ENERGY FINITE $p$-HARMONIC FUNCTIONS ON GRAPHS AND ROUGH ISOMETRIES

SEOK WOO KIM AND YONG HAH LEE

Abstract. We prove that if a graph $G$ of bounded degree has finitely many $p$-hyperbolic ends ($1 < p < \infty$) in which every bounded energy finite $p$-harmonic function is asymptotically constant for almost every path, then the set $\mathcal{HBD}_p(G)$ of all bounded energy finite $p$-harmonic functions on $G$ is in one to one corresponding to $\mathbb{R}^l$, where $l$ is the number of $p$-hyperbolic ends of $G$. Furthermore, we prove that if a graph $G'$ is roughly isometric to $G$, then $\mathcal{HBD}_p(G')$ is also in one to one correspondence with $\mathbb{R}^l$.

1. Introduction

We say that a graph $G$ has the Liouville property if every bounded harmonic function on $G$ is constant. Thus the set of all bounded harmonic functions on $G$ having Liouville property is in one to one correspondence with the real line $\mathbb{R}$. With this view point, given an operator $\mathcal{A}$ on a graph, it seems natural to regard a class $\mathcal{S}$ of solutions of $\mathcal{A}$ which is in an one to one correspondence with the Euclidean space $\mathbb{R}^l$ for some positive integer $l$ as a generalized version of the Liouville property of the pair $(\mathcal{A}, \mathcal{S})$. In this paper, we study case of the $p$-Laplacian operator ($1 < p < \infty$) and the bounded $p$-harmonic functions on a graph $G$ of bounded degree. If $p = 2$, then we obtain harmonic functions on $G$ as a special case. (See [6] and [8]..) In Section 3, we study a sort of an asymptotic behavior of $p$-harmonic functions which enables us to identify a subset of the set of the bounded $p$-harmonic functions on $G$. To be precise, if a graph $G$ has a finite number of $p$-hyperbolic ends and every bounded energy finite $p$-harmonic function on $G$ satisfies such an behavior, then we have the following theorem:

**Theorem 1.1.** Let $G$ be a graph with $l$ ($l \geq 1$) $p$-hyperbolic ends. Suppose that every $p$-harmonic function in $\mathcal{HBD}_p(G)$ is asymptotically constant for $p$-almost every path in each $p$-hyperbolic end, where $\mathcal{HBD}_p(G)$ denotes the set of all

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bounded energy finite p-harmonic functions on $G$. Then given any real numbers $a_1, a_2, \ldots, a_l \in \mathbb{R}$, there exists a unique $p$-harmonic function $v \in \mathcal{HD}_p(G)$ such that

$$v(p) = a_i \text{ for } p\text{-almost every path } p \in \mathcal{P}_{E_i}$$

for each $i = 1, 2, \ldots, l$, where $E_1, E_2, \ldots, E_l$ are $p$-hyperbolic ends of $G$, and $\mathcal{P}_{E_i}$ denotes a family of paths lying in $E_i$ to be explained in Section 3.

In Section 4, we extend our result to graphs being roughly isometric to those satisfying the assumption of Theorem 1.1:

**Theorem 1.2.** Let $G$ be a graph with $l \ (l \geq 1)$ $p$-hyperbolic ends. Suppose that every $p$-harmonic function in $\mathcal{HD}_p(G)$ is asymptotically constant for $p$-almost every path in each $p$-hyperbolic end. Let $G'$ be a graph being roughly isometric to $G$. Then given any real numbers $a_1, a_2, \ldots, a_l \in \mathbb{R}$, there exists a unique $p$-harmonic function $v \in \mathcal{HD}_p(G')$ such that

$$v(p) = a_i \text{ for } p\text{-almost every path } p \in \mathcal{P}_{E_i}$$

for each $i = 1, 2, \ldots, l$, where $E_1, E_2, \ldots, E_l$ are $p$-hyperbolic ends of $G'$.

**2. Preliminaries**

Let $G = (V_G, E_G)$ be a graph, where $V_G$ and $E_G$ denote the vertex set and the edge set, respectively, of $G$. If vertices $x$ and $y$ are the endpoints of the same edge, then we say that $x$ and $y$ are neighbors and write $y \in N_x$ and $x \in N_y$. The degree of $x$ is the number of all neighbors of $x$ and it is denoted by $\|N_x\|$. A graph $G$ is said to be of bounded degree if there exists a number $\nu < \infty$ such that $\|N_x\| \leq \nu$ for all $x \in V_G$. A sequence $x = (x_0, x_1, \ldots, x_r)$ of vertices in $V_G$ is called a path from $x_0$ to $x_r$ with the length $r$ if $x_k$ is an element of $N_{x_{k-1}}$ for each $k = 1, 2, \ldots, r$. We say that a graph $G$ is connected if any two points of $V_G$ can be joined by a path. Throughout this paper, $G$ is a connected infinite graph with no self-loops and is of bounded degree.

