ROUGHNESS IN SUBTRACTION ALGEBRAS

SUN SHIN AHN, YOUNG BAE JUN, AND KYOUNG JA LEE

Abstract. As a generalization of ideals in subtraction algebras, the notion of rough ideals is discussed.

1. Introduction

B. M. Schein [10] considered systems of the form \((\Phi; \circ, \setminus)\), where \(\Phi\) is a set of functions closed under the composition \("\circ\"") of functions (and hence \((\Phi; \circ)\) is a function semigroup) and the set theoretic subtraction \("\setminus\"") (and hence \((\Phi; \setminus)\) is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [11] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [4] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [3], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. Y. B. Jun and K. H. Kim [5] introduced the notion of prime and irreducible ideals of a subtraction algebra, and gave a characterization of a prime ideal. They also provided a condition for an ideal to be a prime/irreducible ideal. In 1982, Pawlak introduced the concept of a rough set (see [8]). This concept is fundamental for the examination of granularity in knowledge. It is a concept which has many applications in data analysis (see [9]). Rough set theory is applied to semigroups and groups (see [6, 7]). In this paper, we apply the rough set theory to subtraction algebras, and we introduce the notion of upper/lower rough subalgebras/ideals which is an extended notion of subalgebras/ideals in a subtraction algebra.
2. Preliminaries

By a subtraction algebra we mean an algebra $(X; -)$ with a single binary operation "-" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on $X$: $a \leq b \iff a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$b \lor c = (b' \wedge c')' = a - ((a - b) \wedge (a - c)) = a - ((a - b) - ((a - b) - (a - c))).$$

In a subtraction algebra, the following are true (see [4, 5]):

(a1) $(x - y) - y = x - y$.
(a2) $x - 0 = x$ and $0 - x = 0$.
(a3) $(x - y) - x = 0$.
(a4) $x - (x - y) \leq y$.
(a5) $(x - y) - (y - x) = x - y$.
(a6) $x - (x - (x - y)) = x - y$.
(a7) $(x - y) - (z - y) \leq x - z$.
(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
(a10) $x, y \leq z$ implies $x - y = x \land (z - y)$.
(a11) $(x \land y) - (x \land z) \leq x \land (y - z)$.

A nonempty subset $S$ of a subtraction algebra $X$ is called a subalgebra of $X$ if $x - y \in S$ whenever $x, y \in S$.

A nonempty subset $A$ of a subtraction algebra $X$ is called an ideal of $X$, denoted by $A \triangleleft X$, if it satisfies

- $0 \in A$
- $(\forall x \in X)(\forall y \in A)(x - y \in A \Rightarrow x \in A)$.

Note that every ideal of a subtraction algebra $X$ is a subalgebra of $X$. 

Lemma 2.1. [5] An ideal $A$ of a subtraction algebra $X$ has the following property:

$$(\forall x \in X)(\forall y \in A)(x \leq y \Rightarrow x \in A).$$

3. Rough sets in subtraction algebras

In what follows let $X$ denote a subtraction algebra unless otherwise specified.

An equivalence relation $\rho$ on $X$ is called a congruence relation on $X$ if whenever $(x, y), (u, v) \in \rho$ then $(x - u, y - v) \in \rho$. We denote by $[a]_\rho$ the $\rho$-congruence class containing the element $a \in X$. Let $X/\rho$ denote the set of all $\rho$-congruence classes on $X$, i.e., $X/\rho := \{[a]_\rho \mid a \in X\}$. For any $[x]_\rho, [y]_\rho \in X/\rho$, if we define $[x]_\rho - [y]_\rho = [x - y]_\rho$, then $(X/\rho, -)$ is a subtraction algebra. Let $\rho$ be an equivalence relation on $X$ and let $\mathcal{P}(X)$ denote the power set of $X$ and $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$. For all $x \in X$, let $[x]_\rho$ denote the equivalence class of $x$ with respect to $\rho$. Define the functions $\rho_*, \rho^* : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ as follows: $\forall S \in \mathcal{P}(X)$,

$$\rho_*(S) = \{x \in X \mid [x]_\rho \subseteq S\} \text{ and } \rho^*(S) = \{x \in X \mid [x]_\rho \cap S \neq \emptyset\}.$$

$\rho_*(S)$ is called the $\rho$-lower approximation of $S$ while $\rho^*(S)$ is called the $\rho$-upper approximation of $S$. For a nonempty subset $S$ of $X$,

$$\rho(S) = (\rho_*(S), \rho^*(S))$$

is called a rough set with respect to $\rho$ of $\mathcal{P}(X) \times \mathcal{P}(X)$ if $\rho_*(S) \neq \rho^*(S)$.

