ON THE EXISTENCE OF SOME TYPES OF LP-SASAKIAN MANIFOLDS

ABSOS A. SHAIKH, KANAK K. BAISHYA, AND SABINA EYASMIN

Abstract. The object of the present paper is to provide the existence of LP-Sasakian manifolds with $\eta$-recurrent, $\eta$-parallel, $\phi$-recurrent, $\phi$-parallel Ricci tensor with several non-trivial examples. Also generalized Ricci recurrent LP-Sasakian manifolds are studied with the existence of various examples.

1. Introduction

In 1989 K. Matsumoto ([4]) introduced the notion of LP-Sasakian manifolds. Then I. Mihai and R. Rosca ([6]) introduced the same notion independently and obtained many interesting results. LP-Sasakian manifolds are also studied by U. C. De, K. Matsumoto and A. A. Shaikh ([2]), I. Mihai, U. C. De and A. A. Shaikh ([5]), A. A. Shaikh and S. Biswas ([8]) and others.

Recently A. A. Shaikh and K. K. Baishya ([7]) introduced the notion of LP-Sasakian manifolds with $\eta$-recurrent, $\phi$-parallel and $\phi$-recurrent Ricci tensor which generalizes the notion of $\eta$-parallel Ricci tensor, introduced by M. Kon ([3]) for a Sasakian manifold.

In the present paper the existence of such notions on LP-Sasakian manifolds are ensured by several non-trivial examples both in odd and even dimensions. Section 2 is concerned with basic identities of LP-Sasakian manifolds. Since the notion of Ricci $\eta$-recurrent is the generalization of Ricci $\eta$-parallelity, natural question arises does there exist LP-Sasakian manifolds with $\eta$-recurrent but not $\eta$-parallel Ricci tensor? The answer is affirmative as shown by several examples in section 3. In section 4, we obtain various examples of LP-Sasakian manifolds with (i) $\phi$-parallel Ricci tensor, (ii) $\phi$-recurrent but not $\phi$-parallel Ricci tensor, (iii) $\phi$-parallel but not $\eta$-parallel Ricci tensor. In ([1]) De et. al introduced the notion of generalized Ricci recurrent Riemannian manifolds. The last section deals with generalized Ricci recurrent LP-Sasakian manifolds and proved that such a manifold is Einstein and the associated 1-forms of the manifold are

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of opposite direction. Also the existence of generalized Ricci recurrent LP-Sasakian manifold is ensured by several non-trivial examples constructed with global vector fields.

2. LP-Sasakian manifolds

An $n$-dimensional differentiable manifold $M$ is said to be an LP-Sasakian manifold ([7], [6]) if it admits a $(1, 1)$ tensor field $\phi$, a unit timelike contravariant vector field $\xi$, a 1-form $\eta$ and a Lorentzian metric $g$ which satisfy

\begin{align}
\eta(\xi) &= -1, \\
g(X, \xi) &= \eta(X), \\
\phi^2 X &= X + \eta(X)\xi,
\end{align}

(2.1) $g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$, $\nabla_X \xi = \phi X,$

\begin{align}
(\nabla_X \phi)(Y) &= g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi,
\end{align}

(2.2) $\nabla_X \eta = \Omega(X, Y)$, $\Omega(X, \xi) = 0$

where $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$. It can be easily seen that in an LP-Sasakian manifold, the following relations hold:

\begin{align}
\phi \xi &= 0, \\
\eta(\phi X) &= 0, \\
\text{rank}\phi &= n - 1.
\end{align}

Again, if we put $\Omega(X, Y) = g(X, \phi Y)$ for any vector field $X, Y$ then the tensor field $\Omega(X, Y)$ is a symmetric $(0, 2)$ tensor field ([4]). Also, since the vector field $\eta$ is closed in an LP-Sasakian manifold ([2],[4]), manifold, we have

\begin{align}
(\nabla_X \eta)(Y) &= \Omega(X, Y), \\
\Omega(X, \xi) &= 0
\end{align}

(2.5) for any vector field $X$ and $Y$.

Let $M$ be an $n$-dimensional LP-Sasakian manifold with structure $(\phi, \xi, \eta, g)$. Then the following relations hold ([7]):

\begin{align}
R(X, Y)\xi &= \eta(Y)X - \eta(X)Y, \\
S(X, \xi) &= (n - 1)\eta(X), \\
S(\phi X, \phi Y) &= S(X, Y) + (n - 1)\eta(X)\eta(Y)
\end{align}

(2.6) (2.7) (2.8) for any vector field $X, Y, Z$ where $R$ is the curvature tensor of the manifold.

3. LP-Sasakian manifolds with $\eta$-recurrent Ricci tensor

**Definition 3.1** ([7]). The Ricci tensor $S$ of an LP-Sasakian manifold is said to be $\eta$-recurrent if it satisfies the following:

\begin{align}
(\nabla_X S)(\phi Y, \phi Z) &= A(X)S(\phi Y, \phi Z)
\end{align}

(3.1) for all $X, Y, Z$ where $A$ is a non-zero 1-form such that $A(X) = g(X, \rho)$, $\rho$ is the associated vector field of the 1-form $A$. 
In particular, if the 1-form $A$ vanishes then the Ricci tensor of the LP-Sasakian manifold is said to be $\eta$-parallel and this notion was first introduced by Kon ([3]) for Sasakian manifolds. Hence the notion of $\eta$-recurrent Ricci tensor generalizes the notion of $\eta$-parallel Ricci tensor.

In ([7]), A. A. Shaikh and K. K. Baishya also studied various properties of LP-Sasakian manifolds with $\eta$-recurrent Ricci tensor. We first construct an example of LP-Sasakian manifold with global vector fields whose Ricci tensor is $\eta$-parallel.