For any vertices $x$ and $y$, we define $d(x, y)$ to be the length of the shortest path joining $x$ to $y$. Then $d$ defines a metric on $V_G$. For this metric $d$ and $r \in \mathbb{N}$, define an $r$-neighborhood $N_r(x) = \{y \in V_G : d(x, y) \leq r\}$ for each $x \in V_G$. Given any subset $S \subset V_G$, the outer boundary $\partial S$ and the inner boundary $\delta S$ of $S$ are defined by

$$\partial S = \{x \in V_G : d(x, S) = 1\} \text{ and } \delta S = \{x \in V_G : d(x, V_G \setminus S) = 1\},$$

respectively.

For each real valued function $u$ on $S \cup \partial S$, define the norm of $p$-gradient, the $p$-Dirichlet sum, and the $p$-Laplacian of $u$ at a point $x \in S$, where $1 < p < \infty$, respectively.
in such a way that

\[ |Du|(x) = \left( \sum_{y \in N_x} |u(y) - u(x)|^p \right)^{1/p}, \]

\[ I_p(u, S) = \sum_{x \in S} |Du|^p(x), \]

\[ \Delta_p u(x) = \sum_{y \in N_x} \text{sign}(u(y) - u(x))|u(y) - u(x)|^{p-1} \]

\[ = \sum_{y \in N_x} |u(y) - u(x)|^{p-2}(u(y) - u(x)), \]

respectively.

We say that \( u \) is \( p \)-harmonic on \( S \) if \( \Delta_p u(x) = 0 \) for all \( x \in S \). We introduce some useful properties of \( p \)-harmonic functions on graphs in [1]. If a subset \( S \subset V_G \) is finite, then the following conditions are equivalent:

(i) A function \( u \) is \( p \)-harmonic on \( S \).

(ii) A function \( u \) satisfies \( p \)-Laplacian equation in a weak form. That is,

\[ \sum_{x \in S} \sum_{y \in N_x} |u(y) - u(x)|^{p-2}(u(y) - u(x))(w(y) - w(x)) = 0 \]

for any real valued function \( w \) on \( S \cup \partial S \) such that \( w = 0 \) on \( \partial S \).

(iii) A function \( u \) is a minimizer of \( p \)-Dirichlet sum \( I_p(\cdot, S) \) among functions on \( S \cup \partial S \) with the same values on \( \partial S \). That is,

\[ \sum_{x \in S} |Du|^p(x) \leq \sum_{x \in S} |Dv|^p(x) \]

for every function \( v \) on \( S \cup \partial S \) such that \( v = u \) on \( \partial S \).

Let us set \( T(u, w; x, y) = |u(y) - u(x)|^{p-2}(u(y) - u(x))(w(y) - w(x)) \) whenever functions \( u \) and \( w \) are defined at \( x \) and \( y \). Then it is easy to check that

\[ T(v, v - u; x, y) \geq T(u, v - u; x, y) \]

if \( u \) and \( v \) are defined at \( x \) and \( y \). The equality occurs only if \( v(x) - u(x) = v(y) - u(y) \). By \( (2) \), the following comparison principle holds on \( S \): Suppose there exist \( p \)-harmonic functions \( u \) and \( v \) on a finite set \( S \subset V_G \) such that \( u \geq v \) on \( \partial S \). Then \( u \geq v \) on \( S \).

Let \( S \) be a finite subset of \( V_G \). Suppose that \( \{u_i\} \) is a sequence of functions on \( S \cup \partial S \) converging to a function \( u \) pointwisely. Then for each point \( x \in S \),

\[ |Du_i|^p(x) \to |Du|^p(x) \]

and

\[ \Delta_p u_i(x) \to \Delta_p u(x) \]

and

\[ I_p(u_i, S) \to I_p(u, S) \]

as \( i \to \infty \). By these facts together with the comparison principle, the following existence and uniqueness result holds: Let \( S \) be a finite subset of \( V_G \). For any
function $v$ on $\partial S$, there exists a unique function on $S \cup \partial S$ which is $p$-harmonic on $S$ and equal to $v$ on $\partial S$.