A subset $S$ of $X$ is said to be definable if $\rho_*(S) = \rho^*(S)$. The pair $(X, \rho)$ is called an approximation space.

The following property is useful for our research (cf. [8]).

Proposition 3.1. Let $\rho$ and $\lambda$ be congruence relations on $X$. Then the following assertions are true.

1. $(\forall F \in \mathcal{P}^*(X)) \ (\rho_*(F) \subseteq F \subseteq \rho^*(F))$,
2. $(\forall F, G \in \mathcal{P}^*(X)) \ (\rho^*(F \cup G) = \rho^*(F) \cup \rho^*(G))$,
3. $(\forall F, G \in \mathcal{P}^*(X)) \ (\rho_*(F \cap G) = \rho_*(F) \cap \rho_*(G))$,
4. $(\forall F, G \in \mathcal{P}^*(X)) \ (F \subseteq G \Rightarrow \rho_*(F) \subseteq \rho_*(G))$,
(5) \((\forall F, G \in \mathcal{P}^*(X)) (F \subseteq G \Rightarrow \rho^*(F) \subseteq \rho^*(G)),\)
(6) \((\forall F, G \in \mathcal{P}^*(X)) (\rho_*(F) \cup \rho_*(G) \subseteq \rho_*(F \cup G)),\)
(7) \((\forall F, G \in \mathcal{P}^*(X)) (\rho^*(F \cap G) \subseteq \rho^*(F) \cap \rho^*(G)),\)
(8) \((\forall F \in \mathcal{P}^*(X)) (\rho \subseteq \lambda \Rightarrow \lambda_*(F) \subseteq \rho_*(F), \ \rho^*(F) \subseteq \lambda^*(F)).\)

**Proof.** Straightforward. \(\square\)

**Corollary 3.2.** If \(\rho\) and \(\lambda\) are congruence relations on \(X\), then

(i) \((\forall F \in \mathcal{P}^*(X)) ((\rho \cap \lambda)^*(F) \subseteq \rho^*(F) \cap \lambda^*(F)).\)

(ii) \((\forall F \in \mathcal{P}^*(X)) (\rho_*(F) \cap \lambda_*(F) \subseteq (\rho \cap \lambda)_*(F)).\)

**Proof.** It follows immediately from Proposition 3.1. \(\square\)

For any \(F, G \in \mathcal{P}^*(X)\), we define \(F - G := \{a - b \mid a \in F, b \in G\}\).

**Theorem 3.3.** If \(\rho\) is a congruence relation on \(X\), then

\[(\forall F, G \in \mathcal{P}^*(X)) (\rho^*(F) - \rho^*(G) \subseteq \rho^*(F - G)).\]

**Proof.** Let \(c \in \rho^*(F) - \rho^*(G)\). Then there exist \(a \in \rho^*(F)\) and \(b \in \rho^*(G)\) such that \(c = a - b\). It follows that \([a]_\rho \cap F \neq \emptyset\) and \([b]_\rho \cap G \neq \emptyset\) so that \(x \in [a]_\rho \cap F\) and \(y \in [b]_\rho \cap G\) for some \(x, y \in X\). Hence \(x - y \in [a]_\rho - [b]_\rho = [a - b]_\rho\) and \(x - y \in F - G\), that is, \(x - y \in [a - b]_\rho \cap (F - G)\). Thus \(c = a - b \in \rho^*(F - G)\), and so \(\rho^*(F) - \rho^*(G) \subseteq \rho^*(F - G)\). \(\square\)

**Theorem 3.4.** If \(\rho\) is a congruence relation on \(X\), then

\[(\forall F, G \in \mathcal{P}^*(X)) (\rho_*(F - G) \neq \emptyset \Rightarrow \rho_*(F) - \rho_*(G) \subseteq \rho_*(F - G)).\]

**Proof.** Let \(c \in \rho_*(F) - \rho_*(G)\). Then \(c = a - b\) for some \(a \in \rho_*(F)\) and \(b \in \rho_*(G)\). Thus we get \([a]_\rho \subseteq F\) and \([b]_\rho \subseteq G\). It follows that

\([a - b]_\rho = [a]_\rho - [b]_\rho \subseteq F - G\)

so that \(c = a - b \in \rho_*(F - G)\). Therefore the result is valid. \(\square\)

The following example shows the condition that \(\rho_*(F - G) \neq \emptyset\) in Theorem 3.4 is necessary.

