UPPER BOUNDS FOR BIVARIATE BONFERRONI-TYPE INEQUALITIES USING CONSECUTIVE EVENTS

MIN-YOUNG LEE

ABSTRACT. Let $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_n$ be two sequences of events on the same probability space. Let $X = X_m(A)$ and $Y = Y_n(B)$, respectively, denote the numbers of those $A_i$'s and $B_j$'s which occur. We establish new bivariate Bonferroni-type inequalities using consecutive events and deduce a known result.

1. Introduction

Let $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_n$ be two sequences of events on the same probability space. Let $X = X_m(A)$ and $Y = Y_n(B)$, respectively, denote the numbers of those $A_i$'s and $B_j$'s which occur. Put $S_{0,0} = 1$ and, for integers $r$ and $t$, set

$$S_{r,t} = \sum \sum P(A_{i_1} A_{i_2} \cdots A_{i_r} B_{j_1} B_{j_2} \cdots B_{j_t}),$$

where the summation is over all subscripts satisfying $1 \leq i_1 < i_2 < \cdots < i_r \leq m$ and $1 \leq j_1 < j_2 < \cdots < j_t \leq n$, $0 \leq r \leq m$ and $0 \leq t \leq n$ (we abbreviate $A \cap B$ as $AB$ and an empty intersection is the sample space). We can easily prove that $S_{r,t}$ at (1) is the binomial moment of the vector $(X,Y)$ and then write the moment form

$$S_{r,t} = E \left[ \binom{X}{r} \binom{Y}{t} \right].$$

We are interested in bivariate Bonferroni-type inequalities which mean bound by linear combinations of the binomial moment $S_{r,t}$. In particular, we want to establish upper bound of $y_{1,1} = P(X_m \geq 1, Y_n \geq 1)$ which appears in many problems in statistics.

Galambos and Xu [3] proved that

$$y_{1,1} = P(\cup_{i=1}^m A_i, \cup_{j=1}^n B_j) \leq S_{1,1} - \frac{2}{m} S_{2,1} - \frac{2}{n} S_{1,2} + \frac{4}{mn} S_{2,2},$$
which insists the best upper bound among all upper bounds of the form $d_1 S_{1,1} + d_2 S_{2,1} + d_3 S_{1,2} + d_4 S_{2,2}$.

The classical lower bound for bivariate probability of degree two is

$$S_{1,1} - S_{1,2} - S_{2,1} \leq P(X_m \geq 1, Y_n \geq 1)$$

and our idea is to reduce the number of terms in binomial moments $S_{1,2}$ and $S_{2,1}$ in order to get an upper bound. For a related idea, see the graph-dependent models of Renyi [5] and Galambos [2].

In this direction, we establish new bivariate Bonferroni-type inequalities using consecutive events and deduce a known result.

**Theorem 1.** For integers $m, n \geq 2$ and $1 \leq i \leq m, \ 1 \leq j \leq n$, then

$$y_{1,1} = P(X_m \geq 1, Y_n \geq 1) \leq S_{1,1} - \sum_{i=1}^{m-1} P(A_i A_{i+1} B_k) - \sum_{j=1}^{n-1} P(A_k B_j B_{j+1}) - \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} P(A_i A_{i+1} B_j B_{j+1}).$$

Taking the averages over $i = 1, \ldots, m$, $j = 1, \ldots, n$ of (2), we get Corollary 1.

**Corollary 1.**

$$y_{1,1} \leq S_{1,1} - \frac{2}{mn} S_{2,1} - \frac{2}{mn} S_{1,2} - \frac{4}{mn} S_{2,2}$$

**Theorem 2.** For integers $m, n \geq 2$ and $1 \leq i \leq m, \ 1 \leq j \leq n$, then

$$y_{1,1} \leq S_{1,1} - \sum_{i=1}^{m-1} \sum_{j=1}^{n} P(A_i A_{i+1} B_j) - \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} P(A_i B_j B_{j+1}) + \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} P(A_i A_{i+1} B_j B_{j+1}).$$

Taking the averages over $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ of (3), we get the following bivariate Bonferroni-type inequality.