**Example 3.1.** We consider a 4-dimensional manifold $M = \{(x, y, z, u) \in \mathbb{R}^4\}$, where $(x, y, z, u)$ are the standard coordinates of $\mathbb{R}^4$. Let $\{E_1, E_2, E_3, E_4\}$ be linearly independent global frame on $M$ given by
\[
E_1 = e^u \frac{\partial}{\partial x}, \quad E_2 = e^u \frac{\partial}{\partial y}, \quad E_3 = e^u \frac{\partial}{\partial z}, \quad E_4 = \frac{\partial}{\partial u}.
\]
Let $g$ be the Lorentzian metric defined by
\[
g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_4) = g(E_2, E_4) = g(E_3, E_4) = g(E_1, E_2) = 0,
\[
g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1, \quad g(E_4, E_4) = -1.
\]
Let $\eta$ be the 1-form defined by $\eta(U) = g(U, E_4)$ for any $U \in \chi(M)$. Let $\phi$ be the $(1, 1)$ tensor field defined by $\phi E_1 = -E_1$, $\phi E_2 = -E_2$, $\phi E_3 = -E_3$, $\phi E_4 = 0$. Then using the linearity of $\phi$ and $g$ we have $\eta(E_4) = -1$, $\phi^2 U = U + \eta(U)E_4$ and $g(\phi U, \phi W) = g(U, W) + \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Thus for $E_4 = \xi$, $(\phi, \xi, \eta, g)$ defines a Lorentzian paracontact structure on $M$.

Let $\nabla$ be the Levi-Civita connection with respect to the Lorentzian metric $g$ and $R$ be the curvature tensor of $g$. Then we have
\[
\]
Taking $E_4 = \xi$ and using Koszul formula for the Lorentzian metric $g$, we can easily calculate
\[
\nabla_{E_1} E_4 = -E_1, \quad \nabla_{E_2} E_4 = -E_4, \quad \nabla_{E_3} E_4 = -E_3, \quad \nabla_{E_4} E_4 = -E_4.
\]
From the above it can be easily seen that $(\phi, \xi, \eta, g)$ is an LP-Sasakian structure on $M$. Consequently $M^3(\phi, \xi, \eta, g)$ is an LP-Sasakian manifold. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:
\[
R(E_1, E_3) E_1 = -E_3, \quad R(E_1, E_3) E_3 = E_1, \quad R(E_1, E_4) E_1 = -E_4,
\]
\[
R(E_1, E_4) E_4 = -E_1, \quad R(E_2, E_3) E_3 = E_2, \quad R(E_2, E_3) E_2 = -E_3,
\]
\[
R(E_2, E_4) E_2 = -E_4, \quad R(E_3, E_4) E_3 = -E_4, \quad R(E_3, E_4) E_4 = -E_3,
\]
\[
R(E_2, E_4) E_4 = -E_2, \quad R(E_1, E_2) E_2 = E_1, \quad R(E_1, E_2) E_1 = -E_2.
\]
and the components which can be obtained from these by the symmetry properties. From the above, we can easily calculate the non-vanishing components of the Ricci tensor $S$ as follows:

$$S(E_1, E_1) = 1, \quad S(E_2, E_2) = 1, \quad S(E_3, E_3) = 1, \quad S(E_4, E_4) = -3.$$ 

Since $\{E_1, E_2, E_3, E_4\}$ forms a basis, any vector field $X, Y \in \chi(M)$ can be written as

$$X = a_1E_1 + b_1E_2 + c_1E_3 + d_1E_4$$

and

$$Y = a_2E_1 + b_2E_2 + c_2E_3 + d_2E_4,$$

where $a_i, b_i, c_i, d_i \in \mathbb{R}^+$ (the set of all positive real numbers), $i = 1, 2$. This implies that

$$\phi X = -a_1E_1 - b_1E_2 - c_1E_3$$

and

$$\phi Y = -a_2E_1 - b_2E_2 - c_2E_3.$$ 

Hence

$$S(\phi X, \phi Y) = (a_1a_2 + b_1b_2 + c_1c_2) \neq 0.$$ 

By virtue of the above we have the following:

$$(\nabla_{E_i} S)(\phi X, \phi Y) = 0 \quad \text{for} \quad i = 1, 2, 3, 4.$$ 

This leads to the following:

**Theorem 3.1.** There exists an LP-Sasakian manifold $(M^4, g)$ with $\eta$-parallel Ricci tensor.

We now construct examples of LP-Sasakian manifolds with $\eta$-recurrent but not $\eta$-parallel Ricci tensor.

**Example 3.2.** We consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3\}$, where $(x, y, z)$ are the standard coordinates of $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame on $M$ given by

$$E_1 = e^z \frac{\partial}{\partial x}, \quad E_2 = e^{-a_2} \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z},$$

where $a$ is a non-zero constant.

Let $g$ be the Lorentzian metric defined by $g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0$, $g(E_1, E_1) = g(E_2, E_2) = 1$, $g(E_3, E_3) = -1$. Let $\eta$ be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \chi(M)$. Let $\phi$ be the $(1, 1)$ tensor field defined by $\phi E_1 = -E_1$, $\phi E_2 = -E_2$, $\phi E_3 = 0$. Then using the linearity of $\phi$ and $g$ we have $\eta(E_3) = -1$, $\phi^2 U = U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U, W) + \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Thus for $E_3 = \xi$, $(\phi, \xi, \eta, g)$ defines a Lorentzian paracontact structure on $M$.

Let $\nabla$ be the Levi-Civita connection with respect to the Lorentzian metric $g$ and $R$ be the curvature tensor of $g$. Then we have

$$[E_1, E_2] = -ae^z E_2, \quad [E_1, E_3] = -E_1, \quad [E_2, E_3] = -E_2.$$
Taking $E_3 = \xi$ and using Koszul formula for the Lorentzian metric $g$, we can easily calculate
\[\nabla_{E_1}E_3 = -E_1, \quad \nabla_{E_2}E_3 = -E_2, \quad \nabla_{E_3}E_3 = 0,\]
\[\nabla_{E_1}E_1 = -E_3, \quad \nabla_{E_2}E_2 = 0, \quad \nabla_{E_3}E_1 = ae^z E_2, \quad \nabla_{E_3}E_2 = -ae^z E_1 - E_3, \quad \nabla_{E_3}E_1 = 0.\]

