BLENDING INSTANTANEOUS AND CONTINUOUS
PHENOMENA IN FEYNMAN’S OPERATIONAL CALCULI:
THE CASE OF TIME DEPENDENT NONCOMMUTING
OPERATORS

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ABSTRACT. Feynman’s operational calculus for noncommuting operators
was studied via measures on the time interval. We investigate some prop-
erties of Feynman’s operational calculi which include a variety of blends
of discrete and continuous measures in the time dependent setting.

1. Introduction

Feynman’s 1951 paper on the operational calculus for noncommuting op-
erators arose out of his ingenious work on quantum electrodynamics and was
inspired in part by his earlier work on the Feynman path integral. Much sur-
prisingly varied work on the subject has been done since by mathematicians
and physicists. References can be found in the recent books of Johnson and
Lapidus [8] and Nazaikinskii, Shatalov and Sternin [12].

A new approach to the mathematically rigorous theory of Feynman’s opera-
tional calculus was begun recently by Jefferies and Johnson [3]-[7]. Each of the
n operators involved has associated with it a measure on an appropriate time
interval, and the resulting n-vector of measures determines a particular oper-
ational calculus. Here we begin the study of a broader theory which includes
a variety of blends of discrete and continuous measure in the time dependent
setting.

We now introduce some notation and begin to our discussion more precise.
Let X be a separable Banach space over the complex numbers and let \( L(X) \)
denote the space of bounded linear operators on X. Fix \( T > 0 \). For \( i = 1, \ldots, n \)
let \( A_i : [0, T] \rightarrow L(X) \) be maps that are measurable in the sense that \( A_i^{-1}(E) \)
is a Borel set in \([0,T]\) for any strong operator open set \( E \subset L(X) \). To each
\( A_i(\cdot) \) we associate a finite Borel measure \( \lambda_i \) on \([0,T]\) and we require that, for

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each $i$,

$$r_i = \int_{[0,T]} ||A_i(s)||\zeta(s)|\lambda_i|(ds) < \infty.$$ 

Given a positive integer $n$ and $n$ positive numbers $r_1, \ldots, r_n$, let $\mathcal{A}(r_1, \ldots, r_n)$ be the space of complex-valued functions of $n$ complex variables $f(z_1, \ldots, z_n)$, which are analytic at $(0, \ldots, 0)$, and are such that their power series expansion

$$(1)\quad f(z_1, \ldots, z_n) = \sum_{m_1, \ldots, m_n = 0}^{\infty} c_{m_1, \ldots, m_n} z_1^{m_1} \cdots z_n^{m_n}$$

converges absolutely, at least on the closed polydisk $|z_1| \leq r_1, \ldots, |z_n| \leq r_n$. Such functions are analytic at least in the open polydisk $|z_1| < r_1, \ldots, |z_n| < r_n$.

For $f \in \mathcal{A}(r_1, \ldots, r_n)$ given by (1), we let

$$(2)\quad ||f|| = ||f||_{\mathcal{A}(r_1, \ldots, r_n)} := \sum_{m_1, \ldots, m_n = 0}^{\infty} |c_{m_1, \ldots, m_n}| r_1^{m_1} \cdots r_n^{m_n}.$$ 

The function on $\mathcal{A}(r_1, \ldots, r_n)$ defined by (2) makes $\mathcal{A}(r_1, \ldots, r_n)$ into a commutative Banach algebra [3].

To the algebra $\mathcal{A}(r_1, \ldots, r_n)$ we associate a disentangling algebra by replacing the $z_i$’s with formal commuting objects $(A_i(\cdot), \lambda_i), \lambda_i = 1, \ldots, n$. Consider the collection $\mathcal{D}((A_1(\cdot), \lambda_1), \ldots, (A_n(\cdot), \lambda_n))$ of all expressions of the form

$$f((A_1(\cdot), \lambda_1), \ldots, (A_n(\cdot), \lambda_n))$$

$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} c_{m_1, \ldots, m_n} ((A_1(\cdot), \lambda_1))^{m_1} \cdots ((A_n(\cdot), \lambda_n))^{m_n},$$

where $c_{m_1, \ldots, m_n} \in \mathbb{C}$ for all $m_1, \ldots, m_n = 0, 1, \ldots,$ and

$$||f((A_1(\cdot), \lambda_1), \ldots, (A_n(\cdot), \lambda_n))||$$

$$= ||f((A_1(\cdot), \lambda_1), \ldots, (A_n(\cdot), \lambda_n))||_{\mathcal{D}((A_1(\cdot), \lambda_1), \ldots, (A_n(\cdot), \lambda_n))}$$

$$(3)\quad := \sum_{m_1, \ldots, m_n = 0}^{\infty} |c_{m_1, \ldots, m_n}| r_1^{m_1} \cdots r_n^{m_n} < \infty,$$

where $r_i = \int_{[0,T]} ||A_i(s)||\zeta(s)|\lambda_i|(ds)$ for $i = 1, 2, \ldots, n$.