**Example 3.5.** Let \(X = \{0, a, b, c\}\) be a subtraction algebra with the following Cayley table:

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<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
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<tr>
<td>c</td>
<td>c</td>
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<td>0</td>
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</table>
Let $\rho$ be a congruence relation on $X$ such that $\{0,a\}$, $\{b\}$, and $\{c\}$ are all $\rho$-congruences of $X$. Taking $F = \{b,c\}$ and $G = \{c\}$, we have $F - G = \{0\}$, $\rho_*(F - G) = \emptyset$, $\rho_*(F) = \{b,c\}$, $\rho_*(G) = \{c\}$, and $\rho_*(F) - \rho_*(G) = \{0\}$.

For any congruence relation $\rho$ on $X$, we note that

- $(\forall F \in \mathcal{P}^*(X)) \ (\rho_*(F) \subseteq F)$,
- $(\forall F, G \in \mathcal{P}^*(X)) \ (F \subseteq G \Rightarrow \rho_*(F) \subseteq \rho_*(G))$,
- $(\forall F \in \mathcal{P}^*(X)) \ (\rho_*(\rho_*(F)) = \rho_*(F))$,

which means that $\rho_*$ is an interior operator on $X$. This operation induces a topology $\mathcal{T}$ on $X$ such that

$$F \in \mathcal{T} \iff \rho_*(F) = F.$$  

**Lemma 3.6.** For any congruence relation $\rho$ on $X$, $\rho^*$ is a closure operator on the topological space $(X, \mathcal{T})$.

**Proof.** For any $F \in \mathcal{P}^*(X)$ we have

$$x \in \rho^*(F) \iff [x]_{\rho} \cap F \neq \emptyset \iff [x]_{\rho} \subseteq F^c \iff x \notin \rho_*(F^c) \iff x \in (\rho_*(F^c))^c,$$

that is, $\rho^*(F) = (\rho_*(F^c))^c$, which completes the proof. \qed

**Lemma 3.7.** For any congruence relation $\rho$ on $X$, we have

(i) $(\forall F \in \mathcal{P}(X)) \ (\rho_*(F) = F \iff \rho^*(F^c) = F^c)$,
(ii) $(\forall F \in \mathcal{P}(X)) \ (\rho_*(F) = F \iff \rho^*(F) = F)$.

**Proof.** Straightforward. \qed

Based on the above two lemmas we have the following result.

**Theorem 3.8.** For any $F \subseteq X$ and a congruence relation $\rho$ on $X$, the following assertions are equivalent.

(i) $F$ is definable with respect to $\rho$.
(ii) $F$ is open in the topological space $(X, \mathcal{T})$.
(iii) $F$ is closed in the topological space $(X, \mathcal{T})$.

According to [7], we say that an open set $F$ of $X$ is said to be free in an approximation space $(X, \rho)$ if $x \notin \rho^*(F \setminus \{x\})$ for all $x \in X$. Since $\rho^*(F \setminus \{x\}) = (\rho_*(F \setminus \{x\}))^c$, a nonempty subset $F$ of $X$ is free if and only if $x \in \rho_*(F^c \cup \{x\})$, i.e., if and only if $[x]_{\rho} \subseteq F^c \cup \{x\}$ for every $x \in F$. Thus for a free subset $F$ and any $(x,y) \in \rho \cap (F \times F)$ we have $y \in F$, which together with $y \in [x]_{\rho} \subseteq F^c \cup \{x\}$ implies that $y = x$. Therefore $\rho \cap (F \times F) = \{(a,a) \mid a \in F\}$. Conversely, let

$$\rho \cap (F \times F) = \{(a,a) \mid a \in F\}$$
and let $y$ be an arbitrary element of $[x]_{\rho}$. If $y \in F$, then $y = x$, i.e., $y \in \{x\} \subseteq F^c \cup \{x\}$. If $y \notin F$, then $y \in F^c \subseteq F^c \cup \{x\}$. Thus, in each case $[x]_{\rho} \subseteq F^c \cup \{x\}$, which means that $F$ is free. Consequently, we obtain the following characterization of free subsets.