From the above it can be easily seen that $(\phi, \xi, \eta, g)$ is an LP-Sasakian structure on $M$. Consequently $M^3(\phi, \xi, \eta, g)$ is an LP-Sasakian manifold. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor $R$ as follows:
\[R(E_2, E_3)E_3 = -E_2, \quad R(E_1, E_3)E_3 = -E_1, \quad R(E_1, E_2)E_2 = (1 - a^2 e^{2z})E_1, \]
\[R(E_2, E_3)E_2 = -E_3, \quad R(E_1, E_3)E_1 = -E_3, \quad R(E_1, E_2)E_1 = -(1 - a^2 e^{2z})E_2\]
and the components which can be obtained from these by the symmetry properties. From the above, we can easily calculate the non-vanishing components of the Ricci tensor $S$ as follows:
\[S(E_1, E_1) = -(ae^z)^2, \quad S(E_2, E_2) = -(ae^z)^2, \quad S(E_3, E_3) = -2.\]

Since $\{E_1, E_2, E_3\}$ forms a basis, any vector field $X, Y \in \chi(M)$ can be written as
\[X = a_1 E_1 + b_1 E_2 + c_1 E_3\]
and
\[Y = a_2 E_1 + b_2 E_2 + c_2 E_3,\]
where $a_i, b_i, c_i \in R^+$ (the set of all positive real numbers), $i = 1, 2$. This implies that
\[\phi X = -a_1 E_1 - b_1 E_2\]
and
\[\phi Y = -a_2 E_1 - b_2 E_2.\]

Hence
\[S(\phi X, \phi Y) = -(a_1 a_2 + b_1 b_2) (ae^z)^2.\]

By virtue of the above we have the following:
\[\nabla_{E_1}S(\phi X, \phi Y) = 0,\]
\[\nabla_{E_2}S(\phi X, \phi Y) = -(a_1 b_2 + a_2 b_1) (ae^z)^3,\]
\[\nabla_{E_3}S(\phi X, \phi Y) = -2(a_1 a_2 + b_1 b_2) (ae^z)^2.\]

Let us now consider the 1-forms
\[A(E_1) = 0, \quad A(E_2) = \frac{(a_1 b_2 + a_2 b_1)}{(a_1 a_2 + b_1 b_2)} (ae^z), \quad A(E_3) = 2.\]
at any point \( p \in M \). In our \( M^3 \), (3.1) reduces with these 1-forms to the following equations:

\[
(\nabla_{E_i} S)(\phi X, \phi Y) = A(E_i) S(\phi X, \phi Y), \quad i = 1, 2, 3.
\]

This implies that the manifold under consideration is an LP-Sasakian manifold with \( \eta \)-recurrent but not \( \eta \)-parallel Ricci tensor. This leads to the following:

**Theorem 3.2.** There exists an LP-Sasakian manifold \((M^3, g)\) with \( \eta \)-recurrent but not \( \eta \)-parallel Ricci tensor.

**Example 3.3.** We consider a 4-dimensional manifold \( M = \{(x, y, z, u) \in \mathbb{R}^4\} \), where \((x, y, z, u)\) are the standard coordinates of \( \mathbb{R}^4 \). Let \( \{E_1, E_2, E_3, E_4\} \) be linearly independent global frame on \( M \) given by

\[
E_1 = y e^{-u} \frac{\partial}{\partial y}, \quad E_2 = y e^{-u} \frac{\partial}{\partial x}, \quad E_3 = e^{-u} \frac{\partial}{\partial z}, \quad E_4 = \frac{\partial}{\partial u}.
\]

Let \( g \) be the Lorentzian metric defined by

\[
g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_4) = g(E_2, E_4) = g(E_3, E_4) = g(E_1, E_2) = 0,
\]

\[
g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1, \quad g(E_4, E_4) = -1.
\]

Let \( \eta \) be the 1-form defined by \( \eta(U) = g(U, E_4) \) for any \( U \in \chi(M) \). Let \( \phi \) be the \((1, 1)\) tensor field defined by \( \phi E_1 = -E_1, \phi E_2 = -E_2, \phi E_3 = -E_3, \phi E_4 = 0 \). Then using the linearity of \( \phi \) and \( g \), we have \( \eta(E_4) = -1, \phi^2 U = U + \eta(U) E_4 \) and \( g(\phi U, \phi W) = g(U, W) + \eta(U) \eta(W) \) for any \( U, W \in \chi(M) \). Thus for \( E_4 = \xi, (\phi, \xi, \eta, g) \) defines a Lorentzian paracontact structure on \( M \).

Let \( \nabla \) be the Levi-Civita connection with respect to the Lorentzian metric \( g \) and \( R \) be the curvature tensor of \( g \). Then we have

\[
\{E_1, E_2\} = e^{-u} E_2, \quad \{E_1, E_4\} = E_1, \quad \{E_2, E_4\} = E_2, \quad \{E_3, E_4\} = E_3.
\]

Taking \( E_1 = \xi \) and using Koszul formula for the Lorentzian metric \( g \), we can easily calculate

\[
\nabla_{E_1} E_4 = E_1, \quad \nabla_{E_2} E_2 = E_4 + e^{-u} E_1, \quad \nabla_{E_3} E_1 = -E_2,
\]

\[
\nabla_{E_3} E_3 = E_3, \quad \nabla_{E_2} E_1 = E_4, \quad \nabla_{E_2} E_4 = E_2, \quad \nabla_{E_3} E_3 = E_4.
\]

From the above it can be easily seen that \((\phi, \xi, \eta, g)\) is an LP-Sasakian structure on \( M \). Consequently \( M^3(\phi, \xi, \eta, g) \) is an LP-Sasakian manifold. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

\[
R(E_1, E_3)E_1 = -E_3, \quad R(E_1, E_4)E_1 = -E_4,
\]

\[
R(E_1, E_4)E_4 = -E_1,
\]

\[
R(E_2, E_4)E_2 = -E_4, \quad R(E_3, E_4)E_3 = -E_4, \quad R(E_3, E_4)E_4 = -E_3,
\]

\[
R(E_2, E_4)E_4 = -E_2, \quad R(E_1, E_2)E_2 = (1 - e^{-2u}) E_1, \quad R(E_1, E_2)E_1 = -(1 - e^{-2u}) E_2.
\]
and the components which can be obtained from these by the symmetry properties. From the above, we can easily calculate the non-vanishing components of the Ricci tensor $S$ as follows:

$$S(E_1, E_1) = (1 - e^{-2u}), \quad S(E_2, E_2) = (1 - e^{-2u}), \quad S(E_3, E_3) = 1, \quad S(E_4, E_4) = -3.$$  

