SOME PROPERTIES OF SYMMETRIC BI-(σ, τ)-DERIVATIONS IN NEAR-RINGS

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Abstract. In this paper, we introduce a symmetric bi-(σ, τ)-derivation in a near-ring and generalize some of the results in [5, 6, 8, 9].

1. Introduction

The concept of a symmetric bi-derivation has been introduced by Maksa in [4]. Some recent results on properties of prime rings, semi-prime rings and near-rings with derivations have been investigated in several ways [1, 2, 3, 6, 9, 8, 10]. In [6], Öztürk and Jun have introduced the concept of a symmetric bi-derivation of a near-ring and studied some properties.

In this note, we introduce the concept of symmetric bi-(σ, τ)-derivation of a near-ring and give some properties.

Throughout this paper, N will be a zero-symmetric left near-ring with multiplicative center Z. Recall that a near ring N is 3-prime if aNb = {0} implies that a = 0 or b = 0. σ and τ will be two near-ring automorphisms of N. For x, y ∈ N, [x, y] , [x, y]σ,τ and (x, y) will denote the commutator xy − yx, xσ(y) − τ(y)x and x + y − x − y respectively. A mapping D : N × N → N is said to be symmetric if D(x, y) = D(y, x) for all x, y ∈ N. A mapping d : N → N defined by d(x) = D(x, x) is called the trace of D where D : N × N → N is a symmetric mapping. It is obvious that, if D : N × N → N is a symmetric mapping which also bi-additive (i.e., additive in both arguments), then the trace of D satisfies the relation d(x + y) = d(x) + 2D(x, y) + d(y) for all x, y ∈ N. A symmetric bi-additive mapping D : N × N → N is called a symmetric bi-derivation if D(xy, z) = D(x, y)y + xD(y, z) is fulfilled for all x, y, z ∈ N. For the terminology used in near-rings, see [7].

2. Results

The following Lemmas are necessary for the paper.
Lemma 1 ([4, Lemma 3]). Let $N$ be a 3-prime near-ring.
   (i) If $z \in Z - \{0\}$, then $z$ is not a zero divisor.
   (ii) If $Z - \{0\}$ contains an element $z$ for which $z + z \in Z$, then $(N, +)$ is abelian.

Lemma 2 ([6, Lemma 3.1]). Let $N$ be a 2-torsion free near-ring, $D$ a symmetric bi-additive mapping on $N$ and $d$ the trace of $D$. If $d(x) = 0$ for all $x \in N$, then $D = 0$.

Firstly, we introduce the definition of symmetric bi-$(\sigma, \tau)$-derivation in a near-ring.

Definition 1. A symmetric bi-additive mapping $D : N \times N \rightarrow N$ is called a symmetric bi-$(\sigma, \tau)$-derivation if there exists automorphism $\sigma, \tau : N \rightarrow N$ such that
   $$D(xy, z) = D(x, z)\sigma(y) + \tau(x)D(y, z)$$
   for all $x, y, z \in N$.

Note that if $\sigma = 1$ and $\tau = 1$ then $D$ is a symmetric bi-derivation.

Lemma 3. Let $N$ be a 2-torsion free 3-prime near-ring, $D$ a symmetric bi-$(\sigma, \tau)$-derivation of $N$ and $d$ the trace of $D$. If $xd(N) = 0$ for all $x \in N$, then $x = 0$ or $D = 0$.

Proof. Since $d(y + z) = d(y) + 2D(y, z) + d(z)$ for all $y, z \in N$, multiplying by $x$ from the left hand side, and using the hypothesis we have $xD(y, z) = 0$. Hence replace $y$ by $yw$, to get $x\tau(y)D(w, z) = 0$ for all $x, y, z, w \in N$. Since $\tau$ is an automorphism of $N$, we get $xD(w, z) = 0$. Again since $N$ is 3-prime near-ring, we have $x = 0$ or $D = 0$. \hfill \Box

Note that in Lemma 3, if $D$ is a nontrivial symmetric bi-$(\sigma, \tau)$-derivation of $N$ and $xd(N) = 0$ for all $x \in N$ then we get only $x = 0$.

Lemma 4. Let $N$ be a near-ring. $D$ is a symmetric bi-$(\sigma, \tau)$-derivation of $N$ if and only if $D(xy, z) = \tau(x)D(y, z) + D(x, z)\sigma(y)$ for all $x, y, z \in N$.

Proof. Let $D$ be a symmetric bi-$(\sigma, \tau)$-derivation of $N$. Since $\sigma$ is an automorphism, we get for all $x, y, z \in N$,
   $$D(x(y + y), z) = D(x, z)\sigma(y + y) + \tau(x)D(y + y, z) = D(x, z)\sigma(y) + D(x, z)\sigma(y) + \tau(x)D(y, z) + \tau(x)D(y, z)$$
and
   $$D(x(y + y), z) = D(xy + xy, z) = D(xy, z) + D(xy, z) = D(x, z)\sigma(y) + \tau(x)D(y, z) + D(x, z)\sigma(y) + \tau(x)D(y, z).$$
Combining the above two equality, we find that
   $$D(x, z)\sigma(y) + \tau(x)D(y, z) = \tau(x)D(y, z) + D(x, z)\sigma(y).$$
Hence we have $D(xy, z) = \tau(x)D(y, z) + D(x, z)\sigma(y)$ for all $x, y, z \in N$. Converse can be prove in a similar way. \hfill \Box
Lemma 5. Let $N$ be a near-ring, $D$ a symmetric bi-$(\sigma, \tau)$-derivation of $N$ and $d$ the trace of $D$. Then, for all $x, y, z, w \in N$,

