COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE OF FOUR MAPPINGS

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Abstract. In this paper, a common fixed point theorem for weakly compatible maps in complete fuzzy metric spaces is proved.

1. Introduction and preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [22] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [5] and Kramosil and Michalek [9] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connections with both string and \( e^{(\infty)} \) theory which were given and studied by El Naschie [1, 2, 3, 4, 19]. Many authors [7, 11, 16, 13, 14, 15] have proved fixed point theorem in fuzzy (probabilistic) metric spaces. Vasuki [20] obtained the fuzzy version of common fixed point theorem which had extra conditions. In fact, Vasuki proved fuzzy common fixed point theorem by a strong definition of Cauchy sequence (see Note 3.13 and Definition 3.15 of [5] also [18, 21]). In this paper, we prove a common fixed point theorem in fuzzy metric spaces for arbitrary \( t \)-norms and modified definition of Cauchy sequence in George and Veeramani’s sense.

Definition 1.1. A binary operation \( * : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is a continuous \( t \)-norm if it satisfies the following conditions

1. \( * \) is associative and commutative,
2. \( * \) is continuous,
3. \( a * 1 = a \) for all \( a \in [0, 1] \),
4. \( a * b \leq c * d \) whenever \( a \leq c \) and \( b \leq d \), for each \( a, b, c, d \in [0, 1] \).

Two typical examples of continuous \( t \)-norm are \( a * b = ab \) and \( a * b = \min(a, b) \).
Definition 1.2. A 3-tuple \((X, M, \ast)\) is called a fuzzy metric space if \(X\) is an arbitrary (non-empty) set, \(\ast\) is a continuous t-norm, and \(M\) is a fuzzy set on \(X^2 \times (0, \infty)\), satisfying the following conditions for each \(x, y, z \in X\) and \(t, s > 0\),

\[
\begin{align*}
(1) & \quad M(x, y, t) > 0, \\
(2) & \quad M(x, y, t) = 1 \text{ if and only if } x = y, \\
(3) & \quad M(x, y, t) = M(y, x, t), \\
(4) & \quad M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s), \\
(5) & \quad M(x, y, \cdot) : (0, \infty) \longrightarrow [0, 1] \text{ is continuous}.
\end{align*}
\]

Let \((X, M, \ast)\) be a fuzzy metric space. For \(t > 0\), the open ball \(B(x, r, t)\) with center \(x \in X\) and radius \(0 < r < 1\) is defined by

\[
B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.
\]

Let \((X, M, \ast)\) be a fuzzy metric space. Let \(\tau\) be the set of all \(A \subset X\) with \(x \in A\) if and only if there exist \(t > 0\) and \(0 < r < 1\) such that \(B(x, r, t) \subset A\). Then \(\tau\) is a topology on \(X\) (induced by the fuzzy metric \(M\)). This topology is Hausdorff and first countable. A sequence \(\{x_n\}\) in \(X\) converges to \(x\) if and only if \(M(x_n, x, t) \rightarrow 1\) as \(n \rightarrow \infty\), for each \(t > 0\). It is called a Cauchy sequence if for each \(0 < \varepsilon < 1\) and \(t > 0\), there exits \(n_0 \in \mathbb{N}\) such that \(M(x_n, x_m, t) > 1 - \varepsilon\) for each \(n, m \geq n_0\). The fuzzy metric space \((X, M, \ast)\) is said to be complete if every Cauchy sequence is convergent. A subset \(A\) of \(X\) is said to be F-bounded if there exists \(t > 0\) and \(0 < r < 1\) such that \(M(x, y, t) > 1 - r\) for all \(x, y \in A\).

Lemma 1.3 ([5]). Let \((X, M, \ast)\) be a fuzzy metric space. Then \(M(x, y, t)\) is nondecreasing with respect to \(t\), for all \(x, y \in X\).

Definition 1.4. Let \((X, M, \ast)\) be a fuzzy metric space. \(M\) is said to be continuous function on \(X^2 \times (0, \infty)\) if

\[
\lim_{n \to \infty} M(x_n, y_n, t_n) = M(x, y, t).
\]

Whenever a sequence \(\{(x_n, y_n, t_n)\}\) in \(X^2 \times (0, \infty)\) converges to a point \((x, y, t) \in X^2 \times (0, \infty)\) i.e.

\[
\lim_{n \to \infty} M(x_n, x, t) = \lim_{n \to \infty} M(y_n, y, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} M(x, y, t_n) = M(x, y, t).
\]

Lemma 1.5. Let \((X, M, \ast)\) be a fuzzy metric space. Then \(M\) is continuous function on \(X^2 \times (0, \infty)\).

