and \( h^{-1} \) are Lipschitz, and \( f \circ h = h \circ g \). Then \( f \) has the \( \mathcal{T}_h \)-weak inverse shadowing property [resp. \( \mathcal{T}_h \)-inverse shadowing property] if and only if \( g \) has the \( \mathcal{T}_h \)-weak inverse shadowing property [resp. \( \mathcal{T}_h \)-inverse shadowing property].

**Proof.** We prove the lemma only for the case of the \( \mathcal{T}_h \)-weak inverse shadowing property.

Assume that \( f \) has the \( \mathcal{T}_h \)-weak inverse shadowing property, and let \( \varepsilon > 0 \) be arbitrary. Find \( \varepsilon_1 > 0 \) such that the inequality \( d(x, y) < \varepsilon_1, \ x, y \in X \), implies that \( d(h^{-1}(x), h^{-1}(y)) < \varepsilon \). Take \( \delta_1 > 0 \) corresponding to \( \varepsilon_1 \) by the assumption of the \( \mathcal{T}_h \)-inverse shadowing property of \( f \), and choose \( \delta > 0 \) such that \( d(x, y) < \delta \) implies \( d(h(x), h(y)) < \delta_1 \).
Let $\tilde{g}$ be a $\delta$-perturbation of $g$, i.e., $d_\infty(\tilde{g}, g) < \delta$, and let $x \in X$. Put $\tilde{f} = h \circ \tilde{g} \circ h^{-1}$. Then $d_\infty(h \circ \tilde{g} \circ h^{-1}, h \circ g \circ h^{-1}) = d_\infty(\tilde{f}, f) < \delta_1$. By the $T_h$-inverse shadowing property of $f$, for the given $h(x)$, there exists a point $y \in X$ such that for any $k \in \mathbb{Z}$, we choose $n(k) \in \mathbb{Z}$ satisfying the inequality
\[d(\tilde{f}^{n(k)}(y), f^k(h(x))) < \varepsilon_1.\]

Here we know that $f \circ h = h \circ g$ implies $h^{-1} \circ f^k = g^k \circ h^{-1}$ and $h^{-1} \circ \tilde{f}^k = \tilde{g}^k \circ h^{-1}$ for any $k \in \mathbb{Z}$.

This shows that for any $k \in \mathbb{Z}$, we can choose $n(k) \in \mathbb{Z}$ satisfying the inequality
\[d(\tilde{g}^{n(k)}(h^{-1}(y)), g^k(x)) < \varepsilon, \quad k \in \mathbb{Z}.\]
This means that $g$ has the $T_h$-weak inverse shadowing property. \hfill $\square$

**Lemma 2.2.** Let $(X, d)$ be a metric space. If the dynamical system $f^m(x) = A^m x$ ($m \in \mathbb{N}$) on $X$ has the $T_h$-weak inverse shadowing property, then the dynamical system $f(x) = Ax$ on $X$ has the $T_h$-weak inverse shadowing property.

**Proof.** Assume that the dynamical system $f^m$ has the $T_h$-weak inverse shadowing property. Let $\varepsilon > 0$ be arbitrary and $L$ be a Lipschitz constant of $f$. Take $0 < \varepsilon_1 < \min\{\frac{\varepsilon}{L}, \varepsilon\}$ such that $d(x, y) < \varepsilon_1 \Rightarrow d(f^i(x), f^i(y)) < \frac{\varepsilon}{m} \quad (1 \leq i \leq m)$.

