OSCULATORY WFI-ALGEBRAS

YOUNG BAE JUN

Abstract. The notions of mote, beam and osculatory WFI-algebra are
introduced, and several properties are investigated. Relations between
osculatory WFI-algebra and associative WFI-algebra are provided. Char-
acterizations of osculatory WFI-algebra are given.

1. Introduction

In 1990, W. M. Wu [7] introduced the notion of fuzzy implication alge-
bra (FI-algebra, for short), and investigated several properties. In [6], Z. Li
and C. Zheng introduced the notion of distributive (resp. regular, commu-
tative) FI-algebra, and investigated the relations between such FI-algebra and
MV-algebra. In [1], Y. B. Jun discussed several aspects of WFI-algebra. He
introduced the notion of associative (resp. normal, medial) WFI-algebra, and
investigated several properties. He gave conditions for a WFI-algebra to be
associative/medial, and provided characterizations of associative/medial WFI-
algebra, and showed that every associative WFI-algebra is a group in which
every element is an involution. He also verified that the class of all medial
WFI-algebras is a variety. Y. B. Jun and S. Z. Song [5] introduced the notions
of simulative and/or mutant WFI-algebra and investigated some properties.
They established characterizations of a simulative WFI-algebra, and gave a
relation between an associative WFI-algebra and a simulative WFI-algebra.
They also found some types for a simulative WFI-algebra to be mutant. Jun
et al. [4] introduced the concept of ideals of WFI-algebra, and gave relations
between a filter and an ideal. Moreover, they provided characterizations of
an ideal, and established an extension property for an ideal. In [2] and [3],
the present author discussed perfect (resp. weak and concrete) filters of WFI-
algebra. In this paper, we introduce the notions of mote, beam and osculatory
WFI-algebra, and investigate several properties. We give relations between
osculatory WFI-algebra and associative WFI-algebra. We provide characteri-
zations of osculatory WFI-algebra.

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2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2,0)$. By a WFI-algebra we mean a system $X := (X, \ominus, 1) \in K(\tau)$ in which the following axioms hold:

(a1) $(x \in X) \ (x \ominus x = 1)$,
(a2) $(x, y \in X) \ (x \ominus y = y \ominus x \Rightarrow x = y)$,
(a3) $(x, y, z \in X) \ (x \ominus (y \ominus z) = y \ominus (x \ominus z))$,
(a4) $(x, y, z \in X) \ ((x \ominus y) \ominus ((y \ominus z) \ominus (x \ominus z)) = 1)$.

We call the special element 1 the unit. For the convenience of notation, we shall write $[x, y_1, y_2, \ldots, y_n]$ for $(\cdots (x \ominus y_1) \ominus y_2) \ominus \cdots \ominus y_n$.

We define $[x, y]^0 = x$, and for $n > 0$, $[x, y]^n = [x, y, y, \ldots, y]$, where $y$ occurs $n$-times. We use the notation $x^n \ominus y$ instead of $x \ominus (\cdots (x \ominus (x \ominus y)) \cdots)$ in which $x$ occurs $n$-times.

**Proposition 2.1 ([1]).** In a WFI-algebra $X$, the following are true:

(b1) $x \ominus [x, y]^2 = 1$,
(b2) $1 \ominus x = 1 \Rightarrow x = 1$,
(b3) $1 \ominus x = x$,
(b4) $x \ominus y = 1 \Rightarrow [y, z, x \ominus z] = 1 \& [z, x, z \ominus y] = 1$,
(b5) $[x, y, 1] = [x, 1, y \ominus 1]$,
(b6) $[x, y]^3 = x \ominus y$.

A nonempty subset $S$ of a WFI-algebra $X$ is called a subalgebra of $X$ if $x \ominus y \in S$ whenever $x, y \in S$. A nonempty subset $F$ of a WFI-algebra $X$ is called a filter of $X$ if it satisfies:

(c1) $1 \in F$,
(c2) $(\forall x \in F) (\forall y \in X) \ (x \ominus y \in F \Rightarrow y \in F)$.

A filter $F$ of a WFI-algebra $X$ is said to be closed [1] if $F$ is also a subalgebra of $X$.