Proof. See Proposition 1 of [10]. \(\square\)

Example 1.6. Let \(X = \mathbb{R}\). Denote \(a \ast b = a \cdot b\) for all \(a, b \in [0, 1]\). For each \(t \in [0, \infty]\), define

\[
M(x, y, t) = \frac{t}{t + |x - y|}
\]

for all \(x, y \in X\).
Definition 1.7. Let $A$ and $S$ be mappings from a fuzzy metric space $(X, M, \ast)$ into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.

Definition 1.8. Let $A$ and $S$ be mappings from a fuzzy metric space $(X, M, \ast)$ into itself. Then the mappings are said to be compatible if

$$\lim_{n \to \infty} M(ASx_n, SAx_n, t) = 1, \forall t > 0$$

whenever $\{x_n\}$ is a sequence in $X$ such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = x \in X.$$  

Proposition 1.9 ([17]). Self-mappings $A$ and $S$ of a fuzzy metric space $(X, M, \ast)$ are compatible, then they are weak compatible.

Lemma 1.10. Let $(X, M, \ast)$ be a fuzzy metric space. If we define $E_{\lambda, M} : X^2 \to \mathbb{R}^+ \cup \{0\}$ by

$$E_{\lambda, M}(x, y) = \inf\{t > 0 : M(x, y, t) > 1 - \lambda\}$$

for $\lambda \in (0, 1)$, then

(i) for each $\mu \in (0, 1)$ there exists $\lambda \in (0, 1)$ such that

$$E_{\mu, M}(x_1, x_n) \leq E_{\lambda, M}(x_1, x_2) + E_{\lambda, M}(x_2, x_3) + \cdots + E_{\lambda, M}(x_{n-1}, x_n)$$

for any $x_1, x_2, \ldots, x_n \in X$

(ii) The sequence $\{x_n\}_{n \in \mathbb{N}}$ is convergent in fuzzy metric space $(X, M, \ast)$ if and only if $E_{\lambda, M}(x_n, x) \to 0$. Also the sequence $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy sequence if and only if it is Cauchy with $E_{\lambda, M}$.

Proof. (i). For every $\mu \in (0, 1)$, we can find a $\lambda \in (0, 1)$ such that

$$\underbrace{(1 - \lambda) \ast (1 - \lambda) \ast \cdots \ast (1 - \lambda)}_n \geq 1 - \mu$$

by triangular inequality we have

$$M(x_1, x_n, E_{\lambda, M}(x_1, x_2) + E_{\lambda, M}(x_2, x_3) + \cdots + E_{\lambda, M}(x_{n-1}, x_n) + n\delta)$$

$$\geq M(x_1, x_2, E_{\lambda, M}(x_1, x_2) + \delta) \ast \cdots \ast M(x_{n-1}, x_n, E_{\lambda, M}(x_{n-1}, x_n) + \delta)$$

$$\geq \underbrace{(1 - \lambda) \ast (1 - \lambda) \ast \cdots \ast (1 - \lambda)}_n \geq 1 - \mu$$

for very $\delta > 0$, which implies that

$$E_{\mu, M}(x_1, x_n) \leq E_{\lambda, M}(x_1, x_2) + E_{\lambda, M}(x_2, x_3) + \cdots + E_{\lambda, M}(x_{n-1}, x_n) + n\delta.$$  

Since $\delta > 0$ is arbitrary, we have

$$E_{\mu, M}(x_1, x_n) \leq E_{\lambda, M}(x_1, x_2) + E_{\lambda, M}(x_2, x_3) + \cdots + E_{\lambda, M}(x_{n-1}, x_n)$$

(ii). Note that since $M$ is continuous in its third place and

$$E_{\lambda, M}(x, y) = \inf\{t > 0 : M(x, y, t) > 1 - \lambda\}.$$


Hence, we have
\[ M(x_n, x, \eta) > 1 - \lambda \iff E_{\lambda, M}(x_n, x) < \eta \]
for every \( \eta > 0 \).