Let $\{S_i\}$ be an increasing sequence of finite connected subsets of $V_G$ and $S = \bigcup S_i$. Let $\{u_i\}$ be a sequence of functions on $S \cup \partial S$ such that each $u_i$ is $p$-harmonic on $S_i$ and $u_i(x) \to u(x) < \infty$ as $i \to \infty$ for all $x \in S \cup \partial S$. Then the limit function $u$ is $p$-harmonic on $S$.

We say that a real valued function $u$ is energy finite if it has finite $p$-Dirichlet sum on the whole set $V_G$, i.e., $I_p(u, V_G) < \infty$. Let $BD_p(G)$ denote the set of all bounded energy finite functions on $V_G$. Then, $BD_p(G)$ is a Banach space with the norm

$$||u||_p = \sup_{V_G} |u| + I_p(u, V_G)^{1/p}. $$

We denote by $BD_{p,0}(G)$ the closure of the set of all finitely supported functions on $V_G$ in $BD_p(G)$ with respect to the norm $|| \cdot ||_p$. The subset of all bounded $p$-harmonic functions in $BD_p(G)$ is denoted by $\mathcal{H}BD_p(G)$.

The subgraph $\Gamma$ induced by a set $S \subset V_G$ is the graph $\Gamma = (S, E_\Gamma)$, where $E_\Gamma$ is the set of all edges in $E_G$ with both ends points in $S$. In particular, that a subset $S \subset V_G$ is connected means that the subgraph $\Gamma = (S, E_\Gamma)$ induced by $S$ is connected. A connected subset $S \subset V_G$ with $\partial S \neq \emptyset$ is called $D_p$-massive if there exists a nonnegative $p$-harmonic function $u$ on $S$ such that $u = 0$ on $\partial S$, $\sup_S u = 1$ and $I_p(u, S) < \infty$. We say that a connected infinite set $S \subset V_G$ is $p$-hyperbolic if there exists a nonempty finite set $A \subset S$ such that

$$\text{Cap}_p(A, \infty, S) = \inf_u I_p(u, S) > 0,$$

where the infimum is taken over all finitely supported function $u$ on $S \cup \partial S$ such that $u = 1$ on $A$. Otherwise, $S$ is called $p$-parabolic.

We now introduce the $p$-Royden decomposition: (See [9].)

**Proposition 2.1.** If a graph $G$ is $p$-hyperbolic, then for each function $u \in BD_p(G)$, there exist unique functions $h \in \mathcal{H}BD_p(G)$ and $g \in BD_{p,0}(G)$ such that $u = h + g$.

For each nonnegative real valued function $w$ on $E_G$, define

$$\mathcal{E}_p(w) = \sum_{e \in E_G} w^p(e).$$

Let $P$ be a family of infinite paths in $G$. The $p$-extremal length $\lambda_p(P)$ of $P$ is defined by

$$\lambda_p(P) = \left(\inf_w \mathcal{E}_p(w)\right)^{-1},$$

where the infimum is taken over the set of all nonnegative functions $w$ on $E_G$ such that $\mathcal{E}_p(w) < \infty$ and $\sum_{e \in E_x} w(e) \geq 1$ for each path $x \in P$, where $E_x$ denotes the edge set of $x$. The following proposition gives some fundamental properties of the extremal length. (See [4].)

**Proposition 2.2.** Let $P_n, n = 1, 2, \ldots,$ be families of paths in a graph $G$. 

(i) If $P_1 \subset P_2$, then $\lambda_p(P_1) \geq \lambda_p(P_2)$.

(ii) $\sum_{n=1}^{\infty} \lambda_p(P_n)^{-1} \geq \lambda_p(\cup_{n=1}^{\infty} P_n)^{-1}$.

On the other hand, the $p$-extremal length is closely related to the $p$-capacity:

Let $S \subset V_G$ be a connected infinite subset. For a nonempty finite subset $A \subset S$, let $P_{S,A}$ be the set of all non-self-intersecting infinite paths in $S$ starting from a vertex in $A$. Then we have

$$(3) \quad \lambda_p(P_{S,A}) = \text{Cap}_p(A, \infty, S)^{-1}.$$  

(See [9] and [7].) Furthermore, if $S \subset V_G$ is $p$-hyperbolic, then by (3),

$$(4) \quad \lambda_p(P_{S,A}) = \text{Cap}_p(A, \infty, S)^{-1} < \infty.$$  

We say that a property holds for $p$-almost every path in $P$ if the subset of all paths for which the property is not true has $p$-extremal length $\infty$.

The following proposition gives some $p$-almost every path properties of energy finite functions: (See [4] and [9].)