Since $\{E_1, E_2, E_3, E_4\}$ forms a basis, any vector field $X, Y \in \chi(M)$ can be written as

$$X = \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4$$

and

$$Y = \beta_1 E_1 + \beta_2 E_2 + \beta_3 E_3 + \beta_4 E_4,$$

where $\alpha_i, \beta_i \in \mathbb{R}^+$ (the set of all positive real numbers), $i = 1, 2, 3, 4$. This implies that

$$\phi X = -\alpha_1 E_1 - \alpha_2 E_2 - \alpha_3 E_3$$

and

$$\phi Y = -\beta_1 E_1 - \beta_2 E_2 - \beta_3 E_3.$$  

Hence

$$S(\phi X, \phi Y) = (\alpha_1 \beta_1 + \alpha_2 \beta_2)(1 - e^{-2u}) + \alpha_3 \beta_3.$$  

By virtue of the above we have the following:

$$(\nabla_{E_1} S)(\phi X, \phi Y) = 0,$$

$$(\nabla_{E_2} S)(\phi X, \phi Y) = -(\alpha_1 \beta_2 + \alpha_2 \beta_1) e^{-3u},$$

$$(\nabla_{E_3} S)(\phi X, \phi Y) = 0,$$

$$(\nabla_{E_4} S)(\phi X, \phi Y) = 2(\alpha_1 \beta_1 + \alpha_2 \beta_2) e^{-2u}.$$  

Let us now consider the 1-forms

$$A(E_1) = 0,$$

$$A(E_2) = -\frac{(\alpha_1 \beta_2 + \alpha_2 \beta_1) e^{-3u}}{(\alpha_1 \beta_1 + \alpha_2 \beta_2)(1 - e^{-2u}) + \alpha_3 \beta_3},$$

$$A(E_3) = 0,$$

$$A(E_4) = \frac{2(\alpha_1 \beta_1 + \alpha_2 \beta_2) e^{-2u}}{(\alpha_1 \beta_1 + \alpha_2 \beta_2)(1 - e^{-2u}) + \alpha_3 \beta_3}$$

at any point $p \in M$. In our $M^4$, (3.1) reduces with these 1-forms to the following equations:

$$(\nabla_{E_i} S)(\phi X, \phi Y) = A(E_i) S(\phi X, \phi Y), \quad i = 1, 2, 3, 4.$$  

This implies that the manifold under consideration is an LP-Sasakian manifold with $\eta$-recurrent but not $\eta$-parallel Ricci tensor. This leads to the following:

**Theorem 3.3.** There exists an LP-Sasakian manifold $(M^4, g)$ with $\eta$-recurrent but not $\eta$-parallel Ricci tensor.
4. LP-Sasakian manifolds with $\phi$-recurrent Ricci tensor

**Definition 4.1** ([7]). The Ricci tensor $S$ of an LP-Sasakian manifold is said to be $\phi$-recurrent if it satisfies

\[(\nabla_{\phi X} S)(\phi Y, \phi Z) = A(\phi X) S(\phi Y, \phi Z)\]

for all $X, Y, Z$ where $A$ is a non-zero 1-form.

In particular, if the 1-form $A$ vanishes then the Ricci tensor of the LP-Sasakian manifold is said to be $\phi$-parallel. We note that the condition of Ricci-$\phi$-parallelity is much more weaker than Ricci-$\eta$-parallelity.

In ([7]), A. A. Shaikh and K. K. Baishya also studied several properties of LP-Sasakian manifolds with $\phi$-recurrent Ricci tensor. We first construct an example of LP-Sasakian manifold with global vector fields whose Ricci tensor is $\phi$-parallel.

**Example 4.1.** We consider a 4-dimensional manifold $M = \{(x, y, z, u) \in R^4\}$, where $(x, y, z, u)$ are the standard coordinates of $R^4$. Let $\{E_1, E_2, E_3, E_4\}$ be linearly independent global frame on $M$ given by

\[E_1 = e^{-u} \frac{\partial}{\partial x}, \quad E_2 = e^{-u} \frac{\partial}{\partial y}, \quad E_3 = e^{-u} \frac{\partial}{\partial z}, \quad E_4 = \frac{\partial}{\partial u}.\]

Let $g$ be the Lorentzian metric defined by

\[g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_4) = g(E_2, E_4) = g(E_3, E_4) = g(E_1, E_2) = 0, \quad g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1, \quad g(E_4, E_4) = -1.\]

Let $\eta$ be the 1-form defined by $\eta(U) = g(U, E_4)$ for any $U \in \chi(M)$. Let $\phi$ be the $(1, 1)$ tensor field defined by $\phi E_1 = -E_1$, $\phi E_2 = -E_2$, $\phi E_3 = -E_3$, $\phi E_4 = 0$. Then using the linearity of $\phi$ and $g$ we have $\eta(E_4) = -1$, $\phi^2 U = U + \eta(U) E_4$ and $g(\phi U, \phi W) = g(U, W) + \eta(U) \eta(W)$ for any $U, W \in \chi(M)$. Thus for $E_4 = \xi$, $(\phi, \xi, \eta, g)$ defines a Lorentzian paracontact structure on $M$.

Let $\nabla$ be the Levi-Civita connection with respect to the Lorentzian metric $g$ and $R$ be the curvature tensor of $g$. Then we have

\[[E_1, E_4] = E_1, \quad [E_2, E_4] = E_2, \quad [E_3, E_4] = E_3.\]

Taking $E_4 = \xi$ and using Koszul formula for the Lorentzian metric $g$, we can easily calculate

\[\nabla_{E_1} E_4 = E_1, \quad \nabla_{E_2} E_4 = E_2, \quad \nabla_{E_3} E_4 = E_3, \quad \nabla_{E_4} E_1 = E_4, \quad \nabla_{E_2} E_2 = E_2, \quad \nabla_{E_3} E_3 = E_3.\]