Choose $\delta_1 > 0$ corresponding to $\varepsilon_1$ by the assumption of the $T_h$-weak inverse shadowing property of $f^m$. Now we find $0 < \delta < \min\{\frac{\varepsilon_1}{m}, \varepsilon_1\}$ such that
\[d_\infty(g, f) < \delta \Rightarrow d_\infty(g^i, f^i) < \frac{\delta_1}{m} \quad (1 \leq i \leq m).\]

Let $g$ be a $\delta$-perturbation of $f$, i.e., $d_\infty(g, f) < \delta$, and let $x \in X$. Then $g^m$ be a $\delta_1$-perturbation of $f^m$. By the $T_h$-weak inverse shadowing property of $f^m$, there exists $y \in X$ such that for any $k \in \mathbb{Z}$, we choose $n(k) \in \mathbb{Z}$ satisfying the inequality
\[d((f^m)^{n(k)}(x), (g^m)^k(y)) < \varepsilon_1.\]

Then for any $k \in \mathbb{Z}$ and $0 \leq j \leq m$,
\[d(f^{m-n(k)+j}(x), g^{m-k+j}(y)) < \varepsilon_1.\]
Hence we can easily show that for any $l \in \mathbb{Z}$, we choose $t(l) \in \mathbb{Z}$ satisfying the inequalities
\[d(f^{t(l)}(x), g^l(y)) < \varepsilon, \quad l \in \mathbb{Z}.\]
This means that $f$ has the $T_h$-weak inverse shadowing property. \hfill \Box

**Lemma 2.3.** [9] Let $A$ be a hyperbolic matrix on $\mathbb{C}^n$. Then there exists $C > 0$, a natural number $m$, $0 < \lambda < 1$, invariant linear subspaces $S(p)$ and $U(p)$ of $T_p\mathbb{C}^n$ for $p \in \mathbb{C}^n$ such that

1. $T_p\mathbb{C}^n = S(p) \oplus U(p)$;
2. $|A^{mk}(v)| < C\lambda^k|v|$, $v \in S(p)$, $k \geq 0$;
3. $|A^{-mk}(v)| < C\lambda^{-k}|v|$, $v \in U(p)$, $k < 0$;
4. If $P(p)$ and $Q(p)$ are the projectors in $T_p\mathbb{C}^n$ onto $S(p)$ parallel to $U(p)$ and onto $U(p)$ parallel to $S(P)$ with the property $P(p) + Q(p) = I(p)$, then

$$||P(p)|| \text{ and } ||Q(p)|| \leq C.$$  

**Lemma 2.4.** [9] Let $A$ be a non-hyperbolic matrix, and $\lambda$ be an eigenvalue of $A$ with $|\lambda| = 1$. Then there exists a nonsingular matrix $T$ such that $J = T^{-1}AT$ is a Jordan form of $A$ and the matrix $J$ has the form

$$
\begin{pmatrix}
B & 0 \\
C & D
\end{pmatrix}
$$

where $B$ is the nonsingular $m \times m$ complex matrix with the form

$$
\begin{pmatrix}
\lambda & 0 & \cdots & 0 & 0 \\
1 & \lambda & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & \lambda
\end{pmatrix}
$$

**Lemma 2.5.** [Schauder-Tychonoff Theorem] Let $\Lambda$ be a closed, convex set in a Banach space and $f : \Lambda \to \Lambda$ a continuous function. If $\bar{f}(\Lambda)$ is compact, then $f$ has a fixed point.

**Theorem 2.6.** For a linear dynamical system $f(x) = Ax$ of $\mathbb{C}^n$, the following conditions are mutually equivalent:

1. $f$ has the $T_h$-inverse shadowing property,
2. $f$ has the $T_h$-orbital inverse shadowing property,
3. $f$ has the $T_h$-weak inverse shadowing property,
4. The matrix $A$ is hyperbolic.

**Proof.** By the definition, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold. We prove that (3) $\Rightarrow$ (4) and that (4) $\Rightarrow$ (1).

Proof of (3) $\Rightarrow$ (4): Assume that $f$ has the $T_h$-weak inverse shadowing property. To obtain a contradiction, assume that the matrix $A$ has an
eigenvalue $\lambda$ such that $|\lambda|=1$. Lemma 2.4 shows that there exists a nonsingular matrix $T$ such that $J = T^{-1}AT$ is a Jordan form of $A$ and the matrix $J$ has the form

\[
\begin{pmatrix}
B & 0 \\
C & D
\end{pmatrix}
\]

where $B$ is the nonsingular $m \times m$ complex matrix with the form

\[
\begin{pmatrix}
\lambda & 0 & \ldots & 0 & 0 \\
1 & \lambda & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & \lambda
\end{pmatrix}
\]