Rather than using the notation $(A_i(\cdot), \lambda_i)$ below, we will often abbreviate to $A_i$, especially when carrying out calculations. We will often write $\mathbb{D}$ in place of $\mathcal{D}((A_1(\cdot), \lambda_1), \ldots, (A_n(\cdot), \lambda_n))$.

Adding and scalar multiplying such expressions coordinatewise, we can easily see that $\mathbb{D}((A_1(\cdot), \lambda_1), \ldots, (A_n(\cdot), \lambda_n))$ is a vector space and that $||\cdot||_{\mathbb{D}}$ defined by (3) is a norm. The normed linear space $(\mathbb{D}((A_1(\cdot), \lambda_1), \ldots, (A_n(\cdot), \lambda_n)), ||\cdot||_{\mathbb{D}})$ can be identified with the weighted $l_1$-space, where the weight at the index $(m_1, \ldots, m_n)$ is $r_1^{m_1} \cdots r_n^{m_n}$. It follows that $\mathbb{D}((A_1(\cdot), \lambda_1), \ldots, (A_n(\cdot), \lambda_n))$ is a commutative Banach algebra with identity [7].
We refer to \( \mathbb{D}(A_1(\cdot), \lambda_1) \) as the disentangling algebra associated with the \( n \)-tuple \((A_1(\cdot), \lambda_1, \ldots, A_n(\cdot), \lambda_n)\).

For \( m = 0, 1, \ldots \), let \( S_m \) denote the set of all permutations of the integers \( \{1, \ldots, m\} \), and given \( \pi \in S_m \), we let
\[
\Delta_m(\pi) = \{(s_1, \ldots, s_m) \in [0, T]^m : 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < T\}.
\]

Now for nonnegative integers \( m_1, \ldots, m_n \) and \( m = m_1 + \cdots + m_n \), we define
\[
C_i(s) = \begin{cases} 
A_1(s), & \text{if } i \in \{1, \ldots, m_1\} \\
A_2(s), & \text{if } i \in \{m_1 + 1, \ldots, m_1 + m_2\} \\
& \vspace{1em} \vdots \\
A_n(s), & \text{if } i \in \{m_1 + \cdots + m_{n-1} + 1, \ldots, m\}
\end{cases}
\]
for \( i = 1, \ldots, m \) and for all \( 0 \leq s \leq T \). Next, in order to accommodate the use of discrete measures, we will need a refined version of the time-ordered framework of the disentangling algebra.

Now we define the map
\[
T_{\lambda_1, \ldots, \lambda_n} \mathbb{D}(A_1(\cdot), \lambda_1) \times \cdots \times \mathbb{D}(A_n(\cdot), \lambda_n) \to \mathbb{D}(A_1(\cdot), \lambda_1) \times \cdots \times \mathbb{D}(A_n(\cdot), \lambda_n)
\]

by
\[
T_{\lambda_1, \ldots, \lambda_n}(A_1(\cdot), \ldots, A_n(\cdot)) = (A_1(\cdot), \ldots, A_n(\cdot)).
\]

**Definition 1.** Let \( P^{m_1, \ldots, m_n}(z_1, \ldots, z_n) = z_1^{m_1} \cdots z_n^{m_n} \). We define the action of the disentangling map on this monomial by
\[
T_{\lambda_1, \ldots, \lambda_n}(A_1(\cdot), \ldots, A_n(\cdot)) = T_{\lambda_1, \ldots, \lambda_n}((A_1(\cdot))^{m_1} \cdots (A_n(\cdot))^{m_n})
\]
Suppose that each of the measures 

\[ \lambda_1, \ldots, \lambda_n \] 
is continuous on \([0, T]\). Then the expression of the disentangling map defined in Definition 1 is identical to that defined in Definition 2.3 of [7].

**Proof.** If each of the measures \( \lambda_1, \ldots, \lambda_n \) is continuous then all of the \( \lambda \)'s have 0 discrete part. So \( q_{12} = \cdots = q_{n2} = 0 \) and \( q_{11} = m_1 \) for \( i = 1, \ldots, n \). Thus 

\[ q_{11} + \cdots + q_{1n} = m_1 + \cdots + m_n = m \] 
and so 
\[ S_{q_{11} + \cdots + q_{1n}} = S_{m_1 + \cdots + m_n} = S_m. \]
Also, \( r_1 = \cdots = r_{h+1} = 0 \) and all of the \( j \)'s are 0. Further, both of the quotients of products of factorials are equal to 1. Hence we have

\[
T_{\lambda_1, \ldots, \lambda_n}^{m_1, \ldots, m_n}(A_1(\cdot), \ldots, A_n(\cdot)) = \sum_{q_{i1} + q_{i2} = m_1} \sum_{q_{j1} + q_{j2} = m_2} \cdots \sum_{q_{n1} + q_{n2} = m_n} \frac{m_1! \cdots m_n!}{q_{i1}! q_{i2}! q_{j1}! q_{j2}! \cdots q_{n1}! q_{n2}!}
\]

