ON THE ORDERED $n$-PRIME IDEALS IN ORDERED $\Gamma$-SEMIGROUPS

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Abstract. The motivation mainly comes from the conditions of the (ordered) ideals to be prime or semiprime that are of importance and interest in (ordered) semigroups and in (ordered) $\Gamma$-semigroups. In 1981, Sen [8] has introduced the concept of the $\Gamma$-semigroups. We can see that any semigroup can be considered as a $\Gamma$-semigroup. The concept of ordered ideal extensions in ordered $\Gamma$-semigroups was introduced in 2007 by Siripitukdet and Iampan [12]. Our purpose in this paper is to introduce the concepts of the ordered $n$-prime ideals and the ordered $n$-semiprime ideals in ordered $\Gamma$-semigroups and to characterize the relationship between the ordered $n$-prime ideals and the ordered ideal extensions in ordered $\Gamma$-semigroups.

1. Preliminaries

In 1981, the concept and notion of the $\Gamma$-semigroups was introduced by Sen [8]. In 1997, Kwon and Lee [5] introduced the concepts of the weakly prime ideals and the weakly semiprime ideals in ordered $\Gamma$-semigroups and gave some characterizations of the weakly prime ideals and the weakly semiprime ideals in ordered $\Gamma$-semigroups analogous to the characterizations of the weakly prime ideals and the weakly semiprime ideals in ordered semigroups considered by Kehayopulu [3]. In 1998, Kwon and Lee [4] introduced the ideals and the weakly prime ideals in ordered $\Gamma$-semigroups and gave some characterizations of the ideals and the weakly prime ideals in ordered $\Gamma$-semigroups analogous to the characterizations of the ideals and the weakly prime ideals in ordered semigroups considered by Kehayopulu [3]. In 1999, Lee and Kwon [6] gave two new characterizations of the weakly prime ideals in ordered semigroups. They proved two theorems as follow: Let $a$ be a quasi-completely regular element of an ordered semigroup $S$. If there exists an ideal not containing $a$, then there exists a weakly prime ideal not containing $a$. Let $P^*$ be the intersection of weakly prime ideals of an ordered semigroup $S$, $a \in P^*$ and $I$ be any proper...
ideal of $S$. Then $a^n \in I$ for some positive integer $n$. $P^*$ is an archimedean subsemigroup of an ordered semigroup $S$. In 2004, Dutta and Adhikari [1] introduced the concepts of the ordered $\Gamma$-semigroups and the intra-regular ordered $\Gamma$-semigroups and the concepts of the left ideals and the right ideals in ordered $\Gamma$-semigroups. The main results of their paper are the following: They proved that for an ordered $\Gamma$-semigroup $M$, the following statements are equivalent:

1. $(A\Gamma] = A$ for each ideal $A$ of $M$.
2. $(A\Gamma B] = A \cap B$ for all ideals $A$ and $B$ of $M$.
3. $a \in (M\Gamma a\Gamma M \Gamma a\Gamma M]$ for all $a \in M$.

Let $M$ be an ordered $\Gamma$-semigroup. The ideals of $M$ are weakly prime if and only if they form a chain and one of the three equivalent conditions (1), (2) and (3) mentioned above holds in $M$. The ideals of $M$ are prime if and only if they form a chain and $M$ is intra-regular. In 2006, Siripitukdet and Iampan [11] characterized the relationship between the (ordered) $s$-prime ideals and the (ordered) semilattice congruences in ordered $\Gamma$-semigroups. They showed that for an ordered $\Gamma$-semigroup $M$, the congruence $n$ on $M$ is the intersection of $\sigma_I$ for all $s$-prime ideals $I$ of $M$ and the congruence $\mathcal{N}$ on $M$ is the intersection of $\sigma_I$ for all ordered $s$-prime ideals $I$ of $M$. In 2007, Siripitukdet and Iampan [12] introduced the concepts of the extensions of ordered $s$-prime ideals, prime ideals, ordered $s$-semiprime ideals and semiprime ideals in ordered $\Gamma$-semigroups and characterize the relationship between the extensions of ordered ideals and some congruences in ordered $\Gamma$-semigroups. They defined the equivalence relations on an ordered $\Gamma$-semigroup $M$ as follows:

\[
\sigma_I := \{(x, y) \in M \times M : x, y \in I \text{ or } x, y \notin I\}, \\
\Phi_I := \{(x, y) \in M \times M : \ll x, I \gg \ll y, I \gg\}, \\
\mathcal{N} := \{(x, y) \in M \times M : N(x) = N(y)\}
\]

and showed that if $I$ is an ordered $s$-prime ideal of $M$, then $\Phi_I = \sigma_I$ and $\mathcal{N} \subseteq \Phi_I$. So the concept of prime is the really interested and important thing about (ordered) semigroups and (ordered) $\Gamma$-semigroups.

Our aim in this paper is fourfold.

1. To generalize the definitions of the ordered prime ideal and the ordered semiprime ideal in ordered $\Gamma$-semigroups.
2. To introduce the concept of the ordered $n$-prime ideals in ordered $\Gamma$-semigroups and to study the ordered $n$-prime ideals in ordered $\Gamma$-semigroups.
3. To generalize the ordered prime ideals in commutative ordered $\Gamma$-semigroups.
4. To characterize the relationship between the ordered $n$-prime ideals and the ordered ideal extensions in commutative ordered $\Gamma$-semigroups.
To present the main theorems we first recall the definition of the $\Gamma$-semigroup which is important here.

Let $\Gamma$ be any nonempty set. A nonempty set $M$ is called a $\Gamma$-semigroup [7, 8, 9] if for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, we have (i) $a \alpha b \in M$ and (ii) $(a \alpha b) \beta c = a \alpha (b \beta c)$. A $\Gamma$-semigroup $M$ is called a commutative $\Gamma$-semigroup if $a \gamma b = b \gamma a$ for all $a, b \in M$ and $\gamma \in \Gamma$. A nonempty subset $K$ of a $\Gamma$-semigroup $M$ is called a sub-$\Gamma$-semigroup of $M$ if $a \gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$.

For examples of $\Gamma$-semigroups, see [2, 10, 11, 12].

A partially ordered $\Gamma$-semigroup $M$ is called an ordered $\Gamma$-semigroup (some author called po-$\Gamma$-semigroup) [5] if for any $a, b, c \in M$ and $\gamma \in \Gamma$, $a \leq b$ implies $a \gamma c \leq b \gamma c$ and $c \gamma a \leq c \gamma b$. An ordered $\Gamma$-semigroup $M$ is called a commutative ordered $\Gamma$-semigroup if $M$ is a commutative $\Gamma$-semigroup. For any nonempty subsets $A$ and $B$ of an ordered $\Gamma$-semigroup $M$ and any nonempty subset $\Gamma'$ of $\Gamma$, let $\Gamma' B := \{a \gamma b : a \in A, b \in B \text{ and } \gamma \in \Gamma'\}$. If $A = \{a\}$, then we also write $\{a\} \Gamma' B$ as $a \Gamma' B$, and similarly if $B = \{b\}$ or $\Gamma' = \{\gamma\}$. A nonempty subset $I$ of an ordered $\Gamma$-semigroup $M$ is called an ordered ideal of $M$ if $M \Gamma I \subseteq I, \Gamma M \subseteq I$ and for all $a \in I$ and $b \in M$, $b \leq a$ implies $b \in I$. An ordered ideal $I$ of an ordered $\Gamma$-semigroup $M$ is called an ordered prime ideal of $M$ if for any $a, b \in M$, $a \Gamma b \subseteq I$ implies $a \in I$ or $b \in I$. Equivalently, for any subsets $A$ and $B$ of $M$, $A \Gamma B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. An ordered ideal $I$ of an ordered $\Gamma$-semigroup $M$ is called an ordered semiprime ideal of $M$ if for any $a \in M$, $a \Gamma \subseteq I$ implies $a \in I$. Equivalently, for any subset $A$ of $M$, $A \Gamma A \subseteq I$ implies $A \subseteq I$. Let $n$ be any integer such that $n \geq 2$. For any subsets $A_1, A_2, \ldots, A_{n-1}$ and $A_n$ of $M$ and let $i$ be an integer such that $2 \leq i \leq n - 1$. We define the symbol as follows:

\[
\hat{A}_{(1:n)} := A_2 \Gamma A_3 \cdots A_{n-1} \Gamma A_n,
\]

\[
\hat{A}_{(i:n)} := A_1 \Gamma A_2 \cdots A_{i-1} \Gamma A_{i+1} \Gamma A_{i+2} \cdots A_{n-1} \Gamma A_n,
\]

\[
\hat{A}_{(n:n)} := A_1 \Gamma A_2 \cdots A_{n-2} \Gamma A_{n-1}.
\]