**Theorem 3.9.** $F \subseteq X$ is free if and only if $\rho \cap (F \times F) = \{(a, a) \mid a \in F\}$.

**Corollary 3.10.** If $X$ is free, then any subset of $X$ is free.

### 4. Roughness of ideals

Let $A$ be an ideal of $X$. Define a relation $\mathcal{R}$ on $X$ by

$$(\forall x, y \in X) \ ((x, y) \in \mathcal{R} \iff x - y \in A, y - x \in A).$$

Then $\mathcal{R}$ is an equivalence relation on $X$ related to an ideal $A$ of $X$. Moreover $\mathcal{R}$ satisfies

$$(\forall x, y, u, v \in X) \ ((x, y) \in \mathcal{R}, (u, v) \in \mathcal{R} \Rightarrow (x - u, y - v) \in \mathcal{R}).$$

Hence $\mathcal{R}$ is a congruence relation on $X$. Let $A_x$ denote the equivalence class of $x$ with respect to the equivalence relation $\mathcal{R}$ related to the ideal $A$ of $X$, and $X/A$ denote the collection of all equivalence classes, that is, $X/A = \{A_x \mid x \in X\}$. Then $A_0 = A$. If $A_x \ominus A_y$ is defined as the class containing $x - y$, that is, $A_x \ominus A_y = A_{x-y}$, then it is easy to verify that $(X/A, -, A_0)$ is a subtraction algebra. Let $\mathcal{R}$ be an equivalence relation on $X$ related to an ideal $A$ of $X$. For any nonempty subset $S$ of $X$, the lower and upper approximations of $S$ are denoted by $\underline{\mathcal{R}}(A; S)$ and $\overline{\mathcal{R}}(A; S)$ respectively, that is,

$$\underline{\mathcal{R}}(A; S) = \{x \in X \mid A_x \subseteq S\} \text{ and } \overline{\mathcal{R}}(A; S) = \{x \in X \mid A_x \cap S \neq \emptyset\}.$$ 

If $A = S$, then $\underline{\mathcal{R}}(A; S)$ and $\overline{\mathcal{R}}(A; S)$ are denoted by $\underline{\mathcal{R}}(A)$ and $\overline{\mathcal{R}}(A)$, respectively.

**Example 4.1.** (1) Let $X = \{0, a, b, c\}$ be a set with the Cayley table as follows:

<table>
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<th>$a$</th>
<th>$b$</th>
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<td>$c$</td>
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</table>
Then \((X, -, 0)\) is a subtraction algebra. Consider an ideal \(A = \{0, a\}\) of \(X\) and let \(\mathcal{R}\) be an equivalence relation on \(X\) related to \(A\). Then \(A_0 = A_3 = A, A_2 = \{b\}\), and \(A_1 = \{c\}\). Hence

\[
\begin{align*}
\mathcal{R}(A; \{0, b\}) &= \{b\} = \mathcal{R}(A; \{b\}), & \mathcal{R}(A; \{0\}) &= \emptyset = \mathcal{R}(A; \{a\}), \\
\mathcal{R}(A; \{0, c\}) &= \{c\} = \mathcal{R}(A; \{c\}), & \mathcal{R}(A; \{0, a, c\}) &= \{0, a, c\} \triangleleft X, \\
\mathcal{R}(A; \{0, a, b\}) &= \{0, a, b\} \triangleleft X, & \mathcal{R}(A; \{0, b, c\}) &= \{b, c\}, \\
\mathcal{R}(A; \{0, a\}) &= \{0, a\} \triangleleft X, & \mathcal{R}(A; \{0, b\}) &= \{0, a, b\} \triangleleft X, \\
\mathcal{R}(A; \{c\}) &= \{0, a, c\} \triangleleft X, & \mathcal{R}(A; \{0, a\}) &= A \triangleleft X, \\
\mathcal{R}(A; \{a\}) &= A \triangleleft X, & \mathcal{R}(A; \{b\}) &= \{b\}.
\end{align*}
\]

(2) Let \(X = \{0, a, b, c, d\}\) be a subtraction algebra with the Cayley table as follows:

\[
\begin{array}{ccccc}
    & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & 0 \\
b & b & b & 0 & 0 \\
c & c & b & a & 0 \\
d & d & d & d & d \\
\end{array}
\]