From the above it can be easily seen that $(\phi, \xi, \eta, g)$ is an LP-Sasakian structure on $M$. Consequently $M^4(\phi, \xi, \eta, g)$ is an LP-Sasakian manifold. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

\[R(E_1, E_3) E_1 = -E_3, \quad R(E_1, E_3) E_3 = E_1, \quad R(E_1, E_4) E_1 = -E_4, \quad R(E_1, E_4) E_4 = -E_1, \quad R(E_2, E_3) E_3 = E_2, \quad R(E_2, E_3) E_2 = -E_3, \quad R(E_2, E_4) E_2 = -E_3, \quad R(E_2, E_4) E_3 = E_2.\]
There exists an LP-Sasakian manifold implies that let $\eta$ be the 1-form defined by $\eta(U) = g(U, E_4)$ for any $U \in \chi(M)$. Let $\phi$ be the $(1, 1)$ tensor field defined by $\phi E_1 = -E_1$, $\phi E_2 = -E_2$, $\phi E_3 = -E_3$, $\phi E_4 = 0$. Then using the linearity of $\phi$ and $g$ we have $\eta(E_4) = -1$, $\phi^2 U = U + \eta(U) E_4$ and $g(\phi U, \phi W) = g(U, W) + \eta(U) \eta(W)$ for any $U, W \in \chi(M)$. Thus for $E_4 = \xi$, $(\phi, \xi, \eta, g)$ defines a Lorentzian paracontact structure on $M$.

By virtue of the above we have the following:

$$S(E_1, E_1) = 1, \quad S(E_2, E_2) = 1, \quad S(E_3, E_3) = 1, \quad S(E_4, E_4) = -3.$$ 

Since $\{E_1, E_2, E_3, E_4\}$ forms a basis of the LP-Sasakian manifold, any vector field $X, Y \in \chi(M)$ can be written as

$$X = a_1 E_1 + b_1 E_2 + c_1 E_3 + d_1 E_4$$

and

$$Y = a_2 E_1 + b_2 E_2 + c_2 E_3 + d_2 E_4,$$

where $a_i, b_i, c_i, d_i \in R^+$ (the set of all positive real numbers), $i = 1, 2$. This implies that

$$\phi X = -a_1 E_1 - b_1 E_2 - c_1 E_3$$

and

$$\phi Y = -a_2 E_1 - b_2 E_2 - c_2 E_3.$$ 

Hence

$$S(\phi X, \phi Y) = (a_1 a_2 + b_1 b_2 + c_1 c_2) \neq 0.$$ 

By virtue of the above we have the following:

$$(\nabla_{\phi E_i} S)(\phi X, \phi Y) = 0 \quad \text{for} \quad i = 1, 2, 3.$$ 

Hence the Ricci tensor of the manifold under consideration is $\phi$-parallel. Thus we can state the following:

**Theorem 4.1.** There exists an LP-Sasakian manifold $(M^4, g)$ with $\phi$-parallel Ricci tensor.

**Example 4.2.** We consider a 4-dimensional manifold $M = \{(x, y, z, u) \in R^4\}$, where $(x, y, z, u)$ are the standard coordinates of $R^4$. Let $\{E_1, E_2, E_3, E_4\}$ be linearly independent global frame on $M$ given by

$$E_1 = e^u \frac{\partial}{\partial x}, \quad E_2 = e^{u-x} \frac{\partial}{\partial y}, \quad E_3 = e^u \frac{\partial}{\partial z}, \quad E_4 = \frac{\partial}{\partial u},$$

Let $g$ be the Lorentzian metric defined by

$$g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_4) = g(E_2, E_4) = g(E_3, E_4) = g(E_1, E_2) = 0,$$

$$g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1, \quad g(E_4, E_4) = -1.$$ 

Let $\eta$ be the 1-form defined by $\eta(U) = g(U, E_4)$ for any $U \in \chi(M)$. Let $\phi$ be the (1, 1) tensor field defined by $\phi E_1 = -E_1$, $\phi E_2 = -E_2$, $\phi E_3 = -E_3$, $\phi E_4 = 0$. Then using the linearity of $\phi$ and $g$ we have $\eta(E_4) = -1$, $\phi^2 U = U + \eta(U) E_4$ and $g(\phi U, \phi W) = g(U, W) + \eta(U) \eta(W)$ for any $U, W \in \chi(M)$. Thus for $E_4 = \xi$, $(\phi, \xi, \eta, g)$ defines a Lorentzian paracontact structure on $M$. 

$$R(E_2, E_4)E_2 = -E_4, \quad R(E_3, E_4)E_3 = -E_4, \quad R(E_3, E_4)E_4 = -E_3,$$

$$R(E_2, E_4)E_4 = -E_2, \quad R(E_1, E_2)E_2 = E_1, \quad R(E_1, E_2)E_1 = -E_2$$

and the components which can be obtained from these by the symmetry properties. From the components of $R$, we can easily calculate the non-vanishing components of the Ricci tensor $S$ as follows:

$$S(E_1, E_1) = 1, \quad S(E_2, E_2) = 1, \quad S(E_3, E_3) = 1, \quad S(E_4, E_4) = -3.$$ 

Ricci tensor.
Let $\nabla$ be the Levi-Civita connection with respect to the Lorentzian metric $g$ and $R$ be the curvature tensor of $g$. Then we have

$$[E_1, E_2] = -e^u E_2, \quad [E_1, E_4] = -E_1, \quad [E_2, E_4] = -E_2, \quad [E_3, E_4] = -E_3.$$ 

Taking $E_4 = \xi$ and using Koszul formula for the Lorentzian metric $g$, we can easily calculate

$$\nabla_{E_1} E_4 = -E_1, \quad \nabla_{E_2} E_2 = -E_4 - e^u E_1, \quad \nabla_{E_2} E_1 = -e^u E_2,$$

$$\nabla_{E_3} E_4 = -E_3, \quad \nabla_{E_1} E_1 = -E_4, \quad \nabla_{E_2} E_4 = -E_2, \quad \nabla_{E_3} E_3 = -E_4.$$ 