\[
\sum_{\pi \in S_{q_{i1} + q_{i2} + \cdots + q_{n1}}} \sum_{j_{i1} + \cdots + j_{i_{h+1}} = q_{i1} + q_{i2} + \cdots + q_{n1}} \sum_{j_{j1} + \cdots + j_{j_{h+1}} = q_{j1} + q_{j2}} \cdots \sum_{j_{n1} + \cdots + j_{n_{h+1}} = q_{n1}} \frac{q_{i1}! q_{i2}! \cdots q_{n1}!}{j_{i1}! j_{i2}! \cdots j_{i_{h+1}}! j_{j1}! \cdots j_{j_{h+1}}! j_{n1}! \cdots j_{n_{h+1}}!}
\]

\[
\int_{\Delta_{q_{i1} + q_{i2} + \cdots + q_{n1}} \times \cdots \times \Delta_{q_{i_{h+1}} + q_{j1} + \cdots + q_{n_{h+1}}}} \left( s_{\pi(q_{i1} + q_{i2} + \cdots + q_{n1})} s_{\pi(q_{j1} + q_{j2})} \cdots s_{\pi(q_{n1})} \right) \left( \mu_1^{q_{i1}} \cdots \mu_1^{q_{i_{h+1}}} \mu_2^{q_{j1}} \cdots \mu_2^{q_{j_{h+1}}} \cdots \mu_n^{q_{n1}} \cdots \mu_n^{q_{n_{h+1}}} \right) ds_{i1}, \ldots, ds_{i_{h+1}} ds_{j1}, \ldots, ds_{j_{h+1}} ds_{n1}, \ldots, ds_{n_{h+1}}
\]

The last equation is the identical expression for the disentangling of

\[
P^{m_1, \ldots, m_n}(A_1(\cdot), \ldots, A_n(\cdot))
\]

in Definition 2.3 of [7].

\[\square\]

**Theorem 2.2.** The disentangling map \( T_{\lambda_1, \ldots, \lambda_n} \) is a bounded linear operator from \( \mathbb{D}((A_1(\cdot), \lambda_1), \ldots, (A_n(\cdot), \lambda_n)) \) to \( \mathcal{L}(X) \). In fact, \( ||T_{\lambda_1, \ldots, \lambda_n}|| \leq 1 \).

**Proof.** The linearity of \( T_{\lambda_1, \ldots, \lambda_n} \) is clear. We have

\[
||T_{\lambda_1, \ldots, \lambda_n} P^{m_1, \ldots, m_n}(A_1(\cdot), \ldots, A_n(\cdot))||
\]
≤ \sum_{q_1+q_2=m_1\atop q_1+q_2=m_2} \cdots \sum_{q_n+q_n=m_n} \left( \frac{m_1! \cdots m_n!}{q_1! q_2! q_2! \cdots q_n! q_n!} \right) \\
\sum_{\pi \in S_{q_1+q_2+\cdots+q_n}} \sum_{j_1+\cdots+j_{n-1}=q_1+j_2+\cdots+j_{n-2}=q_2+\cdots+q_{n-1}+q_n} \sum_{j_1+\cdots+j_{n-1}=q_1+j_2+\cdots+j_{n-2}=q_2+\cdots+q_{n-1}+q_n} \sum_{j_1+\cdots+j_{n-1}=q_1+j_2+\cdots+j_{n-2}=q_2+\cdots+q_{n-1}+q_n} \sum_{j_1+\cdots+j_{n-1}=q_1+j_2+\cdots+j_{n-2}=q_2+\cdots+q_{n-1}+q_n} \frac{q_1 q_2 q_2 ! \cdots q_n}{j_1 \cdots j_{n-1} j_1 \cdots j_{n-1}} \int_{\Delta_{q_1+q_1+\cdots+q_n}} |C_{\pi(j_1+q_1+\cdots+q_1)}(s_\pi(j_1+q_1+\cdots+q_1))| \cdots \\
\left|\sum_{\pi \in S_{q_1+q_2+\cdots+q_n}} \sum_{j_1+\cdots+j_{n-1}=q_1+j_2+\cdots+j_{n-2}=q_2+\cdots+q_{n-1}+q_n} \sum_{j_1+\cdots+j_{n-1}=q_1+j_2+\cdots+j_{n-2}=q_2+\cdots+q_{n-1}+q_n} \sum_{j_1+\cdots+j_{n-1}=q_1+j_2+\cdots+j_{n-2}=q_2+\cdots+q_{n-1}+q_n} \sum_{j_1+\cdots+j_{n-1}=q_1+j_2+\cdots+j_{n-2}=q_2+\cdots+q_{n-1}+q_n} \frac{q_1 q_2 q_2 ! \cdots q_n}{j_1 \cdots j_{n-1} j_1 \cdots j_{n-1}} \int_{\Delta_{q_1+q_1+\cdots+q_n}} |A_1(s_1)| \cdots |A_1(s_{q_1})| \cdots |A_1(s_{q_1+q_2})| \cdots \\
\left|A_n(s_{q_1+\cdots+q_{n-1}+1})| \cdots |A_n(s_{q_1+\cdots+q_{n-1}+q_n})| \int_{[0,T]^{q_1}} |A_1(s_1)| \cdots |A_1(s_{q_1})| \cdots |A_1(s_{q_1+q_2})| |\mu_1(q_{11})| (d_{s_1}, \ldots, d_{s_{q_1}}, \ldots, d_{s_{q_2}}, \ldots, d_{s_{q_1+q_2}}, \ldots, d_{s_{q_1+q_2+\cdots+q_n}})