**Proposition 2.3.** Let $P_o$ be the family of all non-self-intersecting infinite paths from a fixed point $o \in V_G$.

(i) If $u \in BD_p(G)$, then $u(x)$ exists and is finite for $p$-almost every path $x \in P_o$, where $u(x) = \lim u(x)$ as $x \to \infty$ along the vertices of $x$.

(ii) $u \in BD_{p,0}(G)$ if and only if $u(x) = 0$ for $p$-almost every path $x \in P_o$.

3. Asymptotically constant for $p$-almost every path on ends

We now define ends of a graph $G$ with its vertex set $V_G$: Fix a point $o \in V_G$. For each $r \in \mathbb{N}$, we denote by $\#(r)$ the number of infinite connected components of $V_G \setminus N_r(o)$. Let $\lim_{r \to \infty} \#(r) = l$, where $l$ may be infinity, then we say that the number of ends of $G$ is $l$. If $l$ is finite, then we can choose $r_0 \in \mathbb{N}$ such that $\#(r) = l$ for all $r \geq r_0$.

Using the $p$-hyperbolicity, we can divide ends of $G$ into two classes as follows:

An end $E$ of $G$ is called $p$-hyperbolic if

$$\text{Cap}_p(\partial E, \infty, E) = \inf_u I_p(u, E) > 0,$$

where the infimum is taken over all finitely supported function $u$ on $E \cup \partial E$ such that $u = 1$ on $\partial E$. Otherwise, the end is called $p$-parabolic.

From the definition of a $p$-hyperbolic end, we have the following lemma:

**Lemma 3.1.** If $E$ is a $p$-hyperbolic end, then there exists a $p$-harmonic function $u_E$ on $E$, called a $p$-harmonic measure of $E$, with the following properties:

(i) $0 \leq u_E \leq 1$ on $E$;

(ii) $u_E = 0$ on $\partial E$;

(iii) $\limsup_{x \in E} u_E(x) = 1$;

(iv) $u_E$ has finite $p$-Dirichlet sum over $E$. 

Let us denote $P_G$ to be the family of all non-self-intersecting infinite paths lying in $V_G \setminus N_{r_1}(o)$ starting from a vertex in $\delta N_{r_1}(o)$ for some large $r_1 \in \mathbb{N}$. For each end $E$ of $G$, let us denote $P_E \subset P_G$ to be the family of all paths lying in $E \setminus N_{r_1}(o)$ starting from a vertex in $\delta N_{r_1}(o) \cap E$. We say that a real valued function $u$ on $V_G$ is asymptotically constant for $p$-almost every path in $E$ if there exists a constant $c$ such that

$$u(x) = c \text{ for } p\text{-almost every path } x \in P_E,$$

where $u(x) = \lim u(x)$ as $x$ goes to $\infty$ along vertices on $x$.

**Lemma 3.2.** Let $E$ be a $p$-hyperbolic end of a graph $G$ and $u$ be a nonconstant function in $\mathcal{HBD}_p(G)$ such that $0 \leq u \leq 1$. Suppose that $u$ is asymptotically constant for $p$-almost every path in $E$. If $\limsup_{x \to \infty, x \in E} u = 1$, then $u(x) = 1$ for $p$-almost every path $x \in P_E$.

**Proof.** Suppose the lemma is not true. Then by assumption, there exists a constant $c$ such that $u(x) = c$ for $p$-almost every path $x \in P_E$ and $0 \leq c < 1$. Since $u$ is nonconstant, there exists a proper subset $\Omega$ of $E$ such that $\Omega = \{ x \in E : u(x) > 1 - \epsilon \}$, where $\epsilon$ is a positive constant so small that $1 - \epsilon > c$. Clearly, $\Omega$ is a $D_p$-massive subset. By (4), there exists a subfamily $P_{\Omega}$ of $P_E$ such that $\lambda_p(P_{\Omega}) < \infty$. But from the definition of $\Omega$, one can conclude that $u(x) > c$ for all paths $x \in P_{\Omega}$. This contradicts the fact that $u(x) = c$ for $p$-almost every path $x \in P_E$. This completes the proof. $\Box$