From the above it can be easily seen that $(\phi, \xi, \eta, g)$ is an LP-Sasakian manifold. Consequently $M^4(\phi, \xi, \eta, g)$ is an LP-Sasakian structure on $M$. Consequently $M^4(\phi, \xi, \eta, g)$ is an LP-Sasakian manifold. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$R(E_2, E_3)E_3 = E_2, \quad R(E_1, E_3)E_3 = E_1, \quad R(E_1, E_4)E_1 = -E_4,$$

$$R(E_1, E_4)E_4 = -E_1, \quad R(E_2, E_4)E_2 = -E_3, \quad R(E_1, E_3)E_1 = -E_3,$$

$$R(E_2, E_4)E_2 = -E_4, \quad R(E_3, E_4)E_3 = -E_4, \quad R(E_3, E_4)E_4 = -E_3, \quad R(E_2, E_4)E_4 = -E_2, \quad R(E_1, E_2)E_2 = (1-e^{2u})E_1, \quad R(E_1, E_2)E_1 = -(1-e^{2u})E_2$$

and the components which can be obtained from these by the symmetry properties. From the above, we can easily calculate the non-vanishing components of the Ricci tensor $S$ as follows:

$$S(E_1, E_1) = (1-e^{2u}), \quad S(E_2, E_2) = (1-e^{2u}), \quad S(E_3, E_3) = 1, \quad S(E_4, E_4) = -3.$$ 

Since $\{E_1, E_2, E_3, E_4\}$ forms a basis of the LP-Sasakian manifold, any vector field $X, Y \in \chi(M)$ can be written as

$$(4.2) \quad X = \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4 \quad \text{and} \quad Y = \beta_1 E_1 + \beta_2 E_2 + \beta_3 E_3 + \beta_4 E_4,$$

where $\alpha_i, \beta_i \in R^+$ (the set of all positive real numbers), $i = 1, 2, 3, 4$. This implies that

$$\phi X = -\alpha_1 E_1 - \alpha_2 E_2 - \alpha_3 E_3$$

and

$$\phi Y = -\beta_1 E_1 - \beta_2 E_2 - \beta_3 E_3.$$

Hence

$$S(\phi X, \phi Y) = (\alpha_1 \beta_1 + \alpha_2 \beta_2)(1-e^{2u}) + \alpha_3 \beta_3.$$ 

By virtue of the above we have the following:

$$(\nabla_{\phi E_1} S)(\phi X, \phi Y) = 0,$$

$$(\nabla_{\phi E_2} S)(\phi X, \phi Y) = (\alpha_1 \beta_2 + \alpha_2 \beta_1)e^{3u},$$

$$(\nabla_{\phi E_1} S)(\phi X, \phi Y) = 0.$$
Let us now consider the 1-forms

\[ A(E_1) = 0, \]
\[ A(E_2) = \frac{(\alpha_1\beta_2 + \alpha_2\beta_1)e^{3u}}{\alpha_1\beta_1 + \alpha_2\beta_2(1 - e^{2u}) + \alpha_3\beta_3}, \]
\[ A(E_3) = 0 \]

at any point \( p \in M \). In our \( M^4 \), (4.1) reduces with these 1-forms to the following equations:

\[ (\nabla_{\phi E_i} S)(\phi X, \phi Y) = A(E_i)S(\phi X, \phi Y), \quad i = 1, 2, 3. \]

This implies that the Ricci tensor of the manifold under consideration is \( \phi \)-recurrent but not \( \phi \)-parallel. This leads to the following:

**Theorem 4.2.** There exists an LP-Sasakian manifold \( (M^4, g) \) with \( \phi \)-recurrent Ricci tensor but not \( \phi \)-parallel.

However, since \( \{E_1, E_2, E_3, E_4\} \) is a basis of \( M^4 \), if we consider the vector fields \( X, Y \in \chi(M) \) in (4.2) such that \( \alpha_2 = k\alpha_1 \) and \( \beta_2 = -k\beta_1 \) where \( k \in \mathbb{R} - \{1, 0, -1\} \), then we have

\[ \phi X = -\alpha_1E_1 - k\alpha_1E_2 - \alpha_3E_3 \]

and

\[ \phi Y = -\beta_1E_1 + k\beta_1E_2 - \beta_3E_3. \]

Consequently, we get

\[ S(\phi X, \phi Y) = (1 - k^2)(1 - e^{2u})\alpha_1\beta_1 + \alpha_3\beta_3 \neq 0, \]

\[ (\nabla_{\phi E_i} S)(\phi X, \phi Y) = 0, \quad i = 1, 2, 3, 4 \]

and

\[ (\nabla_{E_i} S)(\phi X, \phi Y) = 0 \quad i = 1, 2, 3, \]

\[ = -2(1 - k^2)e^{2u}\alpha_1\beta_1 \quad \text{for} \quad i = 4, \]

for all \( X, Y \in \chi(M) \) and hence the Ricci tensor \( S \) of \( M^4 \) is \( \phi \)-parallel but not \( \eta \)-parallel. This leads to the following:

**Theorem 4.3.** There exists an LP-Sasakian manifold \( (M^4, g) \) with \( \phi \)-parallel Ricci tensor but not \( \eta \)-parallel.

5. Generalized Ricci recurrent LP-Sasakian manifolds

**Definition 5.1** ([1]). An LP-Sasakian manifold is said to be generalized Ricci recurrent if its Ricci tensor \( S \) of type \((0, 2)\) satisfies the condition

\[ (\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(X)g(Y, Z), \]

where \( A \) and \( B \) are two non-zero 1-forms such that \( A(X) = g(X, P) \) and \( B(X) = g(X, L) \), \( P \) and \( L \) being associated vector fields of the 1-form.
Theorem 5.1. In a generalized Ricci recurrent LP-Sasakian manifold the associated 1-forms are linearly dependent and the vector fields of the associated 1-forms are of opposite direction.

Proof. In a generalized Ricci recurrent LP-Sasakian manifold we have the relation (5.1). Setting $Z = \xi$ in (5.1) we have
\[
(\nabla_X S)(Y, \xi) = [(n - 1)A(X) + B(X)]\eta(Y).
\]
Again
\[
(\nabla_X S)(Y, \xi) = \nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi),
\]
which yields by virtue of (2.1), (2.2) and (2.5) that
\[
(\nabla_X S)(Y, \xi) = (n - 1)g(X, \phi Y) - S(\phi X, Y).
\]
From (5.2) and (5.3) it follows that
\[
[n - 1]A(X) + B(X)]\eta(Y) = (n - 1)g(X, \phi Y) - S(\phi X, Y).
\]
Replacing $Y$ by $\xi$ in (5.4) we obtain
\[
(n - 1)A(X) + B(X) = 0.
\]
This proves the Theorem.