**Lemma 1.11.** Let \((X, M, *)\) be a fuzzy metric space. If
\[ M(x_n, x_{n+1}, t) \geq M(x_0, x_1, k^n t) \]
for some \( k > 1 \) and for every \( n \in \mathbb{N} \). Then sequence \( \{x_n\} \) is a Cauchy sequence.

**Proof.** For every \( \lambda \in (0, 1) \) and \( x_n, x_{n+1} \in X \), we have
\[
E_{\lambda, M}(x_{n+1}, x_n) = \inf\{t > 0 : M(x_{n+1}, x_n, t) > 1 - \lambda\}
\leq \inf\{t > 0 : M(x_0, x_1, k^n t) > 1 - \lambda\}
= \inf\{\frac{t}{k^n} : M(x_0, x_1, t) > 1 - \lambda\}
= \frac{1}{k^n} \inf\{t > 0 : M(x_0, x_1, t) > 1 - \lambda\}
= \frac{1}{k^n} E_{\lambda, M}(x_0, x_1).
\]

By Lemma 1.10, for every \( \mu \in (0, 1) \) there exists \( \lambda \in (0, 1) \) such that
\[
E_{\mu, M}(x_n, x_m) \leq E_{\lambda, M}(x_n, x_{n+1}) + E_{\lambda, M}(x_{n+1}, x_{n+2}) + \cdots + E_{\lambda, M}(x_{m-1}, x_m)
\leq \frac{1}{k^n} E_{\lambda, M}(x_0, x_1) + \frac{1}{k^{n+1}} E_{\lambda, M}(x_0, x_1) + \cdots + \frac{1}{k^{m-1}} E_{\lambda, M}(x_0, x_1)
= E_{\lambda, M}(x_0, x_1) \sum_{j=n}^{m-1} \frac{1}{k^j} \to 0.
\]
Hence sequence \( \{x_n\} \) is Cauchy sequence.

\[ \Box \]

**2. The main results**

**A class of implicit relation**

Let \( \Phi \) denotes a family of mappings such that each \( \phi \in \Phi, \phi : [0,1]^3 \to [0,1], \) and \( \phi \) is continuous and increasing in each co-ordinate variable. Also \( \phi(s, s, s) > s \) for every \( s \in [0,1) \).

**Example 2.1.** Let \( \phi : [0,1]^3 \to [0,1] \) is define by

(i) \( \phi(x_1, x_2, x_3) = (\min \{x_i\})^h \) for some \( 0 < h < 1 \).

(ii) \( \phi(x_1, x_2, x_3) = x_i^h \) for some \( 0 < h < 1 \).

(iii) \( \phi(x_1, x_2, x_3) = \max \{x_1^{\alpha_1}, x_2^{\alpha_2}, x_3^{\alpha_3}\} \), where \( 0 < \alpha_i < 1 \) for \( i = 1, 2, 3 \).

In this paper \( p \) is a positive real number and \( \phi^{2p}(s, s, s) = [\phi(s, s, s)]^{2p} \) for every \( s \in [0,1] \). Also
\[
M(Sx, By, t) \vee M(Ty, Ax, t) = \max\{M(Sx, By, t), M(Ty, Ax, t)\}.
\]
Our main result, for a complete fuzzy metric space $X$, reads follows:

**Theorem 2.2.** Let $A, B, S$ and $T$ be a self-mapping of complete fuzzy metric space $(X, M, \ast)$, satisfying the following conditions:

(i) $(A, S)$ and $(B, T)$ are weakly compatible pairs such that $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ also $A(X)$ or $B(X)$ is a closed subset of $X$;

(ii) there exist $\psi, \phi \in \Phi$ such that for all $x, y \in X$,

$$M^{2p}(Ax, By, t) \geq a(s)\phi^{2p}\left(\begin{array}{c} M(Sx, Ty, kt), \\ M(By, Ty, kt) \end{array}\right) + b(s)\psi^{p}\left(\begin{array}{c} M^2(Sx, Ty, kt), \\ M(Sx, By, kt) \end{array}\right) \vee M(Ty, By, kt),$$

for some $k > 1$, where $a, b : [0, 1] \rightarrow [0, 1]$ are two continuous functions such that $a(s) + b(s) = 1$ for every $s = M(x, y, t)$.