**Proposition 2.2 ([1]).** Let $F$ be a filter of a WFI-algebra $X$. Then $F$ is closed if and only if $x \ominus 1 \in F$ for all $x \in F$.

**Proposition 2.3 ([1]).** In a finite WFI-algebra, every filter is closed.

We now define a relation “$\preceq$” on $X$ by $x \preceq y$ if and only if $x \ominus y = 1$. It is easy to verify that a WFI-algebra is a partially ordered set with respect to $\preceq$. A WFI-algebra $X$ is said to be associative [1] if it satisfies $[x, y, z] = x \ominus (y \ominus z)$ for all $x, y, z \in X$. For a WFI-algebra $X$, the set

$$S(X) := \{ x \in X \mid x \preceq 1 \}$$

is called the simulative part of $X$. A WFI-algebra $X$ is said to be simulative [5] if it satisfies

(S) $x \preceq 1 \Rightarrow x = 1$. 
Note that the condition (S) is equivalent to $S(\mathcal{X}) = \{1\}$.  

**Proposition 2.4** ([5]). *The simulative part $S(\mathcal{X})$ of a WFI-algebra $\mathcal{X}$ is a filter of $\mathcal{X}$.***

### 3. Osculatory WFI-algebra

We begin with the following definition.

**Definition 3.1.** A WFI-algebra $\mathcal{X}$ is said to be osculatory if it satisfies:

\[(\forall x, y \in X) (y \odot x = 1 \Rightarrow [x, y]^2 = x).\]

**Example 3.2.** Let $X = \{1, a, b, c\}$ be a set with the following Cayley table.

<table>
<thead>
<tr>
<th>$\ominus$</th>
<th>1</th>
<th>a</th>
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</table>

Then $\mathcal{X} := (X, \ominus, 1)$ is an osculatory WFI-algebra.

Let $\mathcal{X}$ be a WFI-algebra. Consider the following equation:

\[(3.2) [x, y]^2 = [y, x][y, x]^2.\]

**Example 3.3.** Let $X = \{1, a, b\}$ be a set with the following Cayley table.

<table>
<thead>
<tr>
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</thead>
<tbody>
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<td>a</td>
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<td>b</td>
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Then $\mathcal{X} := (X, \ominus, 1)$ is a WFI-algebra which satisfies the equation (3.2). But $\mathcal{X}$ is not associative since $[a, a]^2 = a \neq 1 = a^2 \ominus a$.

**Example 3.4.** Let $X = \{1, a, b\}$ be a set with the following Cayley table.

<table>
<thead>
<tr>
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<tbody>
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Then $\mathcal{X} := (X, \ominus, 1)$ is a WFI-algebra which does not satisfy the equation (3.2) since $[a, b]^2 = a \neq 1 = [b, a][b, a]^2$.

**Lemma 3.5** ([1]). *Let $\mathcal{X}$ be a WFI-algebra. Then the following are equivalent.***

(i) $\mathcal{X}$ is associative.

(ii) $(\forall x \in X) (x \ominus 1 = x)$.

(iii) $(\forall x, y \in X) (x \ominus y = y \ominus x)$.

**Proposition 3.6.** *Every associative WFI-algebra satisfies the equation (3.2).*
Proof. Let $X$ be an associative WFI-algebra and let $x, y \in X$. Using the associativity of $X$, (a1), (b3) and Lemma 3.5, we have
\[
[y, x, [y, x]^2] = [y, x, y \circ x, x] = 1 \circ x = x \\
= x \circ 1 = x \circ (y \circ y) = [x, y]^2.
\]
This completes the proof. □

**Proposition 3.7.** If a WFI-algebra $X$ satisfies the equation (3.2), then $X$ is osculatory.

**Proof.** Let $x, y \in X$ be such that $y \circ x = 1$. Using (b3), it is straightforward. □

**Corollary 3.8.** Every associative WFI-algebra is osculatory.

**Definition 3.9.** If an element $m$ of a WFI-algebra $X$ is maximal in $(X, \preceq)$, we say that $m$ is a mote of $X$.

Denote by $M(X)$ the set of all motes of $X$. For any $m \in M(X)$, the set
\[
B(m) := \{x \in X \mid x \circ m = 1\}
\]
is called a beam of $X$ with respect to $m$ (briefly, $m$-beam of $X$). Obviously, $1 \in M(X)$ and $B(1) = S(X)$.