An ordered ideal $I$ of an ordered $\Gamma$-semigroup $M$ is called an ordered $n$-prime ideal of $M$ if for any subsets $A_1, A_2, \ldots, A_{n-1}$ and $A_n$ of $M$, $A_1 \Gamma A_2 \cdots A_{n-1} \Gamma A_n \subseteq I$ implies that there exists an integer $i$ ($1 \leq i \leq n$) such that

\[
\hat{A}_{(i/n)} \subseteq I.
\]

An ordered ideal $I$ of an ordered $\Gamma$-semigroup $M$ is called an ordered $n$-semiprime ideal of $M$ if for any subsets $A_1, A_2, \ldots, A_{n-1}$ and $A_n$ of $M$ with $A_1 = A_2 = \cdots = A_n, A_1 \Gamma A_2 \cdots A_{n-1} \Gamma A_n \subseteq I$ implies $\hat{A}_{(n:n)} \subseteq I$. Hence we have the following statements for ordered $\Gamma$-semigroups.

1. Every ordered prime ideal is an ordered semiprime ideal.
2. Every $n$-ordered prime ideal is an $n$-ordered semiprime ideal.
3. The ordered prime ideals and the $2$-ordered prime ideals coincide.
(4) The ordered semiprime ideals and the 2-ordered semiprime ideals coincide.

For a subset $H$ of an ordered $\Gamma$-semigroup $M$, we denote $\{H\} := \{t \in M : t \leq h \text{ for some } h \in H\}$. If $H = \{a\}$, then we also write $\{\{a\}\}$ as $\{a\}$. We see that $H \subseteq \{H\}$, $\{(H)\} = \{H\}$ and for any subsets $A$ and $B$ of $M$ with $A \subseteq B$, we have $\{A\} \subseteq \{B\}$. For an ordered ideal $I$ of an ordered $\Gamma$-semigroup $M$ and a subset $A$ of $M$. The set $\langle A, I \rangle := \{x \in M : A\Gamma x \subseteq I\}$ is called the extension $[12]$ of $I$ by $A$. If $A = \{a\}$, then we also write $\langle \{a\}, I \rangle$ as $\langle a, I \rangle$.

We shall assume throughout this paper that $M$ stands for a commutative ordered $\Gamma$-semigroup. Before the characterizations of the relationship between the ordered $n$-prime ideals and ordered ideal extensions in $M$ for the main theorems, we give auxiliary results which are necessary in what follows.

**Lemma 1.1** ([12]). Let $I$ be an ordered ideal of $M$, $A \subseteq M$ and $\gamma \in \Gamma$. Then we have the following statements.

(a) $\langle A, I \rangle$ is an ordered ideal of $M$.

(b) $I \subseteq \langle A, I \rangle \subseteq \langle A\Gamma A, I \rangle \subseteq \langle A\gamma A, I \rangle$.

(c) If $A \subseteq I$, then $\langle A, I \rangle = M$.

**Lemma 1.2** ([12]). Let $I$ be an ordered ideal of $M$ and $A \subseteq M$. Then $\langle A, I \rangle = \bigcap_{a \in A} \langle a, I \rangle = \langle A \setminus I, I \rangle$.

2. Main theorems

In this section, we give the relationship between the ordered $n$-prime ideals and ordered ideal extensions in ordered $\Gamma$-semigroups.

The following theorem shows the important property that hold in every integer $n \geq 3$, the ordered $n$-prime ideals of $M$ are a generalization of ordered $(n-1)$-prime ideals.

**Theorem 2.1.** Every ordered $(n-1)$-prime ideal of $M$ is an ordered $n$-prime ideal of $M$ for all integers $n \geq 3$.