Then, for the dynamical system $g(x) = J(x)$ and the homeomorphism $h(x) = T(x)$, the equality $f \circ h = h \circ g$ holds. Since the homeomorphisms $h$ and $h^{-1}$ are Lipschitz in $\mathbb{C}^n$, Lemma 2.1 implies that $g$ has the $T_h$-weak inverse shadowing property. Let $\delta > 0$ corresponding to $\varepsilon = 1$ by the definition of the $T_h$-weak inverse shadowing property of $g$. Denote by $x_i$ the $i$-th component of a vector $x \in \mathbb{C}^n$. We fix a point $w \in \mathbb{C}^n$ with $|w_1| = 3$ and construct a $\delta$-perturbation $\tilde{g}$ of $g$ as follows:

\[
\tilde{g}(x_1, \ldots, x_n) = \left( \lambda x_1 \left( 1 + \frac{\delta}{2|x_1|} \right), (Jx)_2, \ldots, (Jx)_n \right).
\]

Let $y = (y_1, \ldots, y_n)$ be an arbitrary vector in $\mathbb{C}^n$. Since for $k \to \infty$, $(\tilde{g}(y))_1^k$ leaves on the 1-neighborhood of $S_3 = \{ x_1 \in \mathbb{C} : |x_1| = 3 \}$, there exists $k(y) \in \mathbb{N}$ such that $(\tilde{g}(y))_1^{k(y)}$ leaves on 1-neighborhood of $S_3$. This means that $\tilde{g}^{k(y)}(y)$ leaves on 1-neighborhood of $O(w, g)$. Hence we show that $g$ does not have the $T_h$-weak inverse shadowing property, and so the contradiction completes the proof.

Proof of (4) $\Rightarrow$ (1): Assume that the matrix $A$ is hyperbolic. It suffices to show that $f(x) = Ax$ has the Lipschitz $T_h$-inverse shadowing property, i.e., there exist positive numbers $\delta_0$ and $L$ such that for if $g$ is a $\delta$-perturbation of $f$ with $\delta < \delta_0$, then for any $p \in \mathbb{C}^n$ there exists a point $x_0 \in \mathbb{C}^n$ satisfying the inequalities

\[
|g^k(x_0) - f^k(p)| < L\delta, \quad k \in \mathbb{Z}.
\]

Denote by $S(p)$ the invariant subspace of $T_p\mathbb{C}^n$ corresponding to the eigenvalues $\lambda_j$ of $A$ such that $|\lambda_j| < 1$, and by $U(p)$ the invariant subspace of $T_p\mathbb{C}^n$ corresponding to the eigenvalues $\lambda_j$ of $A$ such that $|\lambda_j| > 1$. By Lemma 2.3, there exist $C > 0$, a natural number $m$,
0 < \lambda < 1$, invariant linear subspaces $S(p)$ and $U(p)$ of $T_p\mathbb{C}^n$ for $p \in \mathbb{C}^n$ such that

(a1) $T_p\mathbb{C}^n = S(p) \oplus U(p)$;

(a2) $|A^{mk}(v)| < C\lambda^k|v|$, $v \in S(p)$, $k \geq 0$;

(a3) $|A^{-mk}(v)| < C\lambda^{-k}|v|$, $v \in U(p)$, $k < 0$;

(a4) If $P(p)$ and $Q(p)$ are the projectors in $T_p\mathbb{C}^n$ onto $S(p)$ parallel to $U(p)$ and onto $U(p)$ parallel to $S(P)$ with the property $P(p) + Q(p) = I(p)$, then

$$||P(p)||, ||Q(p)|| \leq C.$$

By Lemma 2.2, it is enough to show that $f^m(x) = A^n(x)$ has the $T_h$-inverse shadowing property. To simplify the notations, we assume that the inequalities (a2) and (a3) hold with $m = 1$ (another possibility holds similarly.)