Consider \(A = \{0, b, d\} \triangleleft X\) and let \(\mathcal{R}\) be an equivalence relation on \(X\) related to \(A\). Then the equivalence classes are as follows: \(A_0 = A_4 = A_d = A, A_2 = \{a, c, d\}\), and \(A_1 = \{a\}\). Thus

\[
\begin{align*}
\mathcal{R}(A; \{0, a\}) &= \emptyset, & \mathcal{R}(A; \{0, b, c\}) &= \emptyset, \\
\mathcal{R}(A; \{0, a, d\}) &= \emptyset, & \mathcal{R}(A; \{0, a, c\}) &= \{c\}, \\
\mathcal{R}(A; \{0, b, d\}) &= A \triangleleft X, & \mathcal{R}(A; \{0, a, b, c\}) &= \{c\}, \\
\mathcal{R}(A; \{0, b, c, d\}) &= A \triangleleft X, & \mathcal{R}(A; \{0, a\}) &= X, \\
\mathcal{R}(A; \{0, b\}) &= \{0, a, b\} \triangleleft X, & \mathcal{R}(A; \{0, c\}) &= X, \\
\mathcal{R}(A; \{0, d\}) &= \{0, a, b, d\}, & \mathcal{R}(A; \{0, a, d\}) &= X, \\
\mathcal{R}(A; \{b\}) &= A \triangleleft X, & \mathcal{R}(A; \{c\}) &= \{a, c\}, \\
\mathcal{R}(A; \{d\}) &= A \triangleleft X.
\end{align*}
\]

In Example 4.1, we know that there exists a non-ideal \(U\) of \(X\) such that \(\mathcal{R}(A; U) \triangleleft X\); and there exists a non-ideal \(V\) of \(X\) such that \(\mathcal{R}(A; V) \triangleleft X\), where \(\mathcal{R}\) is an equivalence relation on \(X\) related to \(A \triangleleft X\).

**Proposition 4.2.** Let \(\mathcal{R}\) and \(\mathcal{Q}\) be equivalence relations on \(X\) related to ideals \(A\) and \(B\) of \(X\), respectively. If \(A \subseteq B\), then \(\mathcal{R} \subseteq \mathcal{Q}\).

**Proof.** If \((x, y) \in \mathcal{R}\), then \(x - y \in A \subseteq B\) and \(y - x \in A \subseteq B\). Hence \((x, y) \in \mathcal{Q}\), and so \(\mathcal{R} \subseteq \mathcal{Q}\). \(\square\)

**Proposition 4.3.** Let \(\mathcal{R}\) be an equivalence relation on \(X\) related to an ideal \(A\) of \(X\). Then
(1) \((\forall S \in \mathcal{P}(X)) (\overline{\mathcal{R}}(A; S) \subseteq S \subseteq \overline{\mathcal{R}}(A; S))\),
(2) \((\forall S, T \in \mathcal{P}(X)) (\overline{\mathcal{R}}(A; S \cup T) = \overline{\mathcal{R}}(A; S) \cup \overline{\mathcal{R}}(A; T))\),
(3) \((\forall S, T \in \mathcal{P}(X)) (\overline{\mathcal{R}}(A; S \cap T) = \overline{\mathcal{R}}(A; S) \cap \overline{\mathcal{R}}(A; T))\),
(4) \((\forall S, T \in \mathcal{P}(X)) (S \subseteq T \Rightarrow \overline{\mathcal{R}}(A; S) \subseteq \overline{\mathcal{R}}(A; T), \overline{\mathcal{R}}(A; S) \subseteq \overline{\mathcal{R}}(A; T))\),
(5) \((\forall S, T \in \mathcal{P}(X)) (\overline{\mathcal{R}}(A; S \cup T) \supseteq \overline{\mathcal{R}}(A; S) \cup \overline{\mathcal{R}}(A; T))\),
(6) \((\forall S, T \in \mathcal{P}(X)) (\overline{\mathcal{R}}(A; S \cap T) \subseteq \overline{\mathcal{R}}(A; S) \cap \overline{\mathcal{R}}(A; T))\),
(7) If \(\mathcal{Q}\) is an equivalence relation on \(X\) related to an ideal \(B\) of \(X\) and if \(A \subseteq B\), then \(\overline{\mathcal{R}}(A; S) \subseteq \overline{\mathcal{R}}(B; S)\) for all \(S \in \mathcal{P}(X)\).