**Example 3.10.** Let $X = \{1, a, b, c, d\}$ be a set with the following Cayley table.

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</table>

Then $X := (X, \circ, 1)$ is a WFI-algebra and $M(X) = \{1, c\}$. Hence $B(1) = \{1, a, b\}$ and $B(c) = \{c, d\}$.

**Proposition 3.11.** Every mote $m$ of a WFI-algebra $X$ is represented by the following identity:
\[(\forall x \in X) (m = [m, x]^2).\]

**Proof.** Since $m \circ [m, x]^2 = 1$ for all $x \in X$, we have $m = [m, x]^2$ for all $x \in X$. □

We give a condition for an element of $X$ to be a mote of $X$.

**Theorem 3.12.** If an element $m$ of a WFI-algebra $X$ satisfies the following equation:
\[(\forall x, y \in X) (x \circ m = [x, m, y, y])\]
then $m$ is a mote of $X$. 
Proof. Let \( x \in X \) be such that \( m \ominus x = 1 \). Then

\[
m = 1 \ominus m = [1, m, x, x] = [m, x]^2 = 1 \ominus x = x,
\]
and so \( m \) is a mote of \( X \). \( \square \)

The following is more simple condition for an element to be a mote.

**Theorem 3.13.** If an element \( m \) of a WFI-algebra \( X \) satisfies the following identity:

\[
[m, 1]^2 = m,
\]
then \( m \) is a mote of \( X \).

**Proof.** Using (a3), (b1), (b4), (b5) and (3.5), we have

\[
[m, x, 1] = [m, 1, x \ominus 1] = x \ominus [m, 1]^2 = x \ominus m
\]
for all \( x \in X \), and

\[
[x, m, 1, 1] = [x, 1, m \ominus 1] \ominus 1 = [x, 1]^2 \ominus [m, 1]^2 = [x, 1]^2 \ominus m \leq x \ominus m.
\]
for all \( x \in X \). Using (b1), we have \( x \ominus m \leq [x, m, 1, 1] \). Hence

\[
x \ominus m = [x, m, 1, 1],
\]
and so

\[
[x, m, y, y] \leq [y, 1, [x, m, y, 1]] = [y, [x, m, y], 1] = [x, m, y \ominus y, 1] = [x, m, 1, 1] = x \ominus m
\]
for all \( x, y \in X \). Combining (a2), (b1) and (3.9), we get

\[
x \ominus m = [x, m, y, y]
\]
for all \( x, y \in X \). It follows from Theorem 3.12 that \( m \) is a mote of \( X \). \( \square \)

In the following theorem, we show that every mote \( m \) of a WFI-algebra \( X \) satisfies the condition (3.5).

**Theorem 3.14.** Every mote \( m \) of a WFI-algebra \( X \) satisfies the condition (3.5).

**Proof.** Let \( m \) be a mote of a WFI-algebra \( X \). Using (a3) and (3.3), we obtain

\[
[m, x, y \ominus x] = y \ominus [m, x]^2 = y \ominus m
\]
for all \( x, y \in X \). Using (b1), (b4) and (3.11), we have

\[
[m, x, 2] \leq [m, x, [z, x]^2] = [z, x, m].
\]
It follows from (a3) and (b4) that

\[
[m, x, y \ominus z] = y \ominus [m, x, z] \leq y \ominus [z, x, m] = [z, x, y \ominus m].
\]
Using (a1), (b5) and (3.13), we get

\[
[m, 1, x \ominus 1] = [m, x, 1] = [m, x, 1 \ominus 1] \leq [1, x, 1 \ominus m] = x \ominus m.
\]
Obviously, $x \od m \preceq [m, 1, x \od 1]$, and hence
\[(3.14) \quad [m, 1, x \od 1] = x \od m.\]
This implies that $[m, 1]^2 = [m, 1, 1] = 1 \od m = m$. This completes the proof. $\Box$

**Corollary 3.15.** If $m$ is a mote of a WFI-algebra $\mathfrak{X}$, then
\[(3.15) \quad (\forall x \in X) \quad ([x \od m, 1]^2 = [x, 1]^2 \od m).\]