**Corollary 2.**

$$y_{1,1} \leq S_{1,1} - \frac{2}{m} S_{2,1} - \frac{2}{n} S_{1,2} + \frac{4}{mn} S_{2,2}.$$
Theorem 3. For integers $m, n \geq 2$ and $1 \leq i \leq m, 1 \leq j \leq n$, then
\begin{align*}
y_{1,1} &\leq S_{1,1} - \sum_{i=1}^{m-1} P(A_iA_{i+1}B_k) - \sum_{i=1}^{m-2} P(A_iA_{i+2}B_k) - \sum_{j=1}^{n-1} P(A_kB_jB_{j+1}) \\
&- \sum_{j=1}^{n-2} P(A_kB_jB_{j+2}) - \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} P(A_iA_{i+1}B_jB_{j+1}) \\
&+ \sum_{i=1}^{m-2} P(A_iA_{i+1}A_{i+2}B_k) + \sum_{j=1}^{n-2} P(A_kB_jB_{j+1}B_{j+2}).
\end{align*}

Taking the averages over $i = 1, \ldots, m, j = 1, \ldots, n$ of (4), we get Corollary 3.

Corollary 3.
\begin{align*}
y_{1,1} &\leq S_{1,1} - \frac{(2m-3)}{(m^2)n} S_{2,1} - \frac{(2n-3)}{(n^2)m} S_{1,2} - \frac{(m-1)(n-1)}{\binom{m}{2}\binom{n}{2}} S_{2,2} \\
&+ \frac{(m-2)}{(m^3)n} S_{3,1} + \frac{(n-2)}{(n^3)m} S_{1,3}.
\end{align*}

Theorem 4. For integers $m, n \geq 2$ and $1 \leq i \leq m, 1 \leq j \leq n$, then
\begin{align*}
y_{1,1} &\leq S_{1,1} - \sum_{i=1}^{m} \sum_{1 \leq j < k \leq j+2} P(A_iB_jB_k) - \sum_{1 \leq i < l \leq i+2} \sum_{j=1}^{n} P(A_iA_lB_j) \\
&+ \sum_{1 \leq i < l \leq i+2} \sum_{1 \leq j < k \leq j+2} P(A_iA_lB_jB_k) + \sum_{i=1}^{m} \sum_{j=1}^{n-2} P(A_iA_jB_{j+1}B_{j+2}) \\
&+ \sum_{i=1}^{m} \sum_{j=1}^{n-2} P(A_iA_{i+1}A_{i+2}B_j) - \sum_{1 \leq i < l \leq i+2} \sum_{j=1}^{n-2} P(A_iA_lB_jB_{j+1}B_{j+2}) \\
&- \sum_{i=1}^{m} \sum_{1 \leq j < k \leq j+2} P(A_iA_{i+1}A_{i+2}B_jB_k) \\
&+ \sum_{i=1}^{m} \sum_{j=1}^{n-2} P(A_iA_{i+1}A_{i+2}B_jB_{j+1}B_{j+2}).
\end{align*}

Taking the averages over $i = 1, \ldots, m, j = 1, \ldots, n$ of (5), we get Corollary 4.
Corollary 4.

\[ y_{1,1} \leq S_{1,1} - \frac{m(2n - 3)}{\binom{m}{1}\binom{n}{2}} S_{1,2} - \frac{(2m - 3)n}{\binom{m}{2}\binom{n}{1}} S_{2,1} + \frac{(2m - 3)(2n - 3)}{\binom{m}{2}\binom{n}{2}} S_{2,2} \]
\[ + \frac{m(n - 2)}{\binom{m}{1}\binom{n}{3}} S_{1,3} + \frac{(m - 2)n}{\binom{m}{3}\binom{n}{1}} S_{3,1} - \frac{(2m - 3)n}{\binom{m}{2}\binom{n}{3}} S_{2,3} \]
\[ - \frac{(m - 2)(2n - 3)}{\binom{m}{3}\binom{n}{2}} S_{3,2} + \frac{(m - 2)(n - 2)}{\binom{m}{3}\binom{n}{3}} S_{3,3}. \]

2. Proofs

Proof of Theorem 1. We use the method of indicators. Let

\[ I(X \geq 1, Y \geq 1) = \begin{cases} 1, & \text{if } X \geq 1 \text{ and } Y \geq 1 \\ 0, & \text{otherwise}. \end{cases} \]