= \sum_{q_1+q_2=m_1\atop q_1+q_2=m_2} \cdots \sum_{q_n+q_n=m_n} \left( \frac{m_1! \cdots m_n!}{q_1! q_2! q_2! \cdots q_n! q_n!} \right) \\
\sum_{\pi \in S_{q_1+q_2+\cdots+q_n}} \sum_{j_1+\cdots+j_{n-1}=q_1+j_2+\cdots+j_{n-2}=q_2+\cdots+q_{n-1}+q_n} \sum_{j_1+\cdots+j_{n-1}=q_1+j_2+\cdots+j_{n-2}=q_2+\cdots+q_{n-1}+q_n} \sum_{j_1+\cdots+j_{n-1}=q_1+j_2+\cdots+j_{n-2}=q_2+\cdots+q_{n-1}+q_n} \sum_{j_1+\cdots+j_{n-1}=q_1+j_2+\cdots+j_{n-2}=q_2+\cdots+q_{n-1}+q_n} \frac{q_1 q_2 q_2 ! \cdots q_n}{j_1 \cdots j_{n-1} j_1 \cdots j_{n-1}} \int_{\Delta_{q_1+q_1+\cdots+q_n}} |A_1(s_1)| \cdots |A_1(s_{q_1})| \cdots |A_1(s_{q_1+q_2})| |\mu_1(q_{11})| (d_{s_1}, \ldots, d_{s_{q_1}}, \ldots, d_{s_{q_2}}, \ldots, d_{s_{q_1+q_2}}, \ldots, d_{s_{q_1+q_2+\cdots+q_n}})
\[\begin{align*}
\mathbb{T}(\tau_1, \ldots, \tau_n) &= \int_{[0,T]^n} f_1(\tau_1) \cdots f_n(\tau_n) \, d\tau_1 \cdots d\tau_n,
\mathbb{T}(s_{n+1}) &= \int_{[0,T]^{n+1}} f_1(s_1) \cdots f_n(s_n) \, ds_1 \cdots ds_n.
\end{align*}\]

Hence, for \( f(\tilde{s}_1, \ldots, \tilde{s}_n) \in \mathbb{D}(\mathbb{T}_{\lambda_1, \ldots, \lambda_n}) \),
\[
\|f(\tilde{s}_1, \ldots, \tilde{s}_n)\|_{\mathbb{D}(\mathbb{T}_{\lambda_1, \ldots, \lambda_n})} \leq \sum_{m_1, \ldots, m_n=0}^{\infty} |c_{m_1, \ldots, m_n}| \|\mathbb{T}_{\lambda_1, \ldots, \lambda_n} P^{m_1, \ldots, m_n}(\tilde{s}_1, \ldots, \tilde{s}_n)\|_{\mathbb{D}(\mathbb{T}_{\lambda_1, \ldots, \lambda_n})}.
\]

This finishes the proof. \( \square \)

**Theorem 2.3.** Suppose that \( A_i(t)A_j(t) = A_j(t)A_i(t) \) for \( i, j = 1, \ldots, n \) whenever the products are defined, then we have
\[
\mathbb{T}_{\lambda_1, \ldots, \lambda_n} P^{m_1, \ldots, m_n}(A_1(\cdot), \ldots, A_n(\cdot))
\]

\[
= \left[ \int_{[0,T]} A_1(s) \lambda_1(ds) \right]^{m_1} \cdots \left[ \int_{[0,T]} A_n(s) \lambda_n(ds) \right]^{m_n}.
\]
Further, for all \( f(A_1(\cdot), \ldots, A_n(\cdot)) \in \mathbb{D}((A_1(\cdot), \lambda_1; \ldots, (A_n(\cdot), \lambda_n;)) \),

\[
T_{\lambda_1, \ldots, \lambda_n} f(A_1(\cdot), \ldots, A_n(\cdot)) = f \left( \int_{[0,T]} A_1(s)\lambda_1(ds), \ldots, \int_{[0,T]} A_n(s)\lambda_n(ds) \right),
\]

where \( f \) given by

\[
f(z_1, \ldots, z_n) = \sum_{m_1, \ldots, m_n = 0}^{\infty} c_{m_1, \ldots, m_n} z_1^{m_1} \cdots z_n^{m_n}
\]
is an element of \( \mathcal{h}(r_1, \ldots, r_n) \) where

\[
r_i = \int_{[0,T]} ||A_i(s)||_{\mathcal{L}(X)}|\lambda_i|(ds).
\]