Theorem 5.2. A generalized Ricci recurrent LP-Sasakian manifold is Einstein.

Proof. In a generalized Ricci recurrent LP-Sasakian manifold we have the relation (5.4). Hence setting $Y = \phi Y$ in (5.4) and then using (2.8) we have
\[
S(X, Y) = (n - 1)g(X, Y).
\]
This proves the Theorem.

Example 5.1. We consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3\}$, where $(x, y, z)$ are the standard coordinates of $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame on $M$ given by
\[
E_1 = e^z \frac{\partial}{\partial y}, \quad E_2 = e^z \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \quad E_3 = \frac{\partial}{\partial z}.
\]
Let $g$ be the Lorentzian metric defined by $g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0$, $g(E_1, E_1) = g(E_2, E_2) = 1$, $g(E_3, E_3) = -1$. Let $\eta$ be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \chi(M)$. Let $\phi$ be the $(1, 1)$ tensor field defined by $\phi E_1 = -E_1$, $\phi E_2 = -E_2$, $\phi E_3 = 0$. Then using the linearity of $\phi$ and $g$ we have $\eta(E_3) = -1$, $\phi^2 U = U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U, W) + \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Thus for $E_3 = \xi$, $(\phi, \xi, \eta, g)$ defines a Lorentzian paracontact structure on $M$.

Let $\nabla$ be the Levi-Civita connection with respect to the Lorentzian metric $g$ and $R$ be the curvature tensor of $g$. Then we have
\[
[E_1, E_2] = 0, \quad [E_1, E_3] = -E_1, \quad [E_2, E_3] = -E_2.
\]
Taking $E_3 = \xi$ and using Koszul formula for the Lorentzian metric $g$, we can easily calculate
\[
\nabla_{E_1} E_3 = -E_1, \quad \nabla_{E_2} E_3 = -E_2, \quad \nabla_{E_3} E_3 = -E_3, \quad \nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_2} E_1 = 0, \quad \nabla_{E_3} E_2 = 0.
\]
From the above it can be easily seen that $(\phi, \xi, \eta, g)$ is an LP-Sasakian structure on $M$. Consequently $M^3(\phi, \xi, \eta, g)$ is an LP-Sasakian manifold. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:
\[
R(E_2, E_3)E_3 = -E_2, \quad R(E_1, E_3)E_3 = -E_1, \quad R(E_1, E_2)E_2 = -E_1,
\]
and the components which can be obtained from these by the symmetry properties. From the above, we can easily calculate the non-vanishing components of the Ricci tensor $S$ as follows:
\[
S(E_3, E_3) = -2.
\]
Since $\{E_1, E_2, E_3\}$ forms a basis of the LP-Sasakian manifold, any vector field $X, Y \in \chi(M)$ can be written as
\[
X = a_1 E_1 + b_1 E_2 + c_1 E_3
\]
and
\[
Y = a_2 E_1 + b_2 E_2 + c_2 E_3,
\]
where $a_i, b_i, c_i \in \mathbb{R}^+$ (the set of all positive real numbers), $i = 1, 2$. Hence
\[
S(X, Y) = -2c_1 c_2 \quad \text{and} \quad g(X, Y) = a_1 a_2 + b_1 b_2 - c_1 c_2.
\]
By virtue of the above we have the following:
\[
(\nabla_{E_1} S)(X, Y) = -2(a_1 c_2 + a_2 c_1), \quad (\nabla_{E_2} S)(X, Y) = -2(b_1 c_2 + b_2 c_1),
\]
\[
\quad \text{and} \quad (\nabla_{E_3} S)(X, Y) = 0.
\]
Consequently, the manifold under consideration is not Ricci symmetric. Let us now consider the 1-forms
\[
A(E_1) = \frac{(a_1 c_2 + a_2 c_1)}{(a_1 a_2 + b_1 b_2)}, \quad B(E_1) = \frac{-2(a_1 c_2 + a_2 c_1)}{(a_1 a_2 + b_1 b_2)},
\]
\[
A(E_2) = \frac{(b_1 c_2 + b_2 c_1)}{(a_1 a_2 + b_1 b_2)}, \quad B(E_2) = \frac{-2(b_1 c_2 + b_2 c_1)}{(a_1 a_2 + b_1 b_2)},
\]
\[
A(E_3) = 0, \quad B(E_3) = 0
\]
at any point $x \in M$. From (5.1) we have
\[
(\nabla_{E_i} S)(X, Y) = A(E_i) S(X, Y) + B(E_i) g(X, Y), \quad i = 1, 2, 3.
\]
It can be easily shown that the manifold with these 1-forms satisfies the relation (5.7). Hence the manifold under consideration is a generalized Ricci recurrent...
LP-Sasakian manifold which is neither Ricci-symmetric nor Ricci-recurrent. This leads to the following:

**Theorem 5.3.** There exists a generalized Ricci recurrent LP-Sasakian manifold \((M^3, g)\) which is neither Ricci-symmetric nor Ricci-recurrent.

**Example 5.2.** We consider a 4-dimensional manifold \(M = \{(x, y, z, u) \in \mathbb{R}^4\}\), where \((x, y, z, u)\) are the standard coordinates of \(\mathbb{R}^4\). Let \(\{E_1, E_2, E_3, E_4\}\) be linearly independent global frame on \(M\) given by

\[
E_1 = u \frac{\partial}{\partial y}, \quad E_2 = u \left( \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} \right), \quad E_3 = u \frac{\partial}{\partial z}, \quad E_4 = \frac{\partial}{\partial u}.
\]

Let \(g\) be the Lorentzian metric defined by

\[
g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_4) = g(E_2, E_4) = g(E_3, E_4) = g(E_1, E_2) = 0,
\]

\[
g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1, \quad g(E_4, E_4) = -1.
\]

Let \(\eta\) be the 1-form defined by \(\eta(U) = g(U, E_4)\) for any \(U \in \chi(M)\). Let \(\phi\) be the \((1, 1)\) tensor field defined by \(\phi E_1 = -E_1, \phi E_2 = -E_2, \phi E_3 = -E_3, \phi E_4 = 0\). Then using the linearity of \(\phi\) and \(g\) we have \(\eta(E_4) = -1, \phi^2 U = U + \eta(U)E_4\) and \(g(\phi U, \phi W) = g(U, W) + \eta(U)\eta(W)\) for any \(U, W \in \chi(M)\). Thus for \(E_4 = \xi, \phi, \eta, g\) defines a Lorentzian paracontact structure on \(M\).

