FUNCTIONAL CENTRAL LIMIT THEOREMS
FOR MULTIVARIATE LINEAR PROCESSES
GENERATED BY DEPENDENT RANDOM VECTORS

Mi-Hwa Ko

ABSTRACT. Let $X_t$ be an $m$-dimensional linear process defined by
$X_t = \sum_{j=0}^{\infty} A_j Z_{t-j}$, $t = 1, 2, \ldots$, where $\{Z_t\}$ is a sequence of
$m$-dimensional random vectors with mean $0 : m \times 1$ and positive
definite covariance matrix $\Gamma : m \times m$ and $\{A_j\}$ is a sequence of
coefficient matrices. In this paper we give sufficient conditions so
that $\sum_{t=1}^{[n\alpha]} X_t$ (properly normalized) converges weakly to Wiener
measure if the corresponding result for $\sum_{t=1}^{[n\alpha]} Z_t$ is true.

1. Introduction

Consider $m$-dimensional linear process of the form

$$X_t = \sum_{j=0}^{\infty} A_j Z_{t-j},$$

where the innovation $\{Z_t\}$ is a sequence of $m$-dimensional random vec-
tors with mean $0 : m \times 1$ and positive definite covariance matrix $\Gamma : m \times m$. Throughout we shall assume that

$$\sum_{j=0}^{\infty} \|A_j\| < \infty \quad \text{and} \quad \sum_{j=0}^{\infty} A_j \neq O_{m \times m},$$

where for any $m \times m, m \geq 1$, matrix $A = (a_{ij}), \quad i, j = 1, \ldots, m, \quad \|A\| = \sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ij}|$ and $O_{m \times m}$ denotes the $m \times m$ zero matrix. Let $W^m$
denote Wiener measure on $D^m[0, 1]$, the space of all real valued functions.

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on \([0,1]\) that are right continuous and have finite left limits, endowed with
the sup norm (see, e.g., [3], [10]). Further, let

\[
T = \left( \sum_{j=0}^{\infty} A_j \right) \Gamma \left( \sum_{j=0}^{\infty} A_j \right)^{\prime},
\]

\(S_n = \sum_{t=1}^{n} X_t, n \geq 1(S_0 = \emptyset)\) and define for \(n \geq 1\) the stochastic process \(\xi_n\) by

\[
\xi_n(s) = n^{-\frac{1}{2}} S_{[ns]} \quad 0 \leq s \leq 1.
\]

In this paper we give sufficient conditions so that \(\sum_{t=1}^{\lfloor ns \rfloor} X_t\) (properly normalized) converges weakly to Wiener measure if the corresponding result for \(\sum_{t=1}^{\lfloor ns \rfloor} Z_t\) is true. As applications we also discuss functional central limit theorems for linear processes generated by martingale difference and negatively associated random vectors.

2. Main results

**Theorem 2.1.** Let \(X_t\) satisfy model (1.1) and \(d(n)\) be a positive constant sequence satisfying that \(d(n) \to \infty\) as \(n \to \infty\). Assume that \(\{A_j\}\) satisfies (1.2) and \(\{Z_t\}\) is any random vector sequence satisfying

\[
sup_j E \max_{1 \leq m \leq n} \left\| \sum_{k=1}^{m} Z_{k+j} \right\|^2 \leq C d^2(n) \quad \text{for every } n \geq 1
\]

and, as \(n \to \infty\),

\[
\frac{1}{d(n)} \max_{-n \leq k \leq n} \left\| Z_k \right\| \to^p 0.
\]

Then,

\[
\frac{1}{d(n)} \sum_{t=1}^{k_n(s)} X_t \Rightarrow W^m(s) \quad \text{implies} \quad \frac{1}{d(n)} \sum_{t=1}^{k_n(s)} X_t \Rightarrow BW^m(s),
\]

where \(k_n(s) = \sup \{ m : d^2(m) \leq sd^2(n) \}\) and \(B = \sum_{k=0}^{\infty} A_k \).

Theorem 2.1 can be applied to many important cases, such as whether innovation \(\{Z_t\}\) is martingale difference or negatively associated sequence. In the following, we will derive corollaries of Theorem 2.1. We note that Corollary 2.3 below is Theorem 1(i) of [6] and Corollary 2.8 is a new result.
DEFINITION 2.2. Let \( \{Z_k\} \) be a random vector sequence. We say that \( \{Z_k\} \) is a martingale difference sequence if \( E(Z_k|F_{k-1}) = 0 \), a.s. \( k = 0, \pm 1, \pm 2, \ldots \), where \( F_k = \sigma \{Z_i, i \leq k\} \).

COROLLARY 2.3. Define \( X_t \) as in (1.1) and \( \xi_n \) as in (1.4), respectively. Let \( \{Z_t\} \) be a sequence of \( m \)-dimensional martingale difference vectors with \( E(Z_t|F_{t-1}) = 0 \) a.s. and \( \Gamma_t \) denote the conditional covariance matrix of \( Z_t \), \( E(Z_tZ_t'|F_{t-1}) = \Gamma_t \) a.s., such that \( \frac{1}{n} \sum_{t=1}^{n} \Gamma_t \to^p \Gamma \), where \( F_t \) is sub-\( \sigma \)-algebra generated by \( Z_u, u \leq t \) and the prime denotes transpose and \( \Gamma \) is a positive definite(d.f.) non random matrix. Assume that \( \sup \ E\|Z_t\|^2 < \infty \) and \( \frac{1}{n} \sum_{t=1}^{n} E(Z_tZ_t'|I(Z_tZ_t' > n\epsilon)|F_{t-1}) \to^p 0 \) as \( n \to \infty \) for every \( \epsilon > 0 \), where \( I(\cdot) \) denotes the indicator function. Then, \( \xi_n \Rightarrow W^m \), where \( W^m \) is a Wiener measure with covariance matrix \( T = (\sum_{j=0}^{\infty} A_j)\Gamma(\sum_{j=0}^{\infty} A_j)' \).

PROOF. Define for \( n \geq 1 \) the stochastic process \( \eta_n \) by

\[
\eta_n(s) = n^{-\frac{1}{2}} \sum_{i=1}^{[ns]} Z_i, \ 0 \leq s \leq 1.
\]

(2.4)

It follows from the multivariate version of Theorem 1 of [2] or Theorem 2 of [1] that \( \eta_n(s) \) converges weakly to Wiener measure with covariance matrix \( \Gamma \) (c.f. Theorem 3.1 of [8]). On the other hand, it follows from Doob's maximal inequality and \( \sup \ E\|Z_t\|^2 < \infty \) that for every \( n \geq 1 \)

\[
\sup_j \max_{1 \leq m \leq n} \left( \sum_{k=1}^{m} Z_{k+j} \right)^2 \leq C_1 n \sup_j \max_{1 \leq m \leq n} \left( \sum_{k=1}^{m} Z_{k+j} \right)^2 \leq C_2 n
\]

(2.5)

and

\[
\frac{1}{\sqrt{n}} \max_{-n \leq k \leq n} \|Z_k\| \to^p 0.
\]

(2.6)

Hence, corollary 2.3 follows immediately from Theorem 2.1 with \( d(n) = \sqrt{n} \). \( \square \)

DEFINITION 2.4. Let \( \{Z_i, 1 \leq i \leq n\} \) be a sequence of \( m \)-dimensional random vectors. They are said to be negatively associated(NA) for every pair of disjoint subsets \( A \) and \( B \) of \( \{1, \ldots, n\} \) Cov\( f(Z_i, i \in A), g(Z_j, j \in B) \) \( \leq 0 \) whenever \( f \) and \( g \) are coordinatewise increasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated.

LEMMA 2.5. Let \( r \geq 2 \) and let \( \{Z_i, 1 \leq i \leq n\} \) be a sequence of \( m \)-dimensional negatively associated random vectors with \( E Z_i = 0 \) and
$E\|Z_i\|^r < \infty$, where $\|Z_i\| = (Z_{i1}^2 + \cdots + Z_{im}^2)^{\frac{1}{2}}$. Then there exists a constant $0 < A_r < \infty$ such that

$$(2.7) \quad E \max_{1 \leq k \leq n} \| \sum_{i=1}^{k} Z_i \|^r \leq A_r m^r \{ (\sum_{i=1}^{n} E\|Z_i\|^2)^{\frac{r}{2}} + \sum_{i=1}^{n} E\|Z_i\|^r \}. $$

**Proof.** Note that

$$(2.8) \quad \max_{1 \leq k \leq n} \| \sum_{i=1}^{k} Z_i \| \leq \sum_{j=1}^{m} \max_{1 \leq k \leq n} | \sum_{i=1}^{k} Z_{ij} |$$

and by the result in [11] we have

$$(2.9) \quad E \max_{1 \leq k \leq n} | \sum_{i=1}^{k} Z_{ij} |^r \leq A_r \{ (\sum_{i=1}^{n} E(Z_{ij})^2)^{\frac{r}{2}} + \sum_{i=1}^{n} E|Z_{ij}|^r \}$$

Hence, from (2.8) and (2.9) equation (2.7) follows. \qed

**Lemma 2.6.** Let $\{Z_i, 1 \leq i \leq n\}$ be a sequence of $m$-dimensional negatively associated random vectors with $E(Z_i) = 0$ and $E\|Z_i\|^2 < \infty$. Then for all $x > 0$ and $a > 0$,

$$(2.10) \quad P(\max_{1 \leq k \leq n} \| \sum_{i=1}^{k} Z_i \| \geq mx) \leq 2mP(\max_{1 \leq k \leq n} \| Z_k \| > a) + 4m \exp\left(-\frac{x^2}{8 \sum_{i=1}^{n} E\|Z_i\|^2}\right)$$

$$+ 4m\left(\frac{\sum_{i=1}^{n} E\|Z_i\|^2}{4(ax + \sum_{i=1}^{n} E\|Z_i\|^2)}\right)^{x/(12a)}.$$

**Proof.** From (2.8) and the result of [11], (2.10) follows easily. \qed

**Theorem 2.7.** Let $\{Z_i, i \geq 1\}$ be a strictly stationary sequence of $m$-dimensional negatively associated random vectors with $E(Z_1) = 0$ and $E\|Z_1\|^2 < \infty$. Define, for $t \in [0, 1]$, $n \geq 1 \xi_n(t) = n^{-\frac{1}{2}} \sum_{i=1}^{nt} Z_i$. If $E\|Z_1\|^2 + 2 \sum_{i=2}^{\infty} \sum_{j=1}^{m} E(Z_{1j}Z_{ij}) = \sigma^2 < \infty$, then, as $n \to \infty$, $\xi_n \Rightarrow B^m$, where $B^m$ is an $m$-dimensional Wiener measure with covariance matrix $\Gamma = (\sigma_{kj})$ and $\sigma_{kj} = E(Z_{1k}Z_{1j}) + \sum_{i=2}^{\infty} E(Z_{1k}Z_{ij}) + E(Z_{1j}Z_{ik})$.

**Proof.** By means of the simple device due to Cramer Wold (see [3], [4]), from the Newman’s central limit theorem for negatively associated
random variables (see [9]) we obtain $n^{-\frac{1}{2}} \sum_{i=1}^{n} Z_i \rightarrow^{D} N(0, \Gamma)$, where $N(0, \Gamma)$ denotes an $m$-dimensional normal random vector and the symbol $\rightarrow^{D}$ indicates convergence in distribution. Hence, as in the proof of Theorem 2 of [5] on weakly associated random vectors, the limit point of $\xi_n(\cdot)$ is an $m$-dimensional Wiener measure with covariance matrix $\Gamma = (\sigma_{k,l})$. It remains to verify the tightness of $\xi_n(\cdot)$ (see Theorem 15.1 of [3]). By Theorem 8.4 of [3] we only need to show that for any $\varepsilon > 0$, there exist a positive number $\lambda$ and an integer $n$ such that for every $n \geq n_0$

$$P(\max_{1 \leq k \leq n} \| \sum_{i=1}^{k} Z_i \| > \lambda n^{\frac{1}{2}}) \leq m^3 \varepsilon \lambda^{-2}. \quad (2.11)$$

Applying Lemma 2.6 with $\lambda = m\lambda'$, $x = \lambda' n^{\frac{1}{2}}$ and $a = \lambda' n^{\frac{1}{2}}/48$

$$P(\max_{1 \leq k \leq n} \| \sum_{i=1}^{k} Z_i \| > \lambda n^{\frac{1}{2}}) = P(\max_{1 \leq k \leq n} \| \sum_{i=1}^{k} Z_i \| > m\lambda' n^{\frac{1}{2}}) \leq 2mP(\max_{1 \leq k \leq n} \| Z_k \| > \lambda' n^{\frac{1}{2}}/48)

+ 4m \exp\left(-\frac{\lambda'^2 n}{8nE\|Z_1\|^2}\right) + 4m\left(\frac{nE\|Z_1\|^2}{4(nE\|Z_1\|^2 + \lambda'^2 n/48)}\right)^4 \leq m\lambda'^{-2} = m^3 \varepsilon \lambda^{-2}$$

provided that $\lambda$ is sufficiently large. This proves (2.11), and hence the proof of Theorem 2.7 is complete. \hfill \Box

**Corollary 2.8.** Let $\{Z_i, i \geq 1\}$ be a strictly stationary negatively associated sequence of $m$-dimensional random vectors centered at expectations and $E\|Z_1\|^2 < \infty$ and $X_t$ be defined as in (1.1). Let the stochastic process $\xi_n$ be defined as in (1.4). Assume (1.2) and $E\|Z_1\|^2 + 2 \sum_{i=2}^{\infty} \sum_{j=1}^{m} E(Z_{1j}Z_{ij}) = \sigma^2 < \infty$ hold. Then $\xi_n \Rightarrow W^m$.

**Proof.** First note that $\xi_n(s) = n^{-\frac{1}{2}} \sum_{i=1}^{[ns]} Z_i$ converges weakly to Wiener measure $B^m$ with covariance matrix $\Gamma$ by Theorem 2.7. On the
other hand, it follows from Lemma 2.5 and the condition $E\|Z_1\|^2 < \infty$ that (2.5) and (2.6) hold. Hence, Corollary 2.8 follows immediately from Theorem 2.1 with $d(n) = \sqrt{n}$.

\[\square\]

3. Proof of Theorem 2.1

For every fixed $l \geq 1$, put

\[
X_{1j}^{(l)} = \sum_{k=0}^{l} A_k z_{j-k} \quad \text{and} \quad X_{2j}^{(l)} = \sum_{k=l+1}^{\infty} A_k z_{j-k}.
\]

From the idea in [7] (p.320) we obtain that for any $m \geq 1$,

\[
\sum_{j=1}^{m} X_{1j}^{(l)} = \sum_{j=1}^{m} \sum_{k=0}^{l} A_k z_{j-k}
\]

\[
= \sum_{k=0}^{l} A_k \sum_{j=1}^{m} z_{j} + \sum_{s=1}^{l} \sum_{j=s}^{l} A_j + \sum_{s=0}^{l-1} \sum_{j=s+1}^{l} A_j
\]

\[
= \sum_{k=0}^{l} A_k \sum_{j=1}^{m} z_{j} + R(m, l), \quad \text{(say)}.
\]

Therefore, it follows that for every fixed $l \geq 1$,

\[
\frac{1}{d(n)} \sum_{t=1}^{k_n(s)} X_t = \left(\sum_{k=0}^{l} A_k\right) \frac{1}{d(n)} \sum_{j=1}^{k_n(s)} z_{j} + \frac{1}{d(n)} R(k_n(s), l)
\]

\[
\frac{1}{d(n)} \sum_{j=1}^{k_n(s)} X_{2j}^{(l)}.
\]

By (3.3), Theorem 4.1 given in [3] (p.25) and noting that $\sum_{k=0}^{l} \|A_k\| \to B$ as $l \to \infty$, to prove (2.3), it suffices to show that for any $\delta > 0$,

\[
\limsup_{n \to \infty} P\{ \sup_{0 \leq t \leq 1} \|R(k_n(t), l)\| \geq \delta d(n)\} = 0,
\]

for every fixed $l \geq 1$ and

\[
\lim_{l \to \infty} \limsup_{n \to \infty} P\{ \sup_{0 \leq t \leq 1} \|\sum_{j=1}^{k_n(l)} X_{2j}^{(l)}\| \geq \delta d(n)\} = 0.
\]
By condition (2.2) since $\sum_{k=0}^{\infty} \|A_k\| < \infty$, as $n \to \infty$,

$$\frac{1}{d(n)} \sup_{0 \leq t \leq 1} \|R(k_n(s), l)\| \leq \frac{1}{d(n)} \max_{-l \leq j \leq n} \|Z_j\| \sum_{s=0}^{l} \left( \sum_{j=s}^{l} \|A_j\| + \sum_{j=s+1}^{\infty} \|A_u\| \right) \to P_0 0$$

and hence (3.4) holds.

Noting that $\sum_{j=1}^{m} X_{2j}^{(l)} = \sum_{k=l+1}^{\infty} A_k \sum_{j=1}^{m} Z_{j-k}$ for any $m \geq 1$, by applying Hölder inequality and (2.1), we have

$$E \sup_{1 \leq t \leq 1} \|\sum_{j=1}^{k_n(t)} X_{2j}^{(l)}\|^2 \leq (\sum_{k=l+1}^{\infty} \|A_k\|)^2 E \max_{1 \leq m \leq n} \|\sum_{j=1}^{m} Z_{j-k}\|^2 \leq C d^2(n) \left( \sum_{k=l+1}^{\infty} \|A_k\| \right)^2.$$

Hence, (3.5) follows immediately from the Markov inequality and $\sum_{k=l+1}^{\infty} \|A_k\| \to 0$ as $l \to \infty$. The proof of Theorem 2.1 is complete. \(\square\)

References


Department of Mathematics  
WonKwang University  
Jeonbuk 570-749, Korea  
*E-mail*: songhack@wonkwang.ac.kr