*Proof.* Using (b5) and Theorem 3.14, we have
\[\quad [x \od m, 1]^2 = [x, 1]^2 \od [m, 1]^2 = [x, 1]^2 \od m.\]
This completes the proof. $\Box$

**Corollary 3.16.** If $p$ and $q$ are motes of a WFI-algebra $\mathfrak{X}$, then so is $p \od q$.

*Proof.* Let $p$ and $q$ be motes of a WFI-algebra $\mathfrak{X}$. Then $[p, 1]^2 = p$ and $[q, 1]^2 = q$. Hence
\[\quad [p \od q, 1]^2 = [p, 1]^2 \od [q, 1]^2 = p \od q,
\]and so $p \od q$ is a mote of $\mathfrak{X}$ by Theorem 3.13. $\Box$

**Proposition 3.17.** For any element $x$ of a WFI-algebra $\mathfrak{X}$, the element $[x, 1]^2$ is a mote of $\mathfrak{X}$.

*Proof.* Let $x \in X$ and $m = [x, 1]^2$. Then
\[\quad [m, 1]^2 = [[x, 1]^2, 1]^2 = [x, 1]^3 \od 1 = [x, 1]^2 \od m.
\]It follows from Theorem 3.13 that $[x, 1]^2$ is a mote of $\mathfrak{X}$. $\Box$

**Theorem 3.18.** A WFI-algebra $\mathfrak{X}$ is osculatory if and only if it satisfies the following identity:
\[(3.16) \quad (\forall x, y \in X) ([|x, y|]^2, x]^2 = [x, y]^2).\]

*Proof.* Assume that a WFI-algebra $\mathfrak{X}$ is osculatory. Since $x \od |x, y|^2 = 1$, it follows from (3.1) that
\[\quad [x, y]^2 = [|x, y|^2, x]^2\]
which proves (3.16). Now let $\mathfrak{X}$ be a WFI-algebra in which the identity (3.16) is valid. Let $x, y \in X$ be such that $y \od x = 1$. Using (b3) and (3.16), we have
\[\quad [x, y]^2 = [1 \od x, y]^2 = [y, x]^2, y]^2 = [y, x]^2 = 1 \od x = x.
\]Hence $\mathfrak{X}$ is osculatory. $\Box$

**Lemma 3.19.** For any motes $p$ and $q$ of $\mathfrak{X}$, we have
(i) $(\forall x, y \in X) \quad (x \in B(p) \& y \in B(q) \Rightarrow x \od y \in B(p \od q))$.
(ii) $(\forall x, y \in B(p)) \quad ([x, y, 1] = 1)$. 

Proof. (i) Let $x \in B(p)$ and $y \in B(q)$. Then $x \circ p = 1$ and $y \circ q = 1$. Hence
\[
\begin{align*}
[x, y, p \circ q] &= [x, y, [p \circ q, 1]^2] = [p, q, 1] \circ [x, y, 1] \\
&= [p, q, 1] \circ [x, 1, y \circ 1] = [x, 1, y \circ [p \circ q, 1]^2] \\
&= [x, 1, y \circ (p \circ q)] = [x, 1, y \circ (y \circ q)] \\
&= [x, 1, p \circ 1] = [x, p, 1] = 1 \circ 1 = 1,
\end{align*}
\]
and so $x \circ y \in B(p \circ q)$.

(ii) It is direct consequence of (i). \qed

Proposition 3.20. Let $X$ be an osculatory WFI-algebra. Then
\[
(3.17) \quad \forall m \in M(X) \forall x, y \in X \quad (x, y \in B(m) \Rightarrow [x, y]^2 = [y, x]^2).
\]

Proof. Let $m \in M(X)$ and $x, y \in B(m)$. Then $[x, y, 1] = 1$ by Lemma 3.19(ii). Using Theorem 3.18, (a3) and (b6), we have
\[
[y, x]^2 \circ [x, y]^2 = [y, x]^2 \circ [x, y]^2 = [[x, y]^2, x, [y, x]^3]
= [[x, y]^2, x, y \circ x] = y \circ [[x, y]^2, x]^2
= y \circ [x, y]^2 = [x, y, y \circ y]
= [x, y, 1] = 1.
\]
Similarly, $[x, y]^2 \circ [y, x]^2 = 1$. This completes the proof. \qed

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