Let \(\nabla\) be the Levi-Civita connection with respect to the Lorentzian metric \(g\) and \(R\) be the curvature tensor of \(g\). Then we have

\[
\]

Taking \(E_4 = \xi\) and using Koszul formula for the Lorentzian metric \(g\), we can easily calculate

\[
\nabla_{E_1} E_4 = -E_1, \quad \nabla_{E_2} E_4 = -E_2, \quad \nabla_{E_3} E_4 = -E_4 - uE_2, \quad \\
\nabla_{E_3} E_4 = -E_3, \quad \nabla_{E_1} E_4 = -E_4, \quad \nabla_{E_2} E_4 = -uE_2, \quad \nabla_{E_2} E_3 = -E_4.
\]

From the above it can be easily seen that \((\phi, \xi, \eta, g)\) is an LP-Sasakian structure on \(M\). Consequently \(M^4(\phi, \xi, \eta, g)\) is an LP-Sasakian manifold. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

\[
R(E_1, E_2)E_1 = -E_2, \quad R(E_1, E_2)E_2 = E_1, \quad R(E_1, E_3)E_1 = -E_3, \quad R(E_1, E_4)E_1 = -E_1,
\]

\[
R(E_2, E_3)E_4 = E_1, \quad R(E_1, E_4)E_1 = -E_4, \quad R(E_1, E_4)E_4 = -E_1, \quad R(E_2, E_3)E_4 = -E_3,
\]

\[
R(E_2, E_4)E_4 = -E_2, \quad R(E_2, E_3)E_3 = (1-u^2)E_2, \quad R(E_2, E_3)E_2 = -(1-u^2)E_3.
\]

and the components which can be obtained from these by the symmetry properties. From the above, we can easily calculate the non-vanishing components of the Ricci tensor \(S\) as follows:

\[
S(E_1, E_1) = 1, \quad S(E_2, E_2) = (1-u^2), \quad S(E_3, E_3) = (1-u^2), \quad S(E_4, E_4) = -3.
\]
Since \( \{E_1, E_2, E_3\} \) forms a basis of the LP-Sasakian manifold, any vector field \( X, Y \in \chi(M) \) can be written as
\[
X = a_1 E_1 + b_1 E_2 + c_1 E_3 + d_1 E_4
\]
and
\[
Y = a_2 E_1 + b_2 E_2 + c_2 E_3 + d_2 E_4,
\]
where \( a_i, b_i, c_i, d_i \in \mathbb{R}^+ \) (the set of all positive real numbers), \( i = 1, 2 \). Hence
\[
S(X, Y) = (a_1 a_2 - 3 d_1 d_2) + (b_1 b_2 + c_1 c_2)(1 - u^2)
\]
and
\[
g(X, Y) = a_1 a_2 + b_1 b_2 + c_1 c_2 - d_1 d_2.
\]
By virtue of the above we have the following:
\[
(\nabla_{E_i} S)(X, Y) = -2(a_1 d_2 + a_2 d_1)
\]
\[
(\nabla_{E_2} S)(X, Y) = -2(b_1 d_2 + b_2 d_1)
\]
\[
(\nabla_{E_3} S)(X, Y) = -(u^2 + 2)(c_1 d_2 + c_2 d_1)
\]
\[
(\nabla_{E_4} S)(X, Y) = -2a u^2(b_1 b_2 + c_1 c_2).
\]
This implies that the manifold under consideration is not Ricci symmetric. Let us now consider the 1-forms
\[
A(E_1) = \frac{2(a_1 d_2 + a_2 d_1)}{2a_1 a_2 + (u^2 + 2)(b_1 b_2 + c_1 c_2)},
\]
\[
B(E_1) = -\frac{6(a_1 d_2 + a_2 d_1)}{2a_1 a_2 + (u^2 + 2)(b_1 b_2 + c_1 c_2)},
\]
\[
A(E_2) = \frac{2(b_1 d_2 + b_2 d_1)}{2a_1 a_2 + (u^2 + 2)(b_1 b_2 + c_1 c_2)},
\]
\[
B(E_2) = -\frac{6(b_1 d_2 + b_2 d_1)}{2a_1 a_2 + (u^2 + 2)(b_1 b_2 + c_1 c_2)},
\]
\[
A(E_3) = -\frac{(u^2 + 2)(c_1 d_2 + c_2 d_1)}{2a_1 a_2 + (u^2 + 2)(b_1 b_2 + c_1 c_2)},
\]
\[
B(E_3) = \frac{3(u^2 + 2)(c_1 d_2 + c_2 d_1)}{2a_1 a_2 + (u^2 + 2)(b_1 b_2 + c_1 c_2)},
\]
\[
A(E_4) = \frac{2a_1 a_2 + (u^2 + 2)(b_1 b_2 + c_1 c_2)}{2a_1 a_2 + (u^2 + 2)(b_1 b_2 + c_1 c_2)},
\]
\[
B(E_4) = -\frac{6a u^2(b_1 b_2 + c_1 c_2)}{2a_1 a_2 + (u^2 + 2)(b_1 b_2 + c_1 c_2)}.
\]
at any point \( x \in M \). From (5.1) we have
\[
(5.8) \quad (\nabla_{E_i} S)(X, Y) = A(E_i)S(X, Y) + B(E_i)g(X, Y), \quad i = 1, 2, 3, 4.
\]
It can be easily shown that the manifold with the 1-forms under consideration satisfies the relation (5.8). Hence the manifold under consideration is a generalized Ricci recurrent LP-Sasakian manifold which is neither Ricci-symmetric nor Ricci-recurrent. This leads to the following:

**Theorem 5.4.** There exists a generalized Ricci recurrent LP-Sasakian manifold $(M^4, g)$ which is neither Ricci-symmetric nor Ricci-recurrent.

**References**


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