Then $A, B$ and $S, T$ have a unique common fixed point in $X$.

**Proof.** Let $x_0 \in X$ be an arbitrary point as $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, there exist $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$, $Bx_1 = Sx_2$. Inductively, construct sequence $\{y_n\}$ and $\{x_n\}$ in $X$ such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+1} = Bx_2n+1 = Sx_{2n+2}$ for $n = 0, 1, 2, \ldots$.

Now, we prove $\{y_n\}$ is a Cauchy sequence. For simplicity, we set

$$d_n(t) = M(y_n, y_{n+1}, t), \quad n = 0, 1, 2, \ldots$$

Then we have

$$d_{2n}^p(t) \geq a(s)\phi^{2p}\left(\begin{array}{c} M(Sx_{2n}, Tx_{2n+1}, kt), \\ M(Bx_{2n+1}, Tx_{2n+1}, kt) \end{array}\right) + b(s)\psi^p\left(\begin{array}{c} M^2(Sx_{2n}, Tx_{2n+1}, kt), \\ M(Sx_{2n}, Bx_{2n+1}, kt) \end{array}\right) \vee M(Tx_{2n+1}, Bx_{2n+1}, kt)$$

We prove that $d_{2n}(t) \geq d_{2n-1}(t)$. Now, if $d_{2n}(t) < d_{2n-1}(t)$ for some $n \in \mathbb{N}$, since $\phi$ and $\psi$ are increasing functions, then

$$d_{2n}^p(t) \geq a(s)\phi^{2p}(d_{2n-1}(kt), d_{2n-1}(kt), d_{2n}(kt)) + b(s)\psi^p(d_{2n-1}(kt), d_{2n-1}(kt), d_{2n}(kt), 1)$$

$$\geq a(s)\phi^{2p}(d_{2n}(kt), d_{2n}(kt), d_{2n}(kt)) + b(s)\psi^p(d_{2n}(kt), d_{2n}(kt), 1)$$

$$> a(s)d_{2n}^p(kt) + b(s)d_{2n}^p(kt) = d_{2n}^p(kt),$$
hence we have $d_{2n}(t) > d_{2n}(kt)$ is a contradiction. Therefore $d_{2n}(t) \geq d_{2n-1}(t)$. Similarly, one can prove that $d_{2n+1}(t) \geq d_{2n}(t)$ for $n = 0, 1, 2, \ldots$. Consequently, $\{d_n(t)\}$ is a increasing sequence of non-negative real. Thus

\[
d_{2n}^p(t) \\
\geq a(s)\phi_{2p}(d_{2n-1}(kt), d_{2n-1}(kt), d_{2n-1}(kt)) + b(s)\psi^p(d_{2n-1}^2(kt), d_{2n-1}^2(kt), 1) \\
\geq a(s)d_{2n-1}^p(kt) + b(s)d_{2n-1}^2(kt) = d_{2n-1}^p(kt).
\]

That is $d_{2n}(t) \geq d_{2n-1}(kt)$, similarly, we have $d_{2n+1}(t) \geq d_{2n}(kt)$. Thus

\[
d_n(t) \geq d_{n-1}(kt).
\]

That is

\[
M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, kt).
\]

So

\[
M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, kt) \geq \cdots \geq M(y_0, y_1, k^nt).
\]

By Lemma 1.11 sequence $\{y_n\}$ is a Cauchy sequence, then it is converges to $y \in X$. That is

\[
\lim_{n \to \infty} y_n = \lim_{n \to \infty} y_{2n} = \lim_{n \to \infty} y_{2n+1} = \lim_{n \to \infty} Ax_{2n} = \lim_{n \to \infty} Bx_{2n+1} = \lim_{n \to \infty} Sx_{2n} = \lim_{n \to \infty} Tx_{2n+1} = y.
\]