**Proof.** The operator

\[
T_{\lambda_1, \ldots, \lambda_n} f(A_1(\cdot), \ldots, A_n(\cdot))
\]
is given in terms of

\[
T_{\lambda_1, \ldots, \lambda_n} p_{m_1}^{m_1} \cdots m_n (A_1(\cdot), \ldots, A_n(\cdot))
\]
and so it suffices to show equation (4). We have

\[
T_{\lambda_1, \ldots, \lambda_n} p_{m_1}^{m_1} \cdots m_n (A_1(\cdot), \ldots, A_n(\cdot)) = \sum_{q_{11} + q_{21} = m_1}^{\infty} \sum_{q_{12} + q_{22} = m_2}^{\infty} \cdots \sum_{q_{1n} + q_{2n} = m_n}^{\infty} \left( \frac{m_1! \cdots m_n!}{q_{11}! q_{21}! q_{22}! \cdots q_{1n}! q_{2n}!} \right)
\]

\[
\sum_{\pi \in S_{q_{11} + q_{21} + + + q_{1n} r_1 + + + r_n + 1 = q_{11} + q_{21} + + + q_{2n}}^{
\sum_{j_{11} + + + j_{1h} = q_{11} j_{21} + + + j_{2h} = q_{21} j_{n1} + + + j_{nh} = q_{2n}}^{
\int_{0_{q_{11} + q_{21} + + + q_{1n} r_1 + + + r_n + 1}^{
C_{\pi(q_{11} + q_{21} + + + q_{1n})}(s_{\pi(q_{11} + q_{21} + + + q_{1n})}) \cdots
C_{\pi(r_1 + + + r_n + 1)}(s_{\pi(r_1 + + + r_n + 1)}) \left[ p_{n_1} A_1(\tau_1) \right]^{r_1} \cdots
[ p_{2h} A_2(\tau_2) ]^{j_{2h}} \left[ p_{1h} A_1(\tau_1) \right]^{j_{1h}} C_{\pi(r_1 + + + r_n + 1)}(s_{\pi(r_1 + + + r_n + 1)}) \cdots
C_{\pi(r_1 + 1)}(s_{\pi(r_1 + 1)}) \left[ p_{n_1} A_1(\tau_1) \right]^{r_1} \cdots
[ p_{11} A_1(\tau_1) ]^{j_{11}} C_{\pi(r_1)}(s_{\pi(r_1)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \left( \mu_1^{q_{11}} \times \cdots \times \mu_n^{q_{nn}} \right)(ds_1, \ldots, ds_{q_{11} + + + q_{nn}})
We obtain the result.
3. Stability properties

In this section, we obtain stability properties for the disentangling map $T_{\lambda_1, \ldots, \lambda_n}$ which was introduced in the previous section. Let $S$ be a metric space and let $\{\lambda_k\}_{k=1}^{\infty}$ be a sequence of finite Borel measures on $S$. We say that $\lambda_k$ converges weakly to a finite Borel measure $\lambda$ on $S$ and write $\lambda_k \rightharpoonup \lambda$ if for every bounded continuous real-valued function $f$ on $S$ we have $\int_S f(s) \lambda_k(ds) \to \int_S f(s) \lambda(ds)$ as $k \to \infty$. The following result is Lemma 3.1 of [13].

**Lemma 3.1.** Let $\eta = \sum_{i=1}^{h} p_i \delta_{\tau_i}$ be a purely discrete probability measure on $[0, T]$ with finite support. Assume that $0 < \tau_1 < \cdots < \tau_h < T$. Let

$$\alpha_i = \min\{\tau_i - \tau_{i-1}, \tau_{i+1} - \tau_i\}$$

for $i = 1, \ldots, h$ where we take $\tau_0 = 0$ and $\tau_{h+1} = T$. In each interval $(\tau_i - \alpha_i, \tau_i + \alpha_i)$, $i = 1, \ldots, h$ choose sequences $\{\tau_k\}_{k=1}^{\infty}$. For each $i = 1, \ldots, h$ choose a sequence $\{p_k\}_{k=1}^{\infty}$ such that $\eta_k = \sum_{i=1}^{h} p_k \delta_{\tau_k}$ is a probability measure for each $k$. Then $\eta_k \rightharpoonup \eta$ if and only if

$$\begin{cases} p_k \to p_i & \text{and} & \tau_k \to \tau_i & \text{if } p_i \neq 0, \\ p_k \to p_i & \text{and} & \{\tau_k\}_{k=1}^{\infty} \text{bounded} & \text{if } p_i = 0 \end{cases}$$

for $i = 1, \ldots, h$.