By using binomial moments and indicators, the right hand side of (2) becomes

\[ E\left[ XY - \sum_{i=1}^{m-1} I(A_i)I(A_{i+1})I(B_k) - \sum_{j=1}^{n-1} I(A_k)I(B_j)I(B_{j+1}) \right. \]
\[ \left. - \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} I(A_i)I(A_{i+1})I(B_j)I(B_{j+1}) \right]. \]

Then \( E[I(X \geq 1, Y \geq 1)] = P(X \geq 1, Y \geq 1), \) it suffices to show that

\[ I(X \geq 1)I(Y \geq 1) \]
\[ \leq XY - \sum_{i=1}^{m-1} I(A_i)I(A_{i+1})I(B_k) + \sum_{j=1}^{n-1} I(A_k)I(B_j)I(B_{j+1}) \]
\[ + \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} I(A_i)I(A_{i+1})I(B_j)I(B_{j+1}) \].

Note that both sides of (7) are zero if either \( X \) or \( Y \) equals zero, hence, in proving (7) we may assume that \( X \geq 1 \) and \( Y \geq 1 \), in which the left hand side of (7) is identically one. Thus, we have to prove that

\[ u(X, Y) = \text{the right hand side of (7)} \geq 1 \text{ for } 1 \leq X \leq m, \ 1 \leq Y \leq n. \]

We distinguish three cases:

(i) The case \( X = 1, Y = 1; \) that is, there are only two events \( A_i \) and \( B_j \) occur. Then this case is evident, having one on both sides of (8).

(ii) The case \( X = 1, Y = q \) or \( X = p, Y = 1 \) for \( 2 \leq p \leq m, \ 2 \leq q \leq n; \) that is, there are the events that exactly one \( A_i(B_j) \) and at least two more \( B_j's(A_i's) \) occur. Then

\[ u(1, q) = 1 \cdot q - (q - 1) = 1 \text{ and } u(p, 1) = p \cdot 1 - (p - 1) = 1. \]
Hence, we get (8).

(iii) The case \( X = p, Y = q \) for \( 2 \leq p \leq m, \ 2 \leq q \leq n \); that is, there are the events that at least two more \( A_i \)'s and \( B_j \)'s occur. Then

\[
 u(p,q) = p \cdot q - \{(p-1) + (q-1) + (p-1) \cdot (q-1)\} = 1
\]

Hence, we get (8). This completes the proof. \( \Box \)

**Proof of Theorem 2.** We can prove (3) by the same way of proof of Theorem 1. \( \Box \)

**Proof of Theorem 3.** We can prove (4) by the same way of proof of Theorem 1. \( \Box \)

**Proof of Theorem 4.** We use Bonferroni-type inequality of Lee [4], that is,

\[
P(\bigcup_{i=1}^{m} A_i) \leq \sum_{i=1}^{m} P(A_i) - \sum_{i<j \leq i+2}^{m-2} P(A_i A_j) + \sum_{i=1}^{m-2} P(A_i A_{i+1} A_{i+2}).
\]

We consider two univariate Bonferroni-type inequalities.

(9) \[
P(\bigcup_{i=1}^{m} A_i) \leq \sum_{i=1}^{m} P(A_i) - \sum_{i<j \leq i+2}^{m-2} P(A_i A_j) + \sum_{i=1}^{m-2} P(A_i A_{i+1} A_{i+2}),
\]

(10) \[
P(\bigcup_{j=1}^{n} B_i) \leq \sum_{j=1}^{n} P(B_i) - \sum_{j<k \leq j+2}^{n-2} P(B_i B_k) + \sum_{j=1}^{n-2} P(B_j B_{j+1} B_{j+2}).
\]

Turning to indicators, (12) and (13) become

(11) \[
I(X \geq 1) \leq \sum_{i=1}^{m} I(A_i) - \sum_{i<j \leq i+2}^{m-2} I(A_i)I(A_j) + \sum_{i=1}^{m-2} I(A_i)I(A_{i+1})I(A_{i+2}),
\]

(12) \[
I(Y \geq 1) \leq \sum_{j=1}^{n} I(B_j) - \sum_{j<k \leq j+2}^{n-2} I(B_j)I(B_k) + \sum_{j=1}^{n-2} I(B_j)I(B_{j+1})I(B_{j+2}).
\]