(i) $[D(x, z)\sigma(y) + \tau(x)D(y, z)]w = D(x, z)\sigma(y)w + \tau(x)D(y, z)w,$
(ii) $[\tau(x)D(y, z) + D(x, z)\sigma(y)]w = \tau(x)D(y, z)w + D(x, z)\sigma(y)w.$

Proof. Using $\sigma$ and $\tau$ are automorphism and with the technique using in [6, Lemma 3.4], we have the proof. \hfill \Box

Lemma 6. Let $N$ be a 3-prime near-ring, $D$ a nonzero symmetric bi-$(\sigma, \tau)$-derivation of $N$. Then $D(N, N) = 0$ for all $x \in N$ implies $x = 0$ and $xD(N, N) = 0$ implies $x = 0$.

Proof. Suppose $D(y, z)x = 0$ for all $x, y, z \in N$. Then taking $yw$ instead of $y$, using Lemma 5 (i) we have for all $x, y, z \in N$,

$0 = D(yw, z)x = [D(y, z)\sigma(w) + \tau(y)D(w, z)]x$
$= D(y, z)\sigma(w)x + \tau(y)D(w, z)x$
$= D(y, z)\sigma(w)x.$

Since $\sigma$ is an automorphism, we have $D(y, z)Nx = 0$ for all $x, y, z \in N$. Since $N$ is a 3-prime near ring and $D$ is nontrivial, this implies $x = 0$. If $xD(N, N) = 0$ then for all $y, w, z \in N$, we have

$0 = xD(yw, z) = x(D(y, z)\sigma(w) + \tau(y)D(w, z))$
$= xD(y, z)\sigma(w) + x\tau(y)D(w, z)$
$= x\tau(y)D(w, z)$

and with similar argument in the above we have $x = 0$. \hfill \Box

Theorem 1. Let $N$ be a 3-prime near-ring, $D$ a non-zero symmetric bi-$(\sigma, \tau)$-derivation of $N$. If $N$ is 2-torsion free and $D(N, N) \subset Z$, then $N$ is a commutative ring.

Proof. Since $D(N, N) \subset Z$ and $D$ is a non-zero, there exists non-zero elements $x, y \in N$ such that $D(x, y) \in Z - \{0\}$. Then $D(x, y + y) = D(x, y) + D(x, y) \in Z$ and hence $(N, +)$ is abelian by Lemma 1. $D(x, y) \in Z$ for all $x, y \in N$, implies that $zD(x, y) = D(x, y)z$ for all $z \in N$. Hence replace $zw$ with $x$ to get

$z(D(x, y)\sigma(w) + \tau(x)D(w, y)) = (D(x, y)\sigma(w) + \tau(x)D(w, y))z.$

By Lemma 5 (i) and $D(N, N) \subset Z$, we get

$D(x, y)z\sigma(w) + D(w, y)z\tau(x) = D(x, y)\sigma(w)z + \tau(x)D(w, y)z$

or $(N, +)$ abelian we have for all $x, y, z, w \in N$,

$D(x, y)[z, \sigma(w)] = D(w, y)[z, \tau(x)].$

Taking $D(u, v)$ instead of $\sigma(w)$ for all $u, v \in N$ and since $D(u, v) \in Z$ we have

$D(D(u, v), y)[z, \tau(x)] = 0$
for all \( x, y, z, u, v \in N \). Hence by Lemma 3 we have \([z, \tau(x)] = 0\) for all \( x, z \in N \). So \( N \) is a commutative ring since \( \tau \) is an automorphism.

\[ \Box \]

**Theorem 2.** Let \( N \) be a 3-prime near-ring, \( D \) a nontrivial symmetric bi-(\( \sigma, \tau \))-derivation of \( N \) and \( d \) the trace of \( D \). If \([x, d(x)]_{\sigma, \tau} = 0\) then \((N, +)\) is commutative.

**Proof.** Since \( D \) is a symmetric bi-(\( \sigma, \tau \))-derivation we have

\[
D(x(x + y), x) = D(x, x)\sigma(x + y) + \tau(x)D(x + y, x)
\]

\[
= d(x)\sigma(x) + d(x)\sigma(y) + \tau(x)d(x) + \tau(x)D(y, x)
\]

and

\[
D(x(x + y), x) = D(x^2 + xy, x) = D(x^2, x) + D(xy, x)
\]

\[
= d(x)\sigma(x) + \tau(x)d(x) + d(x)\sigma(y) + \tau(x)D(y, x).
\]

Combining the above equality we have

\[
d(x)\sigma(y) + \tau(x)d(x) = \tau(x)d(x) + d(x)\sigma(y).
\]

Since \([d(x), x]_{\sigma, \tau} = 0\), we have

\[
d(x)\sigma(y) + d(x)\sigma(x) - d(x)\sigma(y) - d(x)\sigma(x) = 0
\]

or

\[
d(x)(\sigma(y) + \sigma(x) - \sigma(y) - \sigma(x)) = 0.
\]

Hence we have \(d(x)(\sigma(x), \sigma(y)) = 0\) for all \( x, y \in N \). From Lemma 3 and since \( \sigma \) is an automorphism we get that \((N, +)\) is abelian. \( \Box \)

**References**


SYMOMETRIC BI-(\(\sigma, \tau\))-DERIVATIONS

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