**Proof.** (1) is straightforward.
(2) For any subsets \(S\) and \(T\) of \(X\), we have

\[
x \in \overline{\mathcal{R}}(A; S \cup T) \iff A_x \cap (S \cup T) \neq \emptyset
\iff (A_x \cap S) \cup (A_x \cap T) \neq \emptyset
\iff A_x \cap S \neq \emptyset \text{ or } A_x \cap T \neq \emptyset
\iff x \in \overline{\mathcal{R}}(A; S) \text{ or } x \in \overline{\mathcal{R}}(A; T)
\iff x \in \overline{\mathcal{R}}(A; S) \cup \overline{\mathcal{R}}(A; T),
\]

and hence \(\overline{\mathcal{R}}(A; S \cup T) = \overline{\mathcal{R}}(A; S) \cup \overline{\mathcal{R}}(A; T)\).

(3) For any subsets \(S\) and \(T\) of \(X\) we have

\[
x \in \overline{\mathcal{R}}(A; S \cap T) \iff A_x \subseteq S \cap T
\iff A_x \subseteq S \text{ and } A_x \subseteq T
\iff x \in \overline{\mathcal{R}}(A; S) \text{ and } x \in \overline{\mathcal{R}}(A; T)
\iff x \in \overline{\mathcal{R}}(A; S) \cap \overline{\mathcal{R}}(A; T).
\]

Hence \(\overline{\mathcal{R}}(A; S \cap T) = \overline{\mathcal{R}}(A; S) \cap \overline{\mathcal{R}}(A; T)\).

(4) Let \(S, T \in \mathcal{P}(X)\) be such that \(S \subseteq T\). Then \(S \cap T = S\) and \(S \cup T = T\). It follows from (3) and (2) that

\[
\overline{\mathcal{R}}(A; S) = \overline{\mathcal{R}}(A; S \cap T) = \overline{\mathcal{R}}(A; S) \cap \overline{\mathcal{R}}(A; T)
\]

and

\[
\overline{\mathcal{R}}(A; T) = \overline{\mathcal{R}}(A; S \cup T) = \overline{\mathcal{R}}(A; S) \cup \overline{\mathcal{R}}(A; T),
\]

which yield \(\overline{\mathcal{R}}(A; S) \subseteq \overline{\mathcal{R}}(A; T)\) and \(\overline{\mathcal{R}}(A; S) \subseteq \overline{\mathcal{R}}(A; T)\), respectively.

(5) Since \(S \subseteq S \cup T\) and \(T \subseteq S \cup T\), it follows from (4) that

\[
\overline{\mathcal{R}}(A; S) \subseteq \overline{\mathcal{R}}(A; S \cup T) \text{ and } \overline{\mathcal{R}}(A; T) \subseteq \overline{\mathcal{R}}(A; S \cup T).
\]

Thus \(\overline{\mathcal{R}}(A; S) \cup \overline{\mathcal{R}}(A; T) \subseteq \overline{\mathcal{R}}(A; S \cup T)\).

(6) Since \(S \cap T \subseteq S, T\), it follows from (4) that

\[
\overline{\mathcal{R}}(A; S \cap T) \subseteq \overline{\mathcal{R}}(A; S) \text{ and } \overline{\mathcal{R}}(A; S \cap T) \subseteq \overline{\mathcal{R}}(A; T)
\]

so that \(\overline{\mathcal{R}}(A; S \cap T) \subseteq \overline{\mathcal{R}}(A; S) \cap \overline{\mathcal{R}}(A; T)\).
(7) If \( x \in \mathcal{R}(A; S) \), then \( A_x \cap S \neq \emptyset \), and so there exists \( a \in S \) such that \( a \in A_x \). Hence \( (a, x) \in \mathcal{R} \), that is, \( a - x \in A \) and \( x - a \in A \). Since \( A \subseteq B \), it follows that \( a - x \in B \) and \( x - a \in B \) so that \( (a, x) \in \mathcal{S} \), that is, \( a \in B_x \). Therefore \( a \in B_x \cap S \), which means \( x \in \mathcal{S}(B; S) \). This completes the proof. \( \square \)

**Proposition 4.4.** Let \( \mathcal{R} \) be an equivalence relation on \( X \) related to any ideal \( A \) of \( X \). Then \( \mathcal{R}(A; X) = X = \mathcal{R}(A; X) \), that is, \( X \) is definable.

**Proof.** It is straightforward. \( \square \)

**Proposition 4.5.** Let \( \mathcal{R} \) be an equivalence relation on \( X \) related to the trivial ideal \( \{0\} \) of \( X \). Then \( \mathcal{R}(\{0\}; S) = S = \mathcal{R}(\{0\}; S) \) for every nonempty subset \( S \) of \( X \), that is, every nonempty subset of \( X \) is definable.