By multiplying (11) and (12) and taking expectations, we get Theorem 4. \( \Box \)

3. Numerical examples

**Example 3-1.** Let a machine consist of two pieces of equipments \( A \) and \( B \). Let \( X_i \) be the time to failure of the \( i \)-th component of equipment \( A \) and let \( Y_j \) be the time to failure of the \( j \)-th component of equipment \( B \). Assume that each \( X_i \) and each \( Y_j \) are unit exponential variates, that is, for each \( i, j \),

\[
P(X_i < x) = 1 - e^{-x}, \quad x > 0 \quad \text{and} \quad P(Y_j < y) = 1 - e^{-y}, \quad y > 0.
\]

Consider a group \( A \) of ten components and a group \( B \) of five components. Let \( X_1, X_2, \ldots, X_{10} \) be independent and identically distributed random variables
and let \( Y_1, Y_2, \ldots, Y_5 \) be independent and identically distributed random variables. We assume the structure is such that each \( X_i \) is completely dependent on each \( Y_j \) and it has probability zero that at least one component of equipment \( A(B) \) fails within \( x(y) \) period of time and all components of equipment \( B(A) \) fail after \( y(x) \) period of time, that is, for each \( 1 \leq i \leq 10, 1 \leq j \leq 5 \),

\[
P\left( \bigcup_{i=1}^{10} (X_i < x), \bigcap_{j=1}^{5} (Y_j \geq y) \right) = P\left( \bigcap_{i=1}^{10} (X_i \geq x), \bigcup_{j=1}^{5} (Y_j < y) \right) = 0.
\]

We also specify the bivariate distributions and the trivariate distributions of the combination of \( X_i \) and \( Y_j \). For simplicity, let us use the same bivariate and trivariate distributions for all dependent components. Let, for \( 1 \leq i \leq 10, 1 \leq j \leq 5 \),

\[
P(X_i < x, Y_j < y) = (1 - e^{-x})(1 - e^{-y})(1 - \frac{1}{2} e^{-x-y}),
\]

\[
P(X_{i_1} < x, X_{i_2} < x, Y_j < y) = (1 - e^{-x})^2(1 - e^{-y})(1 - \frac{1}{3} e^{-2x-y}),
\]

\[
P(X_i < x, Y_{j_1} < y, Y_{j_2} < y) = (1 - e^{-x})(1 - e^{-y})^2(1 - \frac{1}{3} e^{-x-2y}),
\]

\[
P(X_{i_1} < x, X_{i_2} < x, X_{i_3} < x, Y_j < y) = (1 - e^{-x})^3(1 - e^{-y})(1 - \frac{1}{4} e^{-3x-y}),
\]

\[
P(X_{i_1} < x, X_{i_2} < x, Y_{j_1} < y, Y_{j_2} < y) = (1 - e^{-x})^2(1 - e^{-y})^2(1 - \frac{1}{4} e^{-2x-2y}),
\]

\[
P(X_{i_1} < x, Y_{j_1} < y, Y_{j_2} < y, Y_{j_3} < y) = (1 - e^{-x})(1 - e^{-y})^3(1 - \frac{1}{4} e^{-x-3y}),
\]

\[
P(X_{i_1} < x, X_{i_2} < x, X_{i_3} < x, Y_{j_1} < y, Y_{j_2} < y)
\]

\[
= (1 - e^{-x})^3(1 - e^{-y})^2(1 - \frac{1}{5} e^{-3x-2y}),
\]

\[
P(X_{i_1} < x, X_{i_2} < x, Y_{j_1} < y, Y_{j_2} < y, Y_{j_3} < y)
\]

\[
= (1 - e^{-x})^2(1 - e^{-y})^3(1 - \frac{1}{5} e^{-2x-3y}),
\]

\[
P(X_{i_1} < x, X_{i_2} < x, X_{i_3} < x, Y_{j_1} < y, Y_{j_2} < y, Y_{j_3} < y)
\]

\[
= (1 - e^{-x})^3(1 - e^{-y})^3(1 - \frac{1}{6} e^{-3x-3y}).
\]