**Proof of Theorem 1.1.** For each $i = 1, 2, \ldots, l$, extend $u_{E_i}$ to be zero outside $E_i$ and then construct a sequence of real valued functions $\{u_{r,i} \}_{r > r_0}$ on $V_G$ such that

$$\begin{cases} 
\Delta_p u_{r,i} = 0 \text{ on } N_r(o); \\
u_{r,i} = u_{E_i} \text{ on } V_G \setminus N_r(o),
\end{cases}$$

where $u_{E_i}$ is a $p$-harmonic measure of $E_i$ constructed in Lemma 3.1 for each $i$. By the comparison principle, $u_{E_i} \leq u_{r,i} \leq 1$ on $N_r(o)$ for each $i$. Thus there exists a convergent subsequence, and its limit function $u_i$ satisfies that

$$\begin{cases} 
\Delta_p u_i = 0 \text{ on } V_G; \\
0 \leq u_i \leq 1; \\
\limsup_{x \to \infty, x \in E_i} u_i = 1.
\end{cases}$$

By the minimizing property of $p$-harmonic functions, $u_i$ is energy finite for each $i$.

Without loss of generality, we may assume that $0 < a_1 \leq a_2 \leq \cdots \leq a_l \leq 2a_1$. Let us construct a sequence of real valued functions $\{v_r \}_{r > r_0}$ such that

$$\begin{cases} 
\Delta_p v_r = 0 \text{ on } N_r(o); \\
v_r = a_i \text{ on } E_i \setminus N_r(o); \\
v_r = 0 \text{ on } V_G \setminus (\bigcup_{k=1}^l E_k \cup N_r(o)),
\end{cases}$$
where $i = 1, 2, \ldots, l$. Then
\[ a_i u_i \leq v_r \leq a_i(2 - u_i) \text{ on } (\delta N_{r_0}(o) \cup \partial N_r(o)) \cap E_i, \]
where $u_i$ is the $p$-harmonic function constructed above. Hence by the comparison principle, we conclude that
\[ a_i u_i \leq v_r \leq a_i(2 - u_i) \text{ on } N_r(o) \cap E_i. \]
There exists a subsequence, denoted by $\{v_{r_m}\}$, converging to a $p$-harmonic function $v$ on $V_G$. By Lemma 3.2, $u_i(x) = 1$ for $p$-almost every path $x \in P_{E_i}$ for each $i$. Hence $v$ satisfies (1). By the minimizing property of $p$-harmonic function, $v$ has finite $p$-Dirichlet sum.

Suppose that there exists a $p$-harmonic function $w \in \mathcal{HBD}_p(G)$ satisfying (1). Put $P_{E_i} = P_{i,v,1} \cup P_{i,v,2}$ for each $i$, where
\[ P_{i,v,1} = \{x \in P_{E_i} : v(x) = a_i\} \text{ and } P_{i,v,2} = \{x \in P_{E_i} : v(x) \neq a_i\}. \]
Then we have $\lambda_p(P_{i,v,1}) < \infty$ and $\lambda_p(P_{i,v,2}) = \infty$ for each $i$. Similarly, let us set $P_{E_i} = P_{i,v,1} \cup P_{i,v,2}$ for each $i$, where
\[ P_{i,v,1} = \{x \in P_{E_i} : v(x) = a_i\} \text{ and } P_{i,v,2} = \{x \in P_{E_i} : v(x) \neq a_i\}. \]
Then we have $\lambda_p(P_{i,v,1}) < \infty$ and $\lambda_p(P_{i,v,2}) = \infty$ for each $i$. From Proposition 2.2 and Proposition 2.3, we conclude that
\[ \lambda_p(P_{E_i} \setminus (P_{i,v,1} \cap P_{i,v,2})) = \lambda_p((P_{E_i} \setminus P_{i,v,1}) \cup (P_{E_i} \setminus P_{i,v,1})) \geq 1/(\lambda_p(P_{E_i} \setminus P_{i,v,1})^{-1} + \lambda_p(P_{E_i} \setminus P_{i,v,1})^{-1}) = \infty \]
for each $i$. This implies that
\[ (v - w)(x) = 0 \text{ for } p\text{-almost every path } x \in P_{E_i}, \]
for each $i = 1, 2, \ldots, l$. On the other hand, since $\lambda_p(P_G \setminus \bigcup_{i=1}^l P_{E_i}) = \infty$, we have
\[ (v - w)(x) = 0 \text{ for } p\text{-almost every path } x \in P_G. \]
Consequently, by Proposition 2.3, we conclude that $v - w \in BD_{p,0}(G)$. Thus there exists a sequence of finitely supported functions converging to $v - w$ in $BD_p(G)$. By this fact together with the Hölder inequality, since $v$ and $w$ are $p$-harmonic functions on $V_G$, it is easy to see that
\[ \sum_{x \in V_G} \sum_{y \in N_x} |v(y) - v(x)|^{p-2}(v(y) - v(x))((v - w)(y) - (v - w)(x)) = 0 \]
and
\[ \sum_{x \in V_G} \sum_{y \in N_x} |w(y) - w(x)|^{p-2}(w(y) - w(x))((v - w)(y) - (v - w)(x)) = 0. \]
Thus by (2), we conclude that $v - w$ is constant function on $N_x$ for all points $x \in V_G$. Since $V_G$ is connected, by (5), we conclude that $v \equiv w$ on $V_G$. \qed
4. Asymptotically constant for $p$-almost every path and rough isometries