**Proof.** Assume that $I$ is an ordered $(n-1)$-prime ideal of $M$. Now, let $A_1, A_2, \ldots, A_n \subseteq M$ be such that $A_1\Gamma A_2 \cdots A_{n-1}\Gamma A_n \subseteq I$. Let $B_1 = A_1\Gamma A_2$ and $B_i = A_{i+1}$ for all $i = 2, 3, \ldots, n-1$. Then $B_1\Gamma B_2 \cdots B_{n-2}\Gamma B_{n-1} \subseteq I$. By hypothesis, it implies that there exists an integer $i$ ($1 \leq i \leq n-1$) such that

\[ \hat{B}(1,n-1), \hat{B}(2,n-1), \ldots, \hat{B}(i-1,n-1), \hat{B}(i+1,n-1), \hat{B}(i+2,n-1), \ldots, \hat{B}(n-1,n-1) \subseteq I. \]

**Case 1:** $\hat{B}(1,n-1) \not\subset I$.

Then $\hat{B}(2,n-1), \hat{B}(3,n-1), \ldots, \hat{B}(n-1,n-1) \not\subset I$, so $\hat{A}(3,n), \hat{A}(4,n), \ldots, \hat{A}(n,n) \not\subset I$. It follows from hypothesis that there exists an integer $j$ ($1 \leq j \leq n-1$) such that

\[ \hat{A}(1,n-1), \hat{A}(2,n-1), \ldots, \hat{A}(j-1,n-1), \hat{A}(j+1,n-1), \hat{A}(j+2,n-1), \ldots, \hat{A}(n-1,n-1) \subseteq I. \]
Let $A_1\Gamma A_2 \cdots A_{n-2} \Gamma A_{n-1} = \hat{A}_{(n,n)} \subseteq I$. Then
\[
A_2 \Gamma A_3 \cdots A_{n-2} \Gamma A_{n-1} = \hat{A}_{(1,n-1)} \subseteq I \text{ or } A_1 \Gamma A_3 \cdots A_{n-2} \Gamma A_{n-1} = \hat{A}_{(2,n-1)} \subseteq I.
\]
Thus, since $I$ is an ordered ideal of $M$,
\[
\hat{A}_{(1,n)} = A_2 \Gamma A_3 \cdots A_{n-1} \Gamma A_n \subseteq I \text{ or } \hat{A}_{(2,n)} = A_1 \Gamma A_3 \cdots A_{n-1} \Gamma A_n \subseteq I.
\]
Hence $\hat{A}_{(1,n)}, \hat{A}_{(3,n)}, \hat{A}_{(4,n)}, \ldots, \hat{A}_{(n,n)} \subseteq I$ or $\hat{A}_{(2,n)}, \hat{A}_{(3,n)}, \ldots, \hat{A}_{(n,n)} \subseteq I$.

**Case 2:** $\hat{B}_{(1,n-1)} \subseteq I$.

Then there exists an integer $j$ ($2 \leq j \leq n-1$) such that
\[
\hat{B}_{(2,n-1)}, \hat{B}_{(3,n-1)}, \ldots, \hat{B}_{(j-1,n-1)}, \hat{B}_{(j+1,n-1)}, \hat{B}_{(j+2,n-1)}, \ldots, \hat{B}_{(n-1,n-1)} \subseteq I.
\]
Thus
\[
\hat{A}_{(3,n)}, \hat{A}_{(4,n)}, \ldots, \hat{A}_{(j,n)}, \hat{A}_{(j+2,n)}, \hat{A}_{(j+3,n)}, \ldots, \hat{A}_{(n,n)} \subseteq I.
\]
Since $A_3 \Gamma A_4 \cdots A_{n-1} \Gamma A_n = \hat{B}_{(1,n-1)} \subseteq I$,
\[
\hat{A}_{(1,n)} = A_2 \Gamma A_3 \cdots A_{n-1} \Gamma A_n \subseteq I \text{ and } \hat{A}_{(2,n)} = A_1 \Gamma A_3 \cdots A_{n-1} \Gamma A_n \subseteq I.
\]
Thus $\hat{A}_{(1,n)}, \hat{A}_{(2,n)}, \ldots, \hat{A}_{(j,n)}, \hat{A}_{(j+2,n)}, \hat{A}_{(j+3,n)}, \ldots, \hat{A}_{(n,n)} \subseteq I$.