No further assumption is made. We would like to estimate \( P(W_X \geq x, W_Y \geq y) \), where \( W_X = \min(X_1, X_2, \ldots, X_{10}) \) and \( W_Y = \min(Y_1, Y_2, \ldots, Y_5) \). Here, of course, the events \( A_i = (X_i < x) \) and \( B_j = (Y_j < y) \) and thus \( V_{10} = 0, U_5 = 0 \) = \( (W_X \geq x, W_Y \geq y) \). We can now compute the following probability. For a numerical calculation, let us choose \( x = 0.1 \) and \( y = 0.2 \). Let \( V_{10} \) be the number of those \( A_i = (X_i < 0.1) \) which occur and let \( U_5 \) be the number of those \( B_j = (Y_j < 0.2) \) which occur.
\[ S_{1,1} = \binom{10}{1} \binom{5}{1} (1 - e^{-0.1})(1 - e^{-0.2})^2 \left(1 - \frac{1}{2} e^{-0.3}\right) = 0.54301, \]
\[
\sum_{i=1}^{9} \sum_{j=1}^{4} P(A_i A_{i+1} B_k) = 9(1 - e^{-0.1})^2(1 - e^{-0.2})(1 - \frac{1}{3} e^{-0.4}) = 0.011472, \]
\[
\sum_{j=1}^{4} P(A_k B_j B_{j+1}) = 4(1 - e^{-0.1})(1 - e^{-0.2})^2 \left(1 - \frac{1}{3} e^{-0.5}\right) = 0.009979, \]
\[
\sum_{i=1}^{9} \sum_{j=1}^{4} P(A_i A_{i+1} B_j B_{j+1}) = 36(1 - e^{-0.1})^2(1 - e^{-0.2})^2 \left(1 - \frac{1}{4} e^{-0.6}\right) = 0.009242, \]
\[
\sum_{i=1}^{9} \sum_{j=1}^{5} P(A_i A_{i+1} B_j) = 45(1 - e^{-0.1})^2(1 - e^{-0.2})(1 - \frac{1}{3} e^{-0.4}) = 0.057362, \]
\[
\sum_{i=1}^{10} \sum_{j=1}^{4} P(A_i B_j B_{j+1}) = 40(1 - e^{-0.1})(1 - e^{-0.2})^2 \left(1 - \frac{1}{3} e^{-0.5}\right) = 0.099787, \]
\[
\sum_{i=1}^{8} P(A_i A_{i+2} B_k) = 8(1 - e^{-0.1})^2(1 - e^{-0.2})(1 - \frac{1}{3} e^{-0.4}) = 0.010198, \]
\[
\sum_{j=1}^{3} P(A_k B_j B_{j+2}) = 3(1 - e^{-0.1})(1 - e^{-0.2})^2 \left(1 - \frac{1}{3} e^{-0.5}\right) = 0.007484, \]
\[
\sum_{i=1}^{8} P(A_i A_{i+1} A_{i+2} B_k) = 8(1 - e^{-0.1})^3(1 - e^{-0.2})(1 - \frac{1}{4} e^{-0.5}) = 0.001060, \]
\[
\sum_{j=1}^{3} P(A_k B_j B_{j+1} B_{j+2}) = 3(1 - e^{-0.1})(1 - e^{-0.2})^3 \left(1 - \frac{1}{4} e^{-0.7}\right) = 0.001489, \]
\[
\sum_{i=1}^{10} \sum_{1 \leq j < k \leq j+2} P(A_i B_j B_k) = 70(1 - e^{-0.1})(1 - e^{-0.2})^2 \left(1 - \frac{1}{3} e^{-0.5}\right) = 0.174627, \]
\[
\sum_{1 \leq i \leq i+2} \sum_{j=1}^{5} P(A_i A_i B_j) = 85(1 - e^{-0.1})^2(1 - e^{-0.2})(1 - \frac{1}{3} e^{-0.4}) = 0.108350, \]
\[
\sum_{1 \leq i \leq i+2} \sum_{1 \leq j < k \leq j+2} \sum_{3} P(A_i A_i B_j B_k) = 119(1 - e^{-0.1})^2(1 - e^{-0.2})^2 \left(1 - \frac{1}{4} e^{-0.6}\right) = 0.030550, \]
\[
\sum_{i=1}^{10} \sum_{j=1}^{3} P(A_i B_j B_{j+1} B_{j+2}) = 30(1 - e^{-0.1})(1 - e^{-0.2})^3 \left(1 - \frac{1}{4} e^{-0.7}\right) = 0.014893, \]
\[\begin{align*}
\sum_{i=1}^{8} \sum_{j=1}^{5} P(A_i A_{i+1} A_{i+2} B_j) &= 40(1 - e^{-0.1})^3(1 - e^{-0.2}) (1 - \frac{1}{4} e^{-0.5}) = 0.005301, \\
\sum_{1 \leq i < l \leq i+2} \sum_{j=1}^{3} P(A_i A_l B_j B_{j+1} B_{j+2}) &= 51(1 - e^{-0.1})^2(1 - e^{-0.2})^3 (1 - \frac{1}{5} e^{-0.8}) = 0.002504, \\
\sum_{i=1}^{8} \sum_{1 \leq j < k \leq j+2} P(A_i A_{i+1} A_{i+2} B_j B_k) &= 56(1 - e^{-0.1})^3(1 - e^{-0.2})^2 (1 - \frac{1}{5} e^{-0.7}) = 0.001428, \\
\sum_{i=1}^{8} \sum_{j=1}^{3} P(A_i A_{i+1} A_{i+2} B_j B_{j+1} B_{j+2}) &= 24(1 - e^{-0.1})^3(1 - e^{-0.2})^3 (1 - \frac{1}{6} e^{-0.9}) = 0.000115.
\end{align*}\]