First we consider the disentangling map for $P_{m_1, \ldots, m_n}(A_1(\cdot), \ldots, A_n(\cdot))$.

**Theorem 3.2.** Let $A_l : [0, T] \to \mathcal{L}(X)$ be continuous with respect to the norm topology on $\mathcal{L}(X)$ for each $l = 1, 2, \ldots, n$. And let $\lambda_1, \ldots, \lambda_n$ be finite Borel measures on $[0, T]$ such that

$$\lambda_l = \mu_l + \eta_l$$

for $l = 1, \ldots, n$ where $\mu_l$ is a continuous probability measure and $\eta_l$ is a finitely supported discrete probability measure for each $l$. Let $\{\tau_1, \ldots, \tau_h\}$ be the set obtained by taking the union of the supports of the discrete measures $\eta_1, \ldots, \eta_n$ and write

$$\eta_l = \sum_{i=1}^{h} p_i \delta_{\tau_i}$$

for each $l = 1, \ldots, n$. Choose sequences $\{\mu_{lk}\}_{k=1}^{\infty}, l = 1, \ldots, n$ of continuous Borel probability measures on $[0, T]$ such that $\mu_{lk} \rightharpoonup \mu_l$. Also choose sequences $\{\eta_{lk}\}_{k=1}^{\infty}, l = 1, \ldots, n$ of discrete probability measures on $[0, T]$ as in Lemma 3.1 such that $\mu_{lk} \rightharpoonup \mu_l$; i.e., write

$$\eta_{lk} = \sum_{i=1}^{h} p_{li} \delta_{\tau_{ik}}$$

where, as in the Lemma 3.1, $p_{li} \to p_i$ and $\tau_{ik} \to \tau_i$ as $k \to \infty$ for all $i, l$ assuming that for $p_{li} \neq 0$ for all $i, l$. Finally let $\lambda_{lk} = \mu_{lk} + \eta_{lk}$ for $l = 1, \ldots, n$. This completes the proof.
Then for any nonnegative integers \( m_1, \ldots, m_n \) and for any \( \Lambda \in \mathcal{L}(X)^* \)

\[
\lim_{k \to \infty} \Lambda(P^{m_1, \ldots, m_n}_{\lambda_{1k}, \ldots, \lambda_{nk}}(A_1(\cdot), \ldots, A_n(\cdot))) = \Lambda(P^{m_1, \ldots, m_n}_{\lambda_{11}, \ldots, \lambda_{nk}}(A_1(\cdot), \ldots, A_n(\cdot))).
\]

**Proof.** We see that for any \( \Lambda \in \mathcal{L}(X)^* \)

\[
|\Lambda(P^{m_1, \ldots, m_n}_{\lambda_{1k}, \ldots, \lambda_{nk}}(A_1(\cdot), \ldots, A_n(\cdot))) - \Lambda(P^{m_1, \ldots, m_n}_{\lambda_{11}, \ldots, \lambda_{nk}}(A_1(\cdot), \ldots, A_n(\cdot)))|
\]

\[
\leq \sum_{q_{11} + q_{22} = m_1} \sum_{q_{21} + q_{22} = m_2} \cdots \sum_{q_{n1} + q_{n2} = m_n} \left( \frac{m_1! \cdots m_n!}{q_{11}! q_{12}! q_{21}! q_{22}! \cdots q_{n1}! q_{n2}!} \right)
\]

\[
\times \sum_{\pi \in S_{q_{11} + q_{22} + \cdots + q_{n1}} \tau_{11} + \cdots + \tau_{nk} = q_{11} + q_{21} + \cdots + q_{n1}} \sum_{j_{11} + \cdots + j_{1k} = q_{11}} \sum_{j_{21} + \cdots + j_{2k} = q_{22}} \cdots \sum_{j_{n1} + \cdots + j_{nk} = q_{n1}} \left( \frac{1}{j_{11}! \cdots j_{11}! j_{21}! \cdots j_{21}! \cdots j_{n1}! \cdots j_{nk}!} \right)
\]

\[
\int_{\Delta_{q_{11} + q_{22} + \cdots + q_{n1}}(\pi)} \Lambda(C_{\pi}(q_{11} + q_{22} + \cdots + q_{n1})) \cdots C_{\pi(\tau_{11} + \cdots + \tau_{nk} + 1)}(s_{\pi(\tau_{11} + \cdots + \tau_{nk} + 1)}) \pi_k^{p_{1k}} \pi_1^{p_{21}} \cdots \pi_n^{p_{nk}} (ds_{11}, \ldots, ds_{q_{11} + q_{21} + \cdots + q_{n1}})
\]

\[
- \int_{\Delta_{q_{11} + q_{22} + \cdots + q_{n1}}(\pi)} \Lambda(C_{\pi(\tau_{11} + \cdots + \tau_{nk} + 1)}(s_{\pi(\tau_{11} + \cdots + \tau_{nk} + 1)}) \pi_k^{p_{1k}} \pi_1^{p_{21}} \cdots \pi_n^{p_{nk}} (ds_{11}, \ldots, ds_{q_{11} + q_{21} + \cdots + q_{n1}})
\]