Therefore $I$ is an ordered $n$-prime ideal of $M$. Hence we complete the proof of the theorem.

The ordered $n$-prime ideals are not ordered $(n-1)$-prime ideals in general for ordered $\Gamma$-semigroups and integers $n \geq 3$. We prove it by the following examples:

**Example 1** ([11]). Let $M = \{a, b, c, d\}$ and $\Gamma = \{\gamma\}$ with the multiplication and the relation $\leq$ on $M$ defined by
\[
x \gamma y = \begin{cases} 
    b & \text{if } x, y \in \{a, b\}, \\
    c & \text{otherwise}.
\end{cases}
\]
\[
\leq: = \{(a, a), (b, b), (c, c), (d, d), (b, c), (b, d), (c, d)\}.
\]
Then $M$ is an ordered $\Gamma$-semigroup and $\{b, c\}$ is an ordered ideal of $M$. We can prove that $\{b, c\}$ is a $3$-prime ideal of $M$ but not a $2$-prime ideal of $M$ since $\{a\} \Gamma \{d\} \subseteq \{b, c\}$ while $\{a\} \not\subseteq \{b, c\}$ and $\{d\} \not\subseteq \{b, c\}$.

**Example 2.** Let $S = \{a, b, c, d\}$ be the ordered semigroup defined by the following multiplication and relation $\leq$ on $S$ as follows:

\[
\begin{array}{cccc}
\ast & a & b & c & d \\
 a & a & b & b & d \\
 b & b & b & b & d \\
 c & d & d & c & d \\
 d & d & d & d & d \\
\end{array}
\]
Let $M = S$ and $\Gamma = \{ \ast \}$. Then $M$ is an ordered $\Gamma$-semigroup and $\{ d \}$ is an ordered ideal of $M$. We can prove that $\{ d \}$ is a 3-prime ideal of $M$ but not a 2-prime ideal of $M$ since $\{ b \} \Gamma \{ c \} \subseteq \{ d \}$ while $b \neq d$ and $c \neq d$.

Immediately from Theorem 2.1, we have Corollary 2.2.

**Corollary 2.2.** Every ordered prime ideal of $M$ is an ordered $n$-prime ideal of $M$ for all integers $n \geq 2$.

**Theorem 2.3.** An ordered ideal $I$ of $M$ is an ordered $n$-prime ideal of $M$ if and only if any extension of $I$ is an ordered $(n - 1)$-prime ideal of $M$ for all integers $n \geq 3$.

**Proof.** Assume that $I$ is an ordered $n$-prime ideal of $M$. By Lemma 1.1 (a), we have that for any subset $A$ of $M$, $\ll A, I \gg$ is ordered ideal of $M$. For any subset $B$ of $M$, let $A_1, A_2, \ldots, A_{n-1} \subseteq M$ be such that $A_1 \Gamma A_2 \cdots A_{n-2} \Gamma A_{n-1} \subseteq \ll B, I \gg$. Then $B \Gamma A_1 \Gamma A_2 \cdots A_{n-2} \Gamma A_{n-1} \subseteq I$. Let $B_i = B$ and $I_i = A_{i-1}$ for $i = 2, 3, \ldots, n$. Then $B_i \Gamma B_{i+1} \cdots B_{n-1} \Gamma B_n \subseteq I$. Since $I$ is an ordered $n$-prime ideal of $M$, there exists an integer $i$ $(1 \leq i \leq n)$ such that

$$\hat{B}_{(1, n)}, \hat{B}_{(2, n)}, \ldots, \hat{B}_{(i-1, n)}, \hat{B}_{(i+1, n)}, \hat{B}_{(i+2, n)}, \ldots, \hat{B}_{(n, n)} \subseteq I.$$

Thus there exists an integer $j$ $(2 \leq j \leq n)$ such that

$$\hat{B}_{(2, n)}, \hat{B}_{(3, n)}, \ldots, \hat{B}_{(j-1, n)}, \hat{B}_{(j+1, n)}, \hat{B}_{(j+2, n)}, \ldots, \hat{B}_{(n, n)} \subseteq I.$$