We begin with introducing rough isometries between metric spaces. A map $\varphi : X \to Y$ is called a rough isometry between metric spaces $X$ and $Y$ if it satisfies the following condition:

\[(R) \quad \text{for some constant } \tau > 0, \text{ the } \tau\text{-neighborhood of the image } \varphi(X) \text{ covers } Y;\]
\[\text{there exist constants } a \geq 1 \text{ and } b \geq 0 \text{ such that }\]
\[a^{-1}d(x_1, x_2) - b \leq d(\varphi(x_1), \varphi(x_2)) \leq ad(x_1, x_2) + b\]
\[\text{for all points } x_1, x_2 \in X, \text{ where } d \text{ denotes the distances of } X \text{ and } Y\]
induced from their metrics, respectively.

If such a map exists, then $X$ is said to be roughly isometric to $Y$. Being roughly isometric is an equivalent relation. (See [2].) In particular, if $\varphi : X \to Y$ is a rough isometry satisfying $(R)$, then for any point $y \in Y$, there exists at least one point $x \in X$ such that $d(\varphi(x), y) < \tau$. If we set $\varphi^{-1}(y) = x$, then $\varphi^-$ satisfies $(R)$ with constants $\tau', a'$ and $b'$, where $\tau' = a(b + \tau), a' = a$ and $b' = a(b + 2\tau)$.

On the other hand, since the vertex set of each graph is a metric space, we can define rough isometries between the vertex sets of graphs similarly as above. Let $G = (V_G, E_G)$ and $G' = (V_{G'}, E_{G'})$ be graphs, and $\varphi : V_{G'} \to V_G$ be a rough isometry. For convenience' sake, we prefer to write the rough isometry $\varphi : G' \to G$ rather than $\varphi : V_{G'} \to V_G$.

Slightly modifying the proof of [5, 3], the number of ends of a graph is a rough isometric invariant. In fact, the rough isometry between graphs gives a one to one correspondence between ends of the graphs and, furthermore, it induces the rough isometry between each end and its corresponding end. On the other hand, the $p$-parabolicity of ends is preserved under rough isometries between ends. Also, we can prove that the property of asymptotically constant for $p$-almost every path is invariant under rough isometries between ends as follows:

**Theorem 4.1.** Let $G$ and $G'$ be graphs with finitely many ends and roughly isometric to each other. Suppose that every $p$-harmonic function in $\mathcal{HBD}_p(G)$ is asymptotically constant for $p$-almost every path in each $p$-hyperbolic end of $G$. Then every $p$-harmonic function in $\mathcal{HBD}_p(G')$ is asymptotically constant for $p$-almost every path in each $p$-hyperbolic end of $G'$.

To prove Theorem 4.1, we need the following lemmas:

**Lemma 4.2.** Let $G$ and $G'$ be graphs with finitely many ends, and $\varphi : G' \to G$ be a rough isometry. Suppose that every $p$-harmonic function in $\mathcal{HBD}_p(G)$ is asymptotically constant for $p$-almost every path in each $p$-hyperbolic end of $G$. Then for each $u \in \mathcal{HBD}_p(G')$, $u \circ \varphi^-$ is asymptotically constant for $p$-almost every path in each $p$-hyperbolic end of $G$. 
Proof. For each \( u \in \mathcal{HBD}_p(G') \), it is easy to check that \( u \circ \varphi^- \in \mathcal{BD}_p(G) \). So, by Proposition 2.1, there exist unique \( h \in \mathcal{HBD}_p(G) \) and \( g \in \mathcal{D}_{p,0}(G) \) such that
\[
u \circ \varphi^- = h + g.
\]
By the assumption, \( h \) is asymptotically constant for \( p \)-almost every path in each \( p \)-hyperbolic end of \( G \). On the other hand, by Proposition 2.3, \( g \) is asymptotically constant \( 0 \) for \( p \)-almost every path in each \( p \)-hyperbolic end of \( G \).