**Proof.** Note that \( \{0\}_x = \{x\} \) for all \( x \in X \), since if \( a \in \{0\}_x \) then \( (a, x) \in \mathcal{R} \) and hence \( a - x = 0 \) and \( x - a = 0 \). It follows that \( a = x \). Hence

\[
\mathcal{R}(\{0\}; S) = \{ x \in X \mid \{0\}_x \subseteq S \} = S
\]

and

\[
\mathcal{R}(\{0\}; S) = \{ x \in X \mid \{0\}_x \cap S \neq \emptyset \} = S.
\]

This completes the proof. \( \square \)

**Remark 4.6.** Let \( \mathcal{R} \) be an equivalence relation on \( X \) related to an ideal \( A \) of \( X \). If \( B \) is an ideal of \( X \) such that \( A \neq B \), then \( \mathcal{R}(A; B) \) is not an ideal of \( X \) in general. For, consider a subtraction algebra \( X \) in Example 4.1(2) and an equivalence relation \( \mathcal{R} \) on \( X \) related to the ideal \( A = \{0, 1, 2\} \). If we take an ideal \( B = \{0, 1, 3\} \) of \( X \), then \( A \neq B \) and \( \mathcal{R}(A; B) = \{3\} \) which is not an ideal of \( X \).

**Definition 4.7.** Let \( \mathcal{R} \) be an equivalence relation on \( X \) related to an ideal \( A \) of \( X \). A nonempty subset \( S \) of \( X \) is called an upper (resp. a lower) rough subalgebra/ideal of \( X \) if the upper (resp. nonempty lower) approximation of \( S \) is a subalgebra/ideal of \( X \). If \( S \) is both an upper and a lower rough subalgebra/ideal of \( X \), we say that \( S \) is a rough subalgebra/ideal of \( X \).

**Theorem 4.8.** Let \( \mathcal{R} \) be an equivalence relation on \( X \) related to an ideal \( A \) of \( X \). Then every subalgebra \( S \) of \( X \) is a rough subalgebra of \( X \).

**Proof.** Let \( x, y \in \mathcal{R}(A; S) \). Then \( A_x \subseteq S \) and \( A_y \subseteq S \). Since \( S \) is a subalgebra of \( X \), it follows that \( A_{x-y} = A_x \cap A_y \subseteq S \) so that \( x - y \in
\( \mathcal{R}(A; S) \). Hence \( \mathcal{R}(A; S) \) is a subalgebra of \( X \). Now if \( x, y \in \mathcal{R}(A; S) \), then \( A_x \cap S \neq \emptyset \) and \( A_y \cap S \neq \emptyset \), and so there exist \( a, b \in S \) such that \( a \in A_x \) and \( b \in A_y \). It follows that \( (a, x) \in \mathcal{R} \) and \( (b, y) \in \mathcal{R} \). Since \( \mathcal{R} \) is a congruence relation on \( X \), we have \( (a - b, x - y) \in \mathcal{R} \). Hence \( a - b \in A_{x-y} \). Since \( S \) is a subalgebra of \( X \), we get \( a - b \in S \), and therefore \( a - b \in A_{x-y} \cap S \), that is, \( A_{x-y} \cap S \neq \emptyset \). This shows that \( x - y \in \mathcal{R}(A; S) \), and consequently \( \mathcal{R}(A; S) \) is a subalgebra of \( X \). This completes the proof. 

COROLLARY 4.9. Let \( \mathcal{R} \) be an equivalence relation on \( X \) related to an ideal \( A \) of \( X \). Then \( \mathcal{R}(A) \) (\( \neq \emptyset \)) and \( \mathcal{R}(A) \) are subalgebras of \( X \), that is, \( A \) is a rough subalgebra of \( X \).

PROOF. It is straightforward. 

THEOREM 4.10. Let \( \mathcal{R} \) be an equivalence relation on \( X \) related to an ideal \( A \) of \( X \). If \( U \) is an ideal of \( X \) containing \( A \), then

1. \( \mathcal{R}(A; U) \) (\( \neq \emptyset \)) is an ideal of \( X \), that is, \( U \) is a lower rough ideal of \( X \).
2. \( \mathcal{R}(A; U) \) is an ideal of \( X \), that is, \( U \) is an upper rough ideal of \( X \).