This implies that there exists an integer $k = j - 1$ $(1 \leq k \leq n - 1)$ such that

$$
\begin{align*}
& B \Gamma \hat{A}_{(1, n-1)}, B \Gamma \hat{A}_{(2, n-1)}, \ldots, B \Gamma \hat{A}_{(k-1, n-1)}, B \Gamma \hat{A}_{(k+1, n-1)}, \\
& \quad B \Gamma \hat{A}_{(k+2, n-1)}, \ldots, B \Gamma \hat{A}_{(n-1, n-1)} \subseteq I.
\end{align*}
$$

Hence

$$
\begin{align*}
& \hat{A}_{(1, n-1)}, \hat{A}_{(2, n-1)}, \ldots, \hat{A}_{(k-1, n-1)}, \hat{A}_{(k+1, n-1)}, \\
& \quad \hat{A}_{(k+2, n-1)}, \ldots, \hat{A}_{(n-1, n-1)} \subseteq \ll B, I \gg.
\end{align*}
$$

Therefore $\ll B, I \gg$ is an ordered $(n - 1)$-prime ideal of $M$.

Conversely, assume that any extension of $I$ is an ordered $(n - 1)$-prime ideal of $M$. Let $A_1, A_2, \ldots, A_n \subseteq M$ be such that $A_1 \Gamma A_2 \cdots A_{n-1} \Gamma A_n \subseteq I$. Then we get $A_1 \Gamma A_2 \cdots A_{n-1} \Gamma A_{n-1} \subseteq \ll A_n, I \gg$. By hypothesis, it implies that there exists an integer $i$ $(1 \leq i \leq n - 1)$ such that

$$
\begin{align*}
& \hat{A}_{(1, n-1)}, \hat{A}_{(2, n-1)}, \ldots, \hat{A}_{(i-1, n-1)}, \hat{A}_{(i+1, n-1)}, \\
& \quad \hat{A}_{(i+2, n-1)}, \ldots, \hat{A}_{(n-1, n-1)} \subseteq \ll A_n, I \gg.
\end{align*}
$$

We consider the following $(n - 1)$ cases. Let $\hat{A}_{(i, n-1)} \not\subseteq A_n, I \gg$. Then

$$
\begin{align*}
& \hat{A}_{(1, n-1)}, \hat{A}_{(2, n-1)}, \ldots, \hat{A}_{(i-1, n-1)}, \hat{A}_{(i+1, n-1)}, \\
& \quad \hat{A}_{(i+2, n-1)}, \ldots, \hat{A}_{(n-1, n-1)} \subseteq \ll A_n, I \gg.
\end{align*}
$$
Thus
\[ \widehat{A}_{(1,n)}, \widehat{A}_{(2,n)}, \ldots, \widehat{A}_{(i-1,n)}, \widehat{A}_{(i+1,n)}, \widehat{A}_{(i+2,n)}, \ldots, \widehat{A}_{(n-1,n)} \subseteq I. \]

We now only prove that \( \widehat{A}_{(i,n)} \subseteq I \) or \( \widehat{A}_{(n,n)} \subseteq I \). For any integer \( 1 \leq j \leq n \) and \( j \neq i \), we have
\[ A_1 \Gamma A_2 \cdots A_{j-1} \Gamma A_{j+1} \Gamma A_{j+2} \cdots A_{n-1} \Gamma A_n \subseteq \ll A_j, I \gg. \]

Let \( B_k = A_k \) for all \( k = 1, 2, \ldots, j-1 \) and \( B_k = A_{k+1} \) for all \( k = j, j+1, \ldots, n-1 \). Then
\[ B_1 \Gamma B_2 \cdots B_{n-2} \Gamma B_{n-1} \subseteq \ll A_j, I \gg. \]

Hence there exists an integer \( k \) (\( 1 \leq k \leq n-1 \)) such that
\[ \widehat{B}_{(1,n-1)}, \widehat{B}_{(2,n-1)}, \ldots, \widehat{B}_{(k-1,n-1)}, \widehat{B}_{(k+1,n-1)}, \widehat{B}_{(k+2,n-1)}, \ldots, \widehat{B}_{(n-1,n-1)} \subseteq \ll A_j, I \gg. \]