For each \( l = 1, \ldots, n, i = 1, \ldots, h, \) \( p_{li}^k \to p_{li} \) and \( \tau_{ik} \to \tau_i \) as \( k \to \infty \). Hence since \( A_i \) is continuous we have

\[
p_{li}^k A_i(\tau_{ik}) \to p_{li} A_i(\tau_i)
\]
as $k \to \infty$. Therefore, we have, for any $\Lambda \in \mathcal{L}(X)^*$

\[
\begin{align*}
&X_{\Delta_{q_1+q_2+\cdots+q_n+1}(\pi)}(\mathcal{C}(\pi_{q_1+q_2+\cdots+q_n+1}) (s_{\pi_{q_1+q_2+\cdots+q_n+1}}) \cdots \\
&C_{\pi_{(r_1+\cdots+r_n+1)}} (s_{\pi_{(r_1+\cdots+r_n+1)}})(p_{n_1}^{k_1} A_{n}(\tau_{h_1}))^{j_{1n}} \cdots [p_{2h}^{k_1} A_{n}(\tau_{h_1})]^{j_{2n}} \\
&[p_{n_1}^{k_1} A_{n}(\tau_{h_1})]^{j_{1n}} \cdots [p_{2n}^{k_1} A_{n}(\tau_{h_1})]^{j_{2n}} [p_{11}^{k_1} A_{n}(\tau_{h_1})]^{j_{11}n} C_{\pi_{(r_1)}} (s_{\pi_{(r_1)}}) \cdots \\
&C_{\pi_{(1)}} (s_{\pi_{(1)}})
\end{align*}
\]

uniformly on $[0,T]^{q_1+q_2+\cdots+q_n+1}$. $\{\mu_{q_1}^{n_1} \times \cdots \times \mu_{q_n}^{n_1}\}$ is a sequence of continuous probability measures on $[0,T]^{q_1+q_2+\cdots+q_n+1}$ since each term in the product is a continuous probability measure. And $[0,T]^{q_1+q_2+\cdots+q_n+1}$ is separable. By Theorem 3.2 of [1] $\mu_{q_1}^{n_1} \times \cdots \times \mu_{q_n}^{n_1} \rightarrow \mu_1^{n_1} \times \cdots \times \mu_n^{n_1}$ since $\mu_k \rightarrow \mu_i$ for each $i$. Hence we have, using Lemma 3.2 of [13],

\[
\begin{align*}
&\lim_{k \to \infty} \int X_{\Delta_{q_1+q_2+\cdots+q_n+1}(\pi)} (\mathcal{C}(\pi_{q_1+q_2+\cdots+q_n+1}) (s_{\pi_{q_1+q_2+\cdots+q_n+1}}) \cdots \\
&C_{\pi_{(r_1+\cdots+r_n+1)}} (s_{\pi_{(r_1+\cdots+r_n+1)}})(p_{n_1}^{k_1} A_{n}(\tau_{h_1}))^{j_{1n}} \cdots [p_{2h}^{k_1} A_{n}(\tau_{h_1})]^{j_{2n}} \\
&[p_{n_1}^{k_1} A_{n}(\tau_{h_1})]^{j_{1n}} \cdots [p_{2n}^{k_1} A_{n}(\tau_{h_1})]^{j_{2n}} [p_{11}^{k_1} A_{n}(\tau_{h_1})]^{j_{11}n} C_{\pi_{(r_1)}} (s_{\pi_{(r_1)}}) \cdots \\
&C_{\pi_{(1)}} (s_{\pi_{(1)}})(\mu_{q_1}^{n_1} \times \cdots \times \mu_{q_n}^{n_1}) (ds_1, \ldots, ds_{q_1+q_2+\cdots+q_n+1})
\end{align*}
\]

Hence the conclusion follows. 

The following results can be obtained easily.

Lemma 3.3. Let $\lambda_1, \ldots, \lambda_n, \lambda_{nk}, k = 1, 2, \ldots$ be finite Borel measures. Suppose for $l = 1, 2, \ldots, n$

$$r_l = \sup \{r_1, r_{11}, \ldots, r_{lk}, \ldots \} < \infty,$$

where $r_l = \int_{[0,T]} \|A_l(s)\| \|\lambda_l(\text{d}s)\|$ and $r_{lk} = \int_{[0,T]} \|A_l(s)\| \|\lambda_{lk}(\text{d}s)\|$. Then for any $f \in \mathcal{A}(r_1, \ldots, r_n)$, $f((A_1(\cdot), \lambda_1)\cdots, (A_n(\cdot), \lambda_n)) \in \mathcal{D}(A_1(\cdot), \lambda_1)\cdots, (A_n(\cdot), \lambda_n)$ and $f((A_1(\cdot), \lambda_{1k})\cdots, (A_n(\cdot), \lambda_{nk})) \in \mathcal{D}(A_1(\cdot), \lambda_{1k})\cdots, (A_n(\cdot), \lambda_{nk})$ for any $k = 1, 2, \ldots$.