As $B(X) \subseteq S(X)$, there exist $u \in X$ such that $Su = y$. So, we have

\[
M^{2p}(Au, Bx_{2n+1}, t) \\
\geq a(s)\phi_{2p} \left( \begin{array}{c} M(Su, Tx_{2n+1}, kt), \\ M(Tx_{2n+1}, Bx_{2n+1}, kt) \end{array} \right) \\
+ b(s)\psi^p \left( \begin{array}{c} M^2(Su, Tx_{2n+1}, kt), M(Su, Au, kt)M(Tx_{2n+1}, Bx_{2n+1}, kt) \\ M(Su, Bx_{2n+1}, kt) \vee M(Tx_{2n+1}, Au, kt) \end{array} \right).
\]

By continuous $M$ and $\phi$, on making $n \to \infty$ the above inequality, we get

\[
M^{2p}(Au, y, t) \geq a(s)\phi_{2p} \left( \begin{array}{c} M(y, y, kt), M(Au, y, kt), M(y, y, kt) \end{array} \right) \\
+ b(s)\psi^p \left( \begin{array}{c} M^2(y, y, kt), M(Au, y, kt)M(y, y, kt) \\ M(y, y, kt) \vee M(y, Au, kt) \end{array} \right),
\]

hence we have

\[
M^{2p}(Au, y, t) \geq a(s)\phi_{2p}(M(Au, y, kt), M(Au, y, kt), M(Au, y, kt)) \\
+ b(s)\psi^p(M^2(Au, y, kt), M(Au, y, kt)M(Au, y, kt), 1).
\]

If $Au \neq y$, by above inequality we get

\[
M^{2p}(Au, y, t) > a(s)M^{2p}(Au, y, kt) + b(s)M^{2p}(Au, y, kt) = M^{2p}(Au, y, kt)
\]

which is contradiction. Hence $M(Au, y, t) = 1$, i.e $Au = y$. Thus $Au = Su = y$. 

As $A(X) \subseteq T(X)$ there exist $v \in X$, such that $Tv = y$. So,

$$M^{2p}(y, Bv, t) = M^{2p}(Au, Bv, t) \geq a(s)\phi^{2p}(M(Su, Tv, kt), M(Au, Su, kt), M(Bv, Tv, kt)) + b(s)\psi^{p}(M^2(Su, Ty, kt), M(Su, Ay, kt)M(Tv, Bv, kt), M(Su, Bv, kt) \vee M(Tv, Au, kt))$$

$$= a(s)\phi^{2p}(1, 1, M(Bv, y, kt)) + b(s)\psi^{p}(1, 1, 1).$$

We claim that $Bv = y$. For if $Bv \neq y$, then $M(Bv, y, t) < 1$.

On the above inequality we get

$$M^{2p}(y, Bv, t) \geq a(s)\phi^{2p}(M(y, Bv, kt), M(y, Bv, kt), M(y, Bv, kt)) + b(s)\psi^{p}(M^2(y, Bv, kt), M^2(y, Bv, kt), M^2(y, Bv, kt))$$

$$> a(s)M^{2p}(y, Bv, kt) + b(s)M^{2p}(y, Bv, kt) = M^{2p}(y, Bv, kt),$$

a contradiction. Hence $Tv = Bv = Au = Su = y$. Since $(A, S)$ is weak compatible, we get that $ASu = SAu$, that is $Ay = Sy$. Since $(B, T)$ is weak compatible, we get that $TBv = BTV$, that is, $Ty = By$. If $Ay \neq y$, then $M(Ay, y, t) < 1$. However

$$M^{2p}(Ay, y, t) = M^{2p}(Ay, Bv, t) \geq a(s)\phi^{2p}(M(Sy, Tv, kt), M(Ay, Sy, kt), M(Bv, Tv, kt)) + b(s)\psi^{p}(M^2(Sy, Ty, kt), M(Sy, Ay, kt)M(Tv, Bv, kt), M(Sy, Bv, kt) \vee M(Tv, Ay, kt))$$

$$= a(s)\phi^{2p}(M(Ay, y, kt), 1, 1) + b(s)\psi^{p}(M^2(Ay, y, kt), 1, M(Ay, y, kt))$$

$$\geq a(s)\phi^{2p}(M(Ay, y, kt), M(Ay, y, kt), M(Ay, y, kt)) + b(s)\psi^{p}(M^2(Ay, y, kt), M^2(Ay, y, kt), M^2(Ay, y, kt))$$