This implies that there exists an integer \( l \) (\( 1 \leq l \leq n \)) and \( l \neq j \) (assume \( l < j \)) such that
\[ \widehat{A}_{(1,n)}, \widehat{A}_{(2,n)}, \ldots, \widehat{A}_{(l-1,n)}, \widehat{A}_{(l+1,n)}, \widehat{A}_{(l+2,n)}, \ldots, \widehat{A}_{(j-1,n)}, \widehat{A}_{(j+1,n)}, \widehat{A}_{(j+2,n)}, \ldots, \widehat{A}_{(n,n)} \subseteq I. \]

Since \( j \neq i \), we get \( \widehat{A}_{(i,n)} \subseteq I \) or \( \widehat{A}_{(n,n)} \subseteq I \). Hence
\[ \widehat{A}_{(1,n)}, \widehat{A}_{(2,n)}, \ldots, \widehat{A}_{(n-2,n)}, \widehat{A}_{(n-1,n)} \subseteq I \]
or
\[ \widehat{A}_{(1,n)}, \widehat{A}_{(2,n)}, \ldots, \widehat{A}_{(i-1,n)}, \widehat{A}_{(i+1,n)}, \widehat{A}_{(i+2,n)}, \ldots, \widehat{A}_{(n,n)} \subseteq I. \]

Therefore \( I \) is an ordered \( n \)-prime ideal of \( M \). Hence the proof of the theorem is completed. \( \square \)

**Theorem 2.4.** If \( a \in (M \Gamma a) \) for all \( a \in M \), then the ordered \( n \)-prime ideals and the ordered \((n-1)\)-prime ideals of \( M \) coincide for all integers \( n \geq 3 \).

**Proof.** Let \( I \) be an ordered \( n \)-prime ideal of \( M \). By Theorem 2.3, \( \ll M, I \gg \) is an ordered \((n-1)\)-prime ideal of \( M \). Let \( a \in \ll M, I \gg \). Then \( a \leq m \gamma a \in I \) for some \( m \in M \) and \( \gamma \in \Gamma \), so \( a \in I \). Thus \( \ll M, I \gg \subseteq I \). By Lemma 1.1 (b), \( \ll M, I \gg = I \). By Lemma 2.1, the proof is completed. \( \square \)

**Theorem 2.5.** If \( I \) is an ordered semiprime ideal of \( M \), then \( I = \ll M, I \gg \).

**Proof.** By Lemma 1.1 (b), \( I \subseteq \ll M, I \gg \). Let \( a \in \ll M, I \gg \). Then \( a \Gamma a \subseteq M \Gamma a \subseteq I \). Since \( I \) is an ordered semiprime ideal of \( M \), \( a \in I \). Hence \( \ll M, I \gg \subseteq I \), so we conclude that \( I = \ll M, I \gg \). \( \square \)
Theorem 2.6. For any integer \( n \geq 3 \), let \( I \) be an ordered semiprime ideal and an ordered \( n \)-prime ideal of \( M \) and let
\[
\mathcal{P} = \{ T : T \text{ is an ordered }(n-1)\text{-prime ideal of } M \text{ and } I \subseteq T \}.
\]
Then \( I = \bigcap_{T \in \mathcal{P}} T \).

Proof. Clearly, \( I \subseteq \bigcap_{T \in \mathcal{P}} T \). By Lemma 1.2 and Theorem 2.5,
\[
I = \bigcap_{x \in M} \ll x, I \gg.
\]
By Lemma 1.1 (b) and Theorem 2.3, \( I \subseteq \ll x, I \gg \) is an ordered \((n-1)\)-prime ideal of \( M \) for all \( x \in M \). Thus \( \ll x, I \gg \in \mathcal{P} \) for all \( x \in M \). Hence
\[
\bigcap_{T \in \mathcal{P}} T \subseteq \bigcap_{x \in M} \ll x, I \gg = I.
\]
Therefore \( I = \bigcap_{T \in \mathcal{P}} T \). Hence the theorem is now completed. \( \square \)

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