Theorem 3.4. Let the hypotheses of Theorem 3.1 be satisfied. Further suppose that for each $l = 1, 2, \ldots, n$ and $k = 1, 2, \ldots$, $\bar{r}_l, r_l, r_{lk}$ are given as in Lemma 3.3. Let $T_{\lambda_{1k}, \ldots, \lambda_{nk}}$ denote the disentangling map corresponding to the $k$th term of sequences of measures. Then for any $f \in \mathcal{A}(\bar{r}_1, \ldots, \bar{r}_n)$, and for any $\Lambda \in \mathcal{L}(X)^n$

$$\lim_{k \to \infty} \Lambda(T_{\lambda_{1k}, \ldots, \lambda_{nk}} f((A_1(\cdot), \lambda_{1k})\cdots, (A_n(\cdot), \lambda_{nk}))) = \Lambda(T_{\lambda_1, \ldots, \lambda_n} f((A_1(\cdot), \lambda_1)\cdots, (A_n(\cdot), \lambda_n))).$$

Proof. We have

$$|\Lambda(T_{\lambda_{1k}, \ldots, \lambda_{nk}} f((A_1(\cdot), \lambda_{1k})\cdots, (A_n(\cdot), \lambda_{nk}))) - \Lambda(T_{\lambda_1, \ldots, \lambda_n} f((A_1(\cdot), \lambda_1)\cdots, (A_n(\cdot), \lambda_n))))| \leq \sum_{m_1, \ldots, m_n=0}^{\infty} |c_{m_1, \ldots, m_n}| |\Lambda(P_{\lambda_{1k}, \ldots, \lambda_{nk}}^{m_1, \ldots, m_n} (A_1(\cdot), \ldots, A_n(\cdot)))| - \Lambda(P_{\lambda_1, \ldots, \lambda_n}^{m_1, \ldots, m_n} (A_1(\cdot), \ldots, A_n(\cdot)))|.$$

Note that

$$\sum_{m_1, \ldots, m_n=0}^{\infty} |c_{m_1, \ldots, m_n}| |\Lambda(P_{\lambda_{1k}, \ldots, \lambda_{nk}}^{m_1, \ldots, m_n} (A_1(\cdot), \ldots, A_n(\cdot)))| - \Lambda(P_{\lambda_1, \ldots, \lambda_n}^{m_1, \ldots, m_n} (A_1(\cdot), \ldots, A_n(\cdot)))| \leq \sum_{m_1, \ldots, m_n=0}^{\infty} |c_{m_1, \ldots, m_n}| |\Lambda| |\left|P_{\lambda_{1k}, \ldots, \lambda_{nk}}^{m_1, \ldots, m_n} (A_1(\cdot), \ldots, A_n(\cdot))\right| |$$

$$+ |\Lambda(P_{\lambda_1, \ldots, \lambda_n}^{m_1, \ldots, m_n} (A_1(\cdot), \ldots, A_n(\cdot)))| |$$

$$\leq |\Lambda| \sum_{m_1, \ldots, m_n=0}^{\infty} |c_{m_1, \ldots, m_n}| \left[ \int_{[0,T]} |A_1(s)| |\lambda_{1k}(\text{d}s)|^{m_1} \cdots \right.$$

$$\left. \int_{[0,T]} |A_n(s)| |\lambda_{nk}(\text{d}s)|^{m_n} + \int_{[0,T]} |A_1(s)| |\lambda_1(\text{d}s)|^{m_1} \cdots \right.$$
\[ = |A| \sum_{m_1, \ldots, m_n = 0}^{\infty} |c_{m_1, \ldots, m_n}| \left[ r_{m_1}^{n_1} \cdots r_{m_n}^{n_k} + r_{m_1}^{n_1} \cdots r_{m_n}^{n_k} \right] \]
\[ \leq 2|A| \sum_{m_1, \ldots, m_n = 0}^{\infty} |c_{m_1, \ldots, m_n}| \bar{r}_{m_1}^{n_1} \cdots \bar{r}_{m_n}^{n_k}. \]

Since \( \sum_{m_1, \ldots, m_n = 0}^{\infty} |c_{m_1, \ldots, m_n}| \bar{r}_{m_1}^{n_1} \cdots \bar{r}_{m_n}^{n_k} < \infty \), by Theorem 3.2 and the Lebesgue Dominated Convergence Theorem, we obtain a result. \( \square \)

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