$$> a(s)M^{2p}(Ay, y, kt) + b(s)M^{2p}(Ay, y, kt) = M^{2p}(Ay, y, kt)$$

a contradiction. Thus $Ay = y$, hence $Ay = Sy = y$. Similarly we prove that $By = y$. For if $By \neq y$, then $M(By, y, t) < 1$, however

$$M^{2p}(y, By, t) = M^{2p}(Ay, By, t) \geq a(s)\phi^{2p}(M(Sy, Ty, kt), M(Ay, Sy, kt), M(By, Ty, kt)) + b(s)\psi^{p}(M^2(Sy, Ty, kt), M(Sy, Ay, kt)M(Ty, By, kt), M(Sy, By, kt) \vee M(Ty, Ay, kt))$$

$$= a(s)\phi^{2p}(M(y, By, kt), M(y, y, kt), M(By, By, kt)) + b(s)\psi^{p}(M^2(y, By, kt), M(y, By, kt), M^2(y, By, kt))$$

$$\geq a(s)\phi^{2p}(M(y, By, kt), M(y, By, kt), M(y, By, kt)) + b(s)\psi^{p}(M^2(y, By, kt), M^2(y, By, kt), M^2(y, By, kt))$$

$$> a(s)M^{2p}(y, By, kt) + b(s)M^{2p}(y, By, kt) = M^{2p}(y, By, kt),$$
a contradiction. Therefore, \( A_y = B_y = S_y = T_y = y \), that is, \( y \) is a common
fixed of \( A, B, S \) and \( T \). Uniqueness, let \( x \) be another common fixed point of
\( A, B, S \) and \( T \). That is, \( x = Ax = Bx = Sx = Tx \). If \( M(x, y, t) < 1 \), then
\[
M^{2p}(y, x, t) = M^{2p}(Ax, Bx, t) \\
\geq a(s)\phi^{2p}(M(Sy, Tx, kt), M(Ay, Sy, kt), M(Bx, Tx, kt)) \\
+ b(s)\psi^{2p}(M^2(Sy, Tx, kt), M(Sy, Ay, kt)M(Tx, Bx, kt), \\
M(Sy, Bx, kt) \vee M(Tx, Ay, kt)) \\
= a(s)\phi^{2p}(M(y, x, kt), 1, 1) + b(s)\psi^{2p}(M^2(y, x, kt), 1, M(y, x, kt)) \\
\geq a(s)\phi^{2p}(M(y, x, kt), M(y, x, kt), M(y, x, kt)) \\
+ b(s)\psi^{2p}(M^2(y, x, kt), M^2(y, x, kt), M^2(y, x, kt)) \\
> a(s)M^{2p}(y, x, kt) + b(s)M^{2p}(y, x, kt) = M^{2p}(y, x, kt),
\]
a contradiction. Therefore, \( y \) is the unique common fixed point of self-maps
\( A, B, S \) and \( T \). \( \square \)

In the following Theorem, function \( \phi: [0, 1]^4 \rightarrow [0, 1] \), is continuous and
increasing in each co-ordinate variable. Also \( \phi(s, s, s, s) > s \) for every \( s \in [0, 1] \).

**Theorem 2.3.** Let \( A, B, S \) and \( T \) be self-mappings of a complete fuzzy metric
space \( (X, *, s) \), satisfying that

(i) \( A(X) \subseteq T(X), \ B(X) \subseteq S(X) \) and \( A(X) \) or \( B(X) \) is a complete subset
of \( X \);

(ii) \( M(Ax, By, t) \geq \phi \left( M(Sx, Ty, kt), M(Ax, Sx, kt), \right. \\
\left. M(By, Ty, kt), M(Ax, Ty, kt) \vee M(By, Sx, kt) \right) \\
for every \( x, y \) in \( X, k > 1 \) and \( \phi \in \Phi \),

(iii) the pairs \( (A, S) \) and \( (B, T) \) are weak compatible.

Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

**Proof.** Let \( x_0 \in X \) be an arbitrary point as \( A(X) \subseteq T(X), \ B(X) \subseteq S(X) \), there
exist \( x_1, x_2 \in X \) such that \( Ax_0 = Tx_1, \ Bx_1 = Sx_2 \). Inductively, construct
sequence \( \{y_n\} \) and \( \{x_n\} \) in \( X \) such that \( y_{2n} = Ax_{2n} = Tx_{2n+1}, \ y_{2n+1} = \\
Bx_{2n+1} = Sx_{2n+2}, \) for \( n = 0, 1, 2, \ldots \).