Now, we can get the upper bounds of \(P(V_{10} \geq 1, U_5 \geq 1)\). Since \(P(W_X \geq 0.1, W_Y \geq 0.2) = 1 - P(V_{10} \geq 1, U_5 \geq 1)\) by our earlier assumption on dependence, we get the following lower bounds of \(P(W_X \geq 0.1, W_Y \geq 0.2)\).

**Lower bounds for \(P(W_X \geq 0.1, W_Y \geq 0.2)\)**

<table>
<thead>
<tr>
<th>inequality</th>
<th>upper bound for (y_{1,1})</th>
<th>lower bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2)</td>
<td>0.512317</td>
<td>0.487683</td>
</tr>
<tr>
<td>(3)</td>
<td>0.395104</td>
<td>0.604896</td>
</tr>
<tr>
<td>(4)</td>
<td>0.497185</td>
<td>0.502815</td>
</tr>
<tr>
<td>(5)</td>
<td>0.306961</td>
<td>0.693039</td>
</tr>
</tbody>
</table>

In the above table, we see that (5) is the best upper bound for \(y_{1,1}\).

**Example 3-2.** Consider a numerical example in the paper of Chen and Seneta [1]. Let \(C_1, \ldots, C_6\) be events with specified probabilities (see table 1 of [1]). Let \(C_1 = A_1, C_2 = A_2, C_3 = A_3, C_4 = B_1, C_5 = B_2, C_6 = B_3\). Then \(S_{1,1} = 1.259, S_{2,1} = 0.225, S_{1,2} = 0.37, S_{2,2} = 0.055, S_{1,3} = S_{2,3} = S_{3,1} = S_{3,2} = S_{3,3} = 0\). The upper bound by Chen and Seneta [1] is following

\[P(m_n \geq a_1, m_N \geq a_2) \leq S_{a_1, a_2} - \left( \frac{a_1 + 1}{n - a_1} - \left( \frac{n}{a_1 + 1} \right)^{-1} \right) S_{a_1 + 1, a_2}
- \left( \frac{a_2 + 1}{N - a_2} - \left( \frac{N}{a_2 + 1} \right)^{-1} \right) S_{a_1, a_2 + 1}
+ \left( \frac{a_1 + 1}{n - a_1} - \left( \frac{n}{a_1 + 1} \right)^{-1} \right) \left( \frac{a_2 + 1}{N - a_2} - \left( \frac{N}{a_2 + 1} \right)^{-1} \right) S_{a_1 + 1, a_2 + 1}.\]
This yields $y_{1,1} \leq 0.887$ (see table 2 of [1]). But Corollary 4 gives $y_{1,1} \leq 0.719$.

References


DEPARTMENT OF MATHEMATICS
DANKOOK UNIVERSITY
CHUNGNA 330-714, KOREA
E-mail address: leemy@dankook.ac.kr