Let \( E_1, E_2, \ldots, E_l \) be \( p \)-hyperbolic ends of \( G \). Then there exist constants \( c_1, c_2, \ldots, c_l \) such that
\[
 h(y) = c_i \quad \text{for \( p \)-almost every path} \quad y \in P_{E_i}
\]
for each \( i = 1, 2, \ldots, l \). Put \( P_{E_i} = P_{i,h,1} \cup P_{i,h,2} \) for each \( i \), where
\[
P_{i,h,1} = \{ y \in P_{E_i} : h(y) = c_i \} \quad \text{and} \quad P_{i,h,2} = \{ y \in P_{E_i} : h(y) \neq c_i \}.
\]
Then we have \( \lambda_p(P_{i,h,1}) < \infty \) and \( \lambda_p(P_{i,h,2}) = \infty \) for each \( i \). Similarly, let us set \( P_{E_i} = P_{i,g,1} \cup P_{i,g,2} \) for each \( i \), where
\[
P_{i,g,1} = \{ y \in P_{E_i} : g(y) = 0 \} \quad \text{and} \quad P_{i,g,2} = \{ y \in P_{E_i} : g(y) \neq 0 \}.
\]
Then, by our claim, we have \( \lambda_p(P_{i,g,1}) < \infty \) and \( \lambda_p(P_{i,g,2}) = \infty \) for each \( i \).

Arguing similarly as in the proof of Theorem 1.1, we have
\[
 \lambda_p(P_{E_i} \setminus (P_{i,h,1} \cap P_{i,g,1})) = \infty
\]
for each \( i \). Hence \( u \circ \varphi^- \) is asymptotically constant \( c_i \) at infinity of \( E_i \) for \( p \)-almost every path \( y \in P_{E_i} \) for each \( i \). This completes the proof. \( \square \)

Lemma 4.3. Let \( G \) and \( G' \) be graphs with finitely many ends and \( \varphi : G' \to G \) be a rough isometry. Let \( u \in \mathcal{HBD}_p(G') \). Suppose that \( u \circ \varphi^- \) is asymptotically constant for \( p \)-almost every path in each \( p \)-hyperbolic end of \( G \). Then \( u \) is asymptotically constant for \( p \)-almost every path in each \( p \)-hyperbolic end of \( G' \).

Proof. Let \( E \) be a \( p \)-hyperbolic end of \( G \) and \( E' \) be the corresponding end of \( G' \) under \( \varphi \). Since \( u \in \mathcal{HBD}_p(G') \), by Proposition 2.3,
\[
u(x) \text{ exists and finite for } p \text{-almost every path} \ x \in P_o.
\]
Put \( P_{E'} = P_1 \cup P_2 \cup P_3 \), where \( P_1 = \{ x \in P_{E'} : u(x) = c \} \), \( P_2 = \{ x \in P_{E'} : u(x) \neq c \} \) and \( P_3 = \{ x \in P_{E'} : u(x) \text{ does not exists.} \} \). Since \( \lambda_p(P_3) = \infty \), we have only to show that \( \lambda_p(P_2) = \infty \).

For each path \( x \in P_2 \), we will assign a suitable path \( y \in P_{2,\varphi^-} \), where \( P_{2,\varphi^-} = \{ y \in P_G : (u \circ \varphi^-)(y) \neq c \} \). Let us choose any path \( x \in P_2 \). We may assume that \( x = (o, x_1, x_2, \ldots, x_n, \ldots) \). By definition of the inverse rough isometry \( \varphi^- \), there exists a point \( y_n \in E \) such that \( d(x_n, \varphi^-(y_n)) < a(b + \tau) \) for each positive integer \( n \). Let us choose a positive constant \( \rho \) in such a way that \( d(y_n, y_{n+1}) \leq \rho \) and \( d(\varphi^-(y_n), \varphi^-(y_{n+1})) \leq \rho \).

For each positive integer \( n \), we can choose a minimal path \( (z^n_0, z^n_1, \ldots, z^n_m, \ldots) \) in such a way that \( z^n_0 = y_n, z^n_m = y_{n+1} \), and \( m_n \leq \rho \). It follows that there exists an infinite path \( y = (o', t_1, t_2, \ldots, t_j, \ldots) \in P_E \) and a nondecreasing sequence of
subscripts \( j(n) \to \infty \) as \( n \to \infty \) such that \( t_{j(n)} = y_n \) and \( j(n + 1) - j(n) \leq \rho \). One can choose a minimal path \((v^0_0, v^1_1, \ldots, v^n_{l_n})\) in such a way that \( s^n_0 = x_n, \ s^n_{l_n} = \varphi^-(t_{j(n)}) \) and \( l_n \leq a(b + \tau) \). Let us observe that

\[
|u(x_n) - u(\varphi^-(t_{j(n)}))| \leq a(b + \tau) \sum_{i=1}^{l_n} |u(s^n_i) - u(s^n_{i-1})| \\
\leq C \sum_{x' \in N_{a(b+\tau)}(x_n)} |Du|(x').
\]

Since \( u \in BD_{p}(E') \), we conclude that

\[
|u(x_n) - u(\varphi^-(t_{j(n)}))|^p \leq C \sum_{x' \in N_{a(b+\tau)}(x_n)} |Du|^p(x') \to 0 \text{ as } n \to \infty.
\]

This implies that \((u \circ \varphi^-)(t_{j(n)}) \to u(y) \neq c \) as \( n \to \infty \). On the other hand, we have

\[
|u(\varphi^-(t_j)) - u(\varphi^-(t_{j(n)}))| \leq \rho \sum_{i=1}^{m_n} |u(\varphi^-(z^n_i)) - u(\varphi^-(z^n_{i-1}))| \\
\leq C \sum_{x' \in N_{a}(x_n)} |Du|(x')
\]

for each subscript \( j \in [j(n), j(n+1)] \). Hence we have

\[
|u(\varphi^-(t_j)) - u(\varphi^-(t_{j(n)}))|^p \leq C \sum_{x' \in N_{a}(x_n)} |Du|^p(x') \to 0 \text{ as } n \to \infty.
\]

Thus \((u \circ \varphi^-)(t_j) \to u(x) \neq c \) as \( j \to \infty \). Hence \( y \) belongs to \( P_{2, \varphi^-} \).

Since \( \lambda_p(P_{2, \varphi^-}) = \infty \), by the equivalent condition for a family of paths to have infinite \( p \)-extremal length \([4]\), there exists a nonnegative function \( w \) on the edge set \( E_E \) of \( E \) such that \( \sum_{\hat{e} \in E_E} w^p(\hat{e}) = E_p(w) < \infty \) and \( \sum_{\hat{e} \in E(y)} w(\hat{e}) = \infty \) for all paths \( y \in P_{2, \varphi^-} \). For each positive integer \( \zeta \) and each edge \( e = [z_1, z_2] \in E_{E'} \), let us define a set \( U(e, \zeta) = \{ \hat{e} = [a_1, a_2] \in E_E : d(z_i, \varphi^-(a_j)) \leq \zeta \text{ for some } i, j = 1, 2 \} \). Let us define a function \( w^* \) on \( E_{E'} \) in the following way: \( w^*(e) = \sup_{\hat{e} \in U(e, \zeta)} w(\hat{e}) \) for all edges \( e \in E_{E'} \). Since \( w^{*\zeta}(e) \leq \sum_{\hat{e} \in U(e, \zeta)} w^{*\zeta}(\hat{e}) \) for each edge \( e \in E_{E'} \), we have

\[
E_p(w^*) \leq C \sum_{\hat{e} \in E_E} w^p(\hat{e}) < \infty,
\]

where \( C \) is a positive constant depending on \( \zeta \). Let us fix a positive integer \( \kappa \) such that \([t_{j-1}, t_j] \in U([x_n, x_{n+1}], \kappa) \) for all \( j(n) \leq j \leq j(n+1) \), where \( y = (o', t_1, t_2, \ldots, t_j, \ldots) \) is a path in \( P_{2, \varphi^-} \) and \( x = (o, x_1, x_2, \ldots, x_n, \ldots) \) is a path in \( P_2 \) which are given above. Then for each path \( x \in P_2 \),

\[
\sum_{e \in E(x)} w^*(e) \geq \frac{1}{\rho} \sum_{\hat{e} \in E(y)} w(\hat{e}) = \infty.
\]
Therefore, we have $\lambda_p(P_2) = \infty$. This completes the proof.

We are now ready to prove Theorem 4.1:

**Proof of Theorem 4.1.** Let $u$ be a $p$-harmonic function in $\mathcal{HD}_p(G')$. By Lemma 4.2, the function $u \circ \phi^-$ is asymptotically constant for $p$-almost every path in each $p$-hyperbolic end of $G$. Then, by Lemma 4.3, the function $u$ is asymptotically constant for $p$-almost every path in each $p$-hyperbolic end of $G'$. This completes the proof.

Combining Theorem 1.1 and Theorem 4.1, we get Theorem 1.2.

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