Now, we prove \( \{y_n\} \) is a Cauchy sequence. Let \( d_m(t) = M(y_m, y_{m+1}, t), t > 0 \) we prove \( \{d_m(t)\} \) is increasing w.r.t \( m \). Set, \( m = 2n \), we have

\[
(2.1) \quad d_{2n}(t) \\
= M(y_{2n}, y_{2n+1}, t) \\
= M(Ax_{2n}, Bx_{2n+1}, t) \\
\geq \phi \left( M(Sx_{2n}, Tx_{2n+1}, kt), M(Ax_{2n}, Sx_{2n}, kt), \right. \\
\left. M(By_{2n+1}, Tx_{2n+1}, kt), M(Ax_{2n}, Ty_{2n+1}, kt) \vee M(By_{2n+1}, Sx_{2n}, kt) \right) \\
= \phi \left( M(y_{2n-1}, y_{2n}, kt), M(y_{2n}, y_{2n-1}, kt), \right. \\
\left. M(y_{2n+1}, y_{2n}, kt), M(y_{2n}, y_{2n}, kt) \vee M(y_{2n+1}, y_{2n-1}, kt) \right)
\]
= \phi(d_{2n-1}(kt), d_{2n-1}(kt), d_{2n}(kt), 1)
\geq \phi(d_{2n-1}(kt), d_{2n-1}(kt), d_{2n}(kt), d_{2n}(kt), 1).

Since, \( \phi \) is an increasing function we claim that for every \( n \in N \), \( d_{2n}(kt) \geq d_{2n-1}(kt) \). For if \( d_{2n}(kt) < d_{2n-1}(kt) \), then in inequality (2.1), we have
\[ d_{2n}(t) \geq \phi(d_{2n}(kt), d_{2n}(kt), d_{2n}(kt), d_{2n}(kt)) > d_{2n}(kt). \]

That is, \( d_{2n}(t) > d_{2n}(kt) \), a contradiction. Hence \( d_{2n}(kt) \geq d_{2n-1}(kt) \) for every \( n \in N \) and \( \forall t > 0 \). Similarly, we have \( d_{2n+1}(kt) \geq d_{2n}(kt) \). Thus \( \{d_{n}(t)\} \) is an increasing sequence in \([0, 1]\). By inequality (2.1) and \( d_{n}(t) \) is an increasing sequence, we get

\[ d_{2n}(t) \geq \phi(d_{2n-1}(kt), d_{2n-1}(kt), d_{2n-1}(kt), d_{2n-1}(kt)) \geq d_{2n-1}(kt). \]

Similarly, we have \( d_{2n+1}(t) \geq d_{2n}(kt) \). Thus \( d_{n}(t) \geq d_{n-1}(kt) \). That is,
\[ M(y_{n}, y_{n+1}, t) \geq M(y_{n-1}, y_{n}, kt) \geq \cdots \geq M(y_{0}, y_{1}, k^{n}t). \]

Hence by Lemma 1.11 \( \{y_{n}\} \) is Cauchy and the completeness of \( X \), \( \{y_{n}\} \) converges to \( y \) in \( X \). That is,
\[ \lim_{n \to \infty} y_{n} = y \Rightarrow \lim_{n \to \infty} y_{2n} = \lim_{n \to \infty} Ax_{2n} = \lim_{n \to \infty} Tx_{2n+1} = \lim_{n \to \infty} Bx_{2n+1} = \lim_{n \to \infty} Sx_{2n+2} = y. \]

As \( B(X) \subseteq S(X) \), there exist \( u \in X \) such that \( Su = y \). So, we have
\[ M(Au, Bx_{2n+1}, t) \geq \phi \left( \begin{array}{c} M(Su, Tx_{2n+1}, kt), \\ M(Au, Su, kt), \\ M(Bx_{2n+1}, Tx_{2n+1}, kt), \\ M(Au, Tx_{2n+1}, kt) \vee M(Bx_{2n+1}, Su, kt) \end{array} \right). \]

If \( Au \neq y \), by continuous \( M \) and \( \phi \), on making \( n \to \infty \) the above inequality, we get
\[ M(Au, y, t) \geq \phi \left( \begin{array}{c} M(y, y, kt), \\ M(Au, y, kt), \\ M(y, y, kt), \\ M(Au, y, kt) \vee M(y, y, kt) \end{array} \right) \]
\[ > M(Au, y, kt). \]

That is, \( M(Au, y, t) > M(Au, y, kt) \) which is contradiction. Hence
\[ M(Au, y, t) = 1, \]
i.e., \( Au = y \). Thus \( Au = Su = y \).