Fix a point $p \in \mathbb{C}^n$ and identify the tangent space $T_p\mathbb{C}^n$ with the linear space of $\mathbb{C}^n$. For a point $x \in \mathbb{C}^n$, we define a mapping $a_p : \mathbb{C}^n \to T_p\mathbb{C}^n$ by $a_p(x) = (x - p)_p$. It is easy to see that the following statements hold:

(b1) the mapping $a_p : \mathbb{C}^n \to T_p\mathbb{C}^n$ is continuous;

(b2) $|a_p(x) - a_p(y)| \leq |x - y|$ for $x, y \in \mathbb{C}^n$;

(b3) there exists a positive number $r'(\text{independent of } p)$ such that $a_p$ is a diffeomorphism of the set

$$B_{r'}(p) = \{ x \in \mathbb{C}^n : |x - p| < r' \}$$

onto its image for which $Da_p(p) = I$ and

$$|a_p^{-1}(v) - a_p^{-1}(v')| \leq 2|v - v'|$$

for $v, v' \in a_p(B_{r'}(p))$.

In formula (2.1) and below, for $v \in a_p(B_{r'}(p))$, we denote by $a_p^{-1}(v)$ the unique point $x \in B_{r'}(p)$ such that $a_p(x) = v$.

Take

$$L = 4L_0 + 1,$$

where $L_0 = C^{2\frac{1}{1-\lambda}}$. For $r > 0$, denote $W_r(p) = \{ v \in T_p\mathbb{C}^n : |v| \leq r \}$. It is easy to see that we can choose a positive number $r < r'$ (where $r'$ is from the property (b3) of the mappings $a_p$) such that, for any $p \in \mathbb{C}^n$, the inclusions $W_r(p) \subset a_p(B_{r'}(p))$ hold, hence the mappings

$$F_p = a_{f(p)} \circ f \circ a_p^{-1}$$

are defined on $W_r(p)$. We assume that, for the chosen $r$, any mapping $F_p$ can be represented as

$$F_p(v) = A(v) + G(v),$$
where

\[(2.3) \quad |G(v)| \leq \frac{1}{2L_0} \quad \text{for} \quad v \in W_r(p).\]

We take

\[\delta < \delta_0 = \frac{r}{2L_0}\]

and fix a \(\delta\)-perturbation \(g\) of \(f\), i.e., \(d_\infty(g, f) < \delta\), and \(p \in \mathbb{C}^n\). We denote \(p_k = f^k(p)\) and \(g_k = g\). We introduce the following mappings defined for \(v \in W_r(p_k); G_k\) are the mappings in the representation (2.2) for the points \(p_k\),

\[\Phi_k = a_{p_{k+1}} \circ f \circ a_{p_k}^{-1} \quad \text{and} \quad \Psi_k = a_{p_{k+1}} \circ g_k \circ a_{p_k}^{-1}.\]

Let \(E\) be the space of sequences

\[V = \{v_k \in T_{p_k} \mathbb{C}^n : k \in \mathbb{Z}\}\]

such that \(||V||_\infty = \sup_{|k| < \infty} |v_k| \leq 2L_0\delta.\]

For a natural number \(m\), we introduce the space \(E_m\) of sequences

\[V = \{v_k \in T_{p_k} \mathbb{C}^n : |k| \leq m\}\]

with the norm

\[||V||_m = \max_{|k| \leq m} |v_k| \leq 2L_0\delta.\]

Denote by \(\pi_m\) and \(\pi_m^l, m \leq l\), the natural projectors of \(E\) to \(E_m\) and of \(E_l\) to \(E_m\), respectively. For a sequence \(V \in E\), let \(Z(V) = \{z_k(V)\}\), where

\[z_{k+1}(V) = G_k(v_k) + \Psi_k(v_k) - \Phi_k(v_k).\]

Since \(|f(x) - g_k(x)| < \delta\) for all \(x\) and \(k\), and \(v_k \in W_r(p_k)\) by the definition of the space \(E\) and by our choice of \(\delta\), it follows from (b2) and (2.3) that

\[(2.4) \quad ||Z(V)||_\infty < \frac{1}{2L_0} ||V||_\infty + d.\]

Define an operator \(R\) on the space \(E\) as follows : \(R(V) = \{w_k\}\), where

\[(2.5) \quad w_k = \sum_{i=-\infty}^{k} A^{k-i}(p_i)P(p_i)z_i(V) - \sum_{i=k+1}^{\infty} A^{k-i}(p_i)Q(p_i)z_i(V).\]

The inequalities (a2)-(a4) show that

\[||R(V)||_\infty \leq L_0 ||Z(V)||_\infty,\]

hence it follows from (2.4) that \(R\) maps \(E\) into itself.

Now it suffices to show that the operator \(R\) has a fixed point in \(E\). Consider the space \(E\) with the topology of uniform convergence on
compact subsets of $Z$. For a natural number $m$, we define the operator $R_m : E \to E_m$ by

$$R_m(V) = \{w_k : |k| \leq m\},$$

where

$$w_k = \sum_{i=-m}^{k} A^{k-i} P(p_i)z_i(V) - \sum_{i=k+1}^{m} A^{k-i} Q(p_i)z_i(V).$$

Since the values $z_k(V), |k| \leq m$, are determined by the values $v_k, |k| \leq m + 1$, each operator $R_m$ is continuous.

The operator $\pi_m R$ maps a sequence $V \in E$ to the sequence $\{w_k : |k| \leq m\}$, where the $w_k$ are given by formula (2.5). Fix a number $l > m$ and consider the operator $\pi_m^l R_l$ mapping a sequence $V \in E$ to the sequence $\{w'_k : |k| \leq m\}$, where

$$w'_k = \sum_{i=-l}^{k} A^{k-i} P(p_i)z_i(V) - \sum_{i=k+1}^{l} A^{k-i} Q(p_i)z_i(V).$$

Let us estimate

$$||\pi_m R(V) - \pi_m^l R_l(V)||_m = \max_{|k| \leq m} |w_k - w'_k|$$

$$\leq 2L_0 C^2 d \max_{|k| \leq m} \left( \sum_{i=-\infty}^{-l-1} \lambda^{k-i} + \sum_{i=l+1}^{\infty} \lambda^{i-k} \right)$$

$$\leq 4L_0 C^2 d \lambda^{1-m} \frac{1 - \lambda^l}{1 - \lambda} \lambda^l.$$

This estimate implies that the operator $\pi_m R$ is the uniform limit (as $l \to \infty$) of the continuous operators $\pi_m^l R_l$, hence the operator $\pi_m R$ is continuous. It follows from our choice of topology of the space $E$ that the operator $R$ is continuous. It is easy to see that the image $R(E)$ is relatively compact in $E$. Since $R$ maps $E$ into itself, Lemma 2.5 implies the existence of a fixed point of $R$ in $E$.

If $V = R(V)$ for some $V \in E$, then

$$v_{k+1} = Av_k + z_{k+1}(V) = Av_k + G_k(v_k) + \Psi_k(v_k) - \Phi_k(v_k),$$

i.e., $v_{k+1} = \Psi_k(v_k)$. This means that, for the sequence of points $x_k = a_{p_k}^{-1}(v_k)$, the equalities $x_{k+1} = g_k(x_k)$ hold. The inclusion $V \in E$ and
the property (b3) of the mappings $a_p$ imply the inequalities
\[
|g^k(x_0) - f^k(p)| = |x_k - p_k| = |a^{-1}_{p_k}(v_k) - a^{-1}_{p_k}(0_{p_k})| \\
\leq |v_k - 0_{p_k}| \leq 4L_0\delta < L\delta.
\]

Therefore, $f$ has the Lipschitz $T_n$-inverse shadowing property, and so completes the proof. \qed

**Remark 2.7.** Remark 2.1 in [11] and Theorem 3.2.1 in [9] say that, for a linear dynamical system $f(x) = Ax$ of $\mathbb{C}^n$, the following conditions are mutually equivalent:

1. $f$ has the shadowing property,
2. $f$ has the orbital shadowing property,
3. $f$ has the weak shadowing property,
4. the matrix $A$ is hyperbolic.

**References**


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