PROOF. Let \( U \) be an ideal of \( X \) containing \( A \). Let \( x \in A_0 \). Then \( x \in A \subseteq U \), and so \( A_0 \subseteq U \). Hence \( 0 \in \mathcal{R}(A; U) \). Let \( x, y \in X \) be such that \( y \in \mathcal{R}(A; U) \) and \( x - y \in \mathcal{R}(A : U) \). Then \( A_y \subseteq U \) and \( A_x \cap A_y = A_{x-y} \subseteq U \). Let \( a \in A_x \) and \( b \in A_y \). Then \( (a, x) \in \mathcal{R} \) and \( (b, y) \in \mathcal{R} \), which implies \( (a - b, x - y) \in \mathcal{R} \). Hence \( a - b \in A_{x-y} \subseteq U \). Since \( b \in A_y \subseteq U \) and \( U \) is an ideal, it follows that \( a \in U \), so that \( A_x \subseteq U \). Thus \( x \in \mathcal{R}(A; U) \). This shows that \( \mathcal{R}(A; U) \) is an ideal of \( X \), that is, \( U \) is a lower rough ideal of \( X \). Now, obviously \( 0 \in \mathcal{R}(A; U) \). Let \( x, y \in X \) be such that \( y \in \mathcal{R}(A; U) \) and \( x - y \in \mathcal{R}(A; U) \). Then \( A_y \cap U \neq \emptyset \) and \( A_{x-y} \cap U \neq \emptyset \), and so there exist \( a, b \in U \) such that \( a \in A_y \) and \( b \in A_{x-y} \). Hence \( (a, y) \in \mathcal{R} \) and \( (b, x - y) \in \mathcal{R} \), which implies \( y - a \in A \subseteq U \) and \( (x - y) - b \in A \subseteq U \). Since \( a, b \in U \) and \( U \) is an ideal, we get \( y \in U \) and \( x - y \in U \); hence \( x \in U \). Note that \( x \in A_x \), thus \( x \in A_x \cap U \), that is, \( A_x \cap U \neq \emptyset \). Therefore \( x \in \mathcal{R}(A; U) \), and consequently \( U \) is an upper rough ideal of \( X \).

COROLLARY 4.11. Let \( \mathcal{R} \) be an equivalence relation on \( X \) related to an ideal \( A \) of \( X \). Then \( \mathcal{R}(A) \) (\( \neq \emptyset \)) and \( \mathcal{R}(A) \) are ideals of \( X \), that is, \( A \) is a rough ideal of \( X \).

Theorem 4.10 shows that the notion of an upper (resp. a lower) rough ideal is an extended notion of an ideal in a subtraction algebra.
The following example shows that if $A$ and $U$ are ideals of $X$ such that $A \not\subseteq U$, then $\mathcal{R}(A; U)$ may not be an ideal of $X$.

**Example 4.12.** (1) Let $X = \{0, a, b, c, d\}$ be a subtraction algebra described in Example 4.1(2). Consider two ideals $A = \{0, b\}$ and $U = \{0, d\}$ of $X$. Then $\mathcal{R}(A; U) = \{d\}$ which is not an ideal of $X$.

(2) Let $X = \{0, a, b, c, d\}$ be a subtraction algebra with the Cayley table as follows:

<table>
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<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
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</thead>
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<td>a</td>
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<tr>
<td>b</td>
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<td>0</td>
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</tbody>
</table>

Consider $A = \{0, a, b\} \triangleleft X$ and let $\mathcal{R}$ be an equivalence relation on $X$ related to $A$. Then the equivalence classes are as follows: $A_0 = A_a = A_b = A$, $A_c = \{c\}$, and $A_d = \{d\}$. Then $U = \{0, a, c\}$ is an ideal of $X$ which does not contain $A$, and $\mathcal{R}(A; U) = \{c\}$ which is not an ideal of $X$.

**References**


Sun Shin Ahn
Department of Mathematics Education
Dongguk University
Seoul 100-715, Korea
E-mail: sunshine@dongguk.ac.kr

Young Bae Jun
Department of Mathematics Education (and RINS)
Gyeongsang National University
Chinju 660-701, Korea
E-mail: ybjun@gnu.ac.kr or jamjana@korea.com

Kyoung Ja Lee
School of General Education
Kookmin University
Seoul 136-702, Korea
E-mail: lsj1109@kookmin.ac.kr