As \( A(X) \subseteq T(X) \) there exist \( v \in X \), such that \( Tv = y \). So,
\[ M(y, Bv, t) = M(Au, Bv, t) \geq \phi \left( \begin{array}{c} M(Su, Tv, kt), \\ M(Au, Su, kt), \\ M(Bv, Tv, kt), \\ M(Au, Tv, kt) \vee M(Bv, Su, kt) \end{array} \right) \]
\[ = \phi \left( \begin{array}{c} 1, \\ M(Bv, y, kt) \end{array} \right). \]
We claim that \( Bv = y \). For if \( Bv \neq y \), then \( M(Bv, y, t) < 1 \).

On the above inequality we get

\[
M(y, Bv, t) \geq \phi \left( \frac{M(y, Bv, kt), M(y, Bv, kt)}{M(y, Bv, kt), M(y, Bv, kt)} \right)
\]

\[
> M(y, Bv, kt),
\]

a contradiction. Hence \( Tv = Bv = Au = Su = y \). Since \((A, S)\) is weak compatible, we get that \( ASu = SAu \), that is \( Ay = Sy \).

Since \((B, T)\) is weak compatible, we get that \( TBv = BTv \), that is \( Ty = By \). If \( Ay \neq y \), then \( M(Ay, y, t) < 1 \). However

\[
M(Ay, y, t) = M(Ay, Bv, t)
\]

\[
\geq \phi \left( \frac{M(Sy, Ty, kt), M(Ay, Sy, kt)}{M(Bv, Ty, kt), M(Ay, Ty, kt) \lor M(Bv, Sy, kt)} \right)
\]

\[
\geq \phi(M(Ay, y, kt), 1, 1, M(Ay, y, kt))
\]

\[
\geq \phi \left( \frac{M(Ay, y, kt), M(Ay, y, kt)}{M(Ay, y, kt), M(Ay, y, kt)} \right)
\]

\[
> M(Ay, y, kt)
\]

a contradiction. Thus \( Ay = y \), hence \( Ay = Sy = y \). Similarly we prove that \( By = y \). For if \( By \neq y \), then \( M(By, y, t) < 1 \), however

\[
M(y, By, t) = M(Ay, By, t)
\]

\[
\geq \phi \left( \frac{M(Sy, Ty, kt), M(Ay, Sy, kt)}{M(By, Ty, kt), M(Ay, Ty, kt) \lor M(By, Sy, kt)} \right)
\]

\[
\geq \phi(M(y, By, kt), M(y, By, kt), M(y, By, kt), M(y, By, kt))
\]

\[
> M(y, By, kt)
\]

a contradiction. Therefore, \( Ay = By = Sy = Ty = y \), that is, \( y \) is a common fixed of \( A, B, S \) and \( T \). Uniqueness, let \( x \) be another common fixed point of \( A, B, S \) and \( T \). That is \( x = Ax = Bx = Sx = Tx \). If \( M(x, y, t) < 1 \), then

\[
M(y, x, t) = M(Ay, Bx, t)
\]

\[
\geq \phi \left( \frac{M(Sy, Tx, kt), M(Ay, Sy, kt)}{M(Bx, Tx, kt), M(Ay, Tx, kt) \lor M(Bx, Sy, kt)} \right)
\]

\[
= \phi \left( \frac{M(y, x, kt), 1}{1, M(y, x, kt) \lor M(x, y, kt)} \right)
\]

\[
\geq \phi(M(y, x, kt), M(y, x, kt), M(y, x, kt), M(y, x, kt))
\]

\[
> M(y, x, kt)
\]

a contradiction. Therefore, \( y \) is the unique common fixed point of self-maps \( A, B, S